On the quantitative asymptotic behavior of strongly nonexpansive mappings in Banach and geodesic spaces

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Abstract

We give explicit rates of asymptotic regularity for iterations of strongly nonexpansive mappings T in general Banach spaces as well as rates of metastability (in the sense of Tao) in the context of uniformly convex Banach spaces when T is odd. This, in particular, applies to linear norm-one projections as well as to sunny nonexpansive retractions. The asymptotic regularity results even hold for strongly quasi-nonexpansive mappings (in the sense of Bruck), the addition of error terms and very general metric settings. In particular, we get the first quantitative results on iterations (with errors) of compositions of metric projections in CAT(κ)-spaces ($\kappa > 0$). Under an additional compactness assumption we obtain, moreover, a rate of metastability for the strong convergence of such iterations.

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1 Introduction

In this paper we give quantitative forms of asymptotic results on the iterations of strongly nonexpansive operators in Banach spaces as well as in more general metric settings. Strongly nonexpansive operators were introduced by Bruck and Reich [9] and are in many ways much better behaved than nonexpansive ones. Some of the most important mappings used in nonlinear analysis are strongly nonexpansive, e.g., the resolvent of a maximal monotone operator as well as metric projections onto closed convex sets C in Hilbert spaces and sunny nonexpansive retractions onto C in uniformly convex Banach spaces. All these examples are, in fact, even so-called firmly nonexpansive which implies being strongly nonexpansive in uniformly convex Banach spaces. In contrast, however, to the class of firmly nonexpansive mappings (introduced by Browder [6] for Hilbert spaces and by Bruck [7] for general Banach spaces), the class of strongly nonexpansive ones is closed under composition.

One of the central facts about strongly nonexpansive mappings $T : S \to S$ ($S \subseteq X$ some subset of a normed space X) is the property of asymptotic regularity

$$\|T^{n+1}x - T^n x\| \to 0$$

which holds whenever T has a fixed point ([9]). In the context of uniformly convex Banach spaces one gets from this even the strong convergence of $(T^n x)$ provided that S is a symmetric convex closed subset and T is odd ([4]) (in fact a much more general condition suffices as shown in [16]). All this can be applied, in particular, to compositions of norm-one projections onto subspaces as well as to sunny nonexpansive retractions onto closed subsets of a closed convex set ([9]).

In this paper, we give explicit quantitative forms of all these results. In the case of asymptotic regularity theorems, these mostly come in the form of full rates of convergence. For the strongly convergent case, computable rates of convergence can be excluded on general grounds from computability theory but we get explicit effective rates Φ of so-called metastability (in the sense of T. Tao)

$$\forall \varepsilon > 0 \, \forall g \in \mathbb{N}^{\mathbb{N}} \, \exists n \leq \Phi(\varepsilon, g) \, \forall i, j \in [n, n + g(n)] \; (\|T^i x - T^j x\| < \varepsilon),$$

where here $[n, n + g(n)]$ denotes the set $\{n, n + 1, \ldots, n + g(n)\}$.

In the final section we show that the quantitative asymptotic regularity results can largely be generalized to the setting of arbitrary metric spaces and strongly quasi-nonexpansive mappings SQNE in the sense of [8] (to be distinguished from other more restricted concepts with the same name). This latter class is very well-behaved in geodesic settings and e.g. in CAT(κ)-spaces ($\kappa > 0$) metric projections onto closed and convex sets are SQNE and Lipschitzian (see [1]) but not nonexpansive. As a consequence of this we obtain quantitative results on the asymptotic regularity of iterations (with error terms) of compositions of metric projections in CAT(κ)-spaces. Under an additional compactness condition we also get an explicit rate of metastability for the strong convergence of such iterations. This is based on the fact that the sequence satisfies a uniform version of being quasi-Fejér monotone which allows one to apply the general quantitative results for such sequences from [18].

Finally, we show that firmly (quasi-)nonexpansive mappings in uniformly convex hyperbolic spaces (so-called UCW-spaces, see [22]) are SQNE with an explicit 'SQNE-modulus'. Putting this together with our rates of asymptotic regularity for SQNE-mappings we obtain (as an instance of a more general result) back precisely the rates of asymptotic regularity which recently have been extracted for firmly nonexpansive mappings in [2]. We also show that being SQNE with an explicit modulus is implied (in general metric spaces) by the so-called property (P_1) recently introduced in [3].

The approach in this paper is part of the so-called 'proof-mining' paradigm where tools from mathematical logic ('proof interpretations') are used to convert prima facie nonquantitative proofs in such a way that new quantitative information can be read-off. Though the proofs in this paper make no reference to logic, they can be viewed as instances of this general logical methodology and we occasionally make remarks referring to some logically relevant points. For details on 'proof mining' see [15].

2 Rates of asymptotic regularity in Banach spaces

In this paper $\mathbb{N}$ always denotes the set $\{0, 1, 2, \ldots\}$, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Definition 2.1. *Let X be a Banach space and $S \subseteq X$. A nonexpansive mapping $T : S \to X$ is called strongly nonexpansive (SNE) if for all sequences $(x_n), (y_n)$ in S the following implication is true:*

$$\text{if } \left((x_n - y_n) \text{ bounded } \wedge \|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0\right), \text{then } (x_n - y_n) - (Tx_n - Ty_n) \to 0.$$

Lemma 2.2. *A mapping $T : S \to X$ is strongly nonexpansive iff T satisfies*

$$(*) \begin{cases} \forall c, k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall x, y \in S \\ \qquad \left(\|x - y\| \leq c \wedge \|x - y\| - \|Tx - Ty\| < 2^{-n} \to \|(x - y) - (Tx - Ty)\| < 2^{-k}\right). \end{cases}$$

Proof: The converse implication '$\Leftarrow$' is trivial once one observes that $(*)$, in particular, implies that T is nonexpansive. So let's prove '$\Rightarrow$' by contraposition: suppose $(*)$ is wrong. Then there are $c, k \in \mathbb{N}$ such that the following holds

$$\begin{aligned} &\forall n \in \mathbb{N}\, \exists x_n, y_n \in S \\ &(\|x_n - y_n\| \leq c \wedge \|x_n - y_n\| - \|Tx_n - Ty_n\| < 2^{-n} \wedge \|(x_n - y_n) - (Tx_n - Ty_n)\| \geq 2^{-k}) \end{aligned}$$

which contradicts T being SNE. $\square$

Remark 2.3. *Note that the proof of '$\Rightarrow$' is noneffective by the use of contraposition together with countable choice.*

The lemma above shows that the strong nonexpansivity of $T : S \to X$ is nothing else but a uniform version of being strictly nonexpansive in the sense of (see [10])

$$\forall x, y \in S\ (x - y \neq Tx - Ty \to \|Tx - Ty\| < \|x - y\|)$$

which can easily be seen to be equivalent to

$$\forall x, y \in S\, \forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\ (\|x - y\| - \|Tx - Ty\| \leq 2^{-n} \to \|(x - y) - (Tx - Ty)\| < 2^{-k}).$$

It is now easy to see (as was originally observed by S. Reich, see [10]) that for compact S every strictly nonexpansive mapping $T : S \to X$ is strongly nonexpansive.
From general logical metatheorems due to the author (see [15]) it, moreover, follows (note that '$(\ldots)$' in the formula above can be written in purely existential form) that from a proof (even if noneffective) of the fact that a class of operators T is strictly nonexpansive one can extract a proof of T being SNE together with an explicit effective bound ω in the sense of the next definition provided that the proof is carried out for classes of spaces X and mappings T that are allowed in these metatheorems. This, e.g., is the case for the class of uniformly convex Banach spaces and for the class of firmly nonexpansive mappings and so Proposition 2.17 proven below can be seen as an (simple) instance of this general phenomenon.

Definition 2.4. *A function $\omega : \mathbb{N}^2 \to \mathbb{N}$ witnessing '$\exists n$' in $(*)$ above, i.e.*

$$(**) \begin{cases} \forall c, k \in \mathbb{N}\, \forall x, y \in S \\ \big(\|x-y\| \le c \wedge \|x-y\| - \|Tx - Ty\| < 2^{-\omega(c,k)} \to \|(x-y) - (Tx - Ty)\| < 2^{-k}\big), \end{cases}$$

is called an SNE-modulus of T.

So a mapping $T : S \to X$ is strongly nonexpansive iff it possesses an SNE-modulus.

Remark 2.5. *Of course, it is an inessential variation of $(**)$ to replace in the conclusion '$<$' by '$\le$' and in the premise '$\le c$' by '$< c$' (shifting in one direction from ω to $\omega'(c,k) := \omega(c/2, k+1)$). Then with the representation of real numbers as in [15], $(**)$ becomes a purely universal statement (if S is treated as an abstract set just as the abstract convex sets C in [15]). As a result of this, all the general logical bound extraction theorems from [15] for nonexpansive mappings are also true if 'nonexpansive' is replaced by 'SNE with modulus ω' with the only difference that now the extracted bound will additionally depend on ω.*

A computationally weaker form of quantitatively witnessing strong nonexpansivity is the following:

Definition 2.6. *Let T be SNE. A modulus of metastability for T is a functional $\Omega : \mathbb{N} \times \mathbb{N}^{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}} \to \mathbb{N}^{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}}$ such that for every bound $c \in \mathbb{N}$ on $(x_n - y_n)$ and every rate of metastability φ for $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$, i.e.*

$$\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \le \varphi(k,g)\, \forall i \in [n, n+g(n)]\ (\|x_i - y_i\| - \|Tx_i - Ty_i\| < 2^{-k}),$$

$\psi := \Omega(c, \varphi)$ is a rate of metastability for $(x_n - y_n) - (Tx_n - Ty_n) \to 0$, i.e.

$$\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \le \psi(k,g)\, \forall i \in [n, n+g(n)]\ (\|(x_i - y_i) - (Tx_i - Ty_i)\| < 2^{-k}).$$

ω provides a stronger quantitative information than Ω: the latter is definable from the former by $[\Omega(c,\varphi)](k,g) := \varphi(\omega(c,k), g)$, but it does not seem to be easily possible to convert a modulus Ω into ω.

While for logical considerations, it is convenient to work with $2^{-n}, 2^{-k}$ rather than with $\varepsilon, \delta > 0$ (since then quantification over positive reals is replaced by just quantifying over natural numbers n, k and also computability theory can be directly applied to n, k), things become much more readable by shifting back to the ε/δ-formulation which we do for much of the rest of this paper.
We first aim at a quantitative version of the following theorem which goes back to [9]:

Theorem 2.7 ([9])**.** *Let $T : S \to S$ be SNE and let T possess a fixed point $p \in S$. Then*

$$\|T^{n+1}x - T^n x\| \to 0, \quad \text{for all } x \in S,$$

i.e. T is asymptotically regular.

In the following, for a function $f : \mathbb{N} \to \mathbb{N}$, we denote the n-th iteration of f starting from 0 by $f^{(n)}(0)$. We now give a quantitative version of Theorem 2.7:

Theorem 2.8. *Let T, x and p be as in Theorem 2.7. Let $d \in \mathbb{N}$ be such that $d \geq \|x-p\|$ and Ω be a modulus of metastability for T. Define $\varphi(\varepsilon, g) := \tilde{g}^{(\lceil d/\varepsilon \rceil)}(0)$, where $\tilde{g}(n) := n + g(n) + 1$, and $\psi_d(\varepsilon) := (\Omega(d, \varphi))(\varepsilon, 0)$. Then ψ_d is a rate of asymptotic regularity, i.e.*

$$\forall \varepsilon > 0 \, \forall n \geq \psi_d(\varepsilon) \, \left(\|T^{n+1}x - T^n x\| < \varepsilon \right).$$

If ω is an SNE-modulus for T, then we obtain as rate of asymptotic regularity $\psi_d(\varepsilon) := \lceil d/\omega(d, \varepsilon) \rceil$.

Proof: For $x \in S$ define $x_n := T^n x$. Let p be a fixed point of T and $d \geq \|x-p\|$. Since T, in particular, is (quasi-)nonexpansive, the sequence $(\|x_n - p\|)$ is nonincreasing and bounded by d. From [15][Proposition 2.27 and Remark 2.29] it follows that

$$\forall \varepsilon > 0 \, \forall g \in \mathbb{N}^{\mathbb{N}} \, \exists n \leq \varphi(\varepsilon, g) \, \forall i, j \in [n, n + g(n) + 1] \;\; (|\|x_i - p\| - \|x_j - p\|| < \varepsilon)$$

and so, in particular,

$$\forall \varepsilon > 0 \, \forall g \in \mathbb{N}^{\mathbb{N}} \, \exists n \leq \varphi(\varepsilon, g) \, \forall i \in [n, n + g(n)] \;\; (\|x_i - p\| - \|x_{i+1} - p\| < \varepsilon).$$

Since $\|x_n - p\| - \|Tx_n - Tp\| = \|x_n - p\| - \|x_{n+1} - p\|$, this means that φ is a rate of metastability for $\|x_n - p\| - \|Tx_n - Tp\| \to 0$. Hence by the assumption on T being SNE with modulus of metastability Ω we get that $\Omega(d, \varphi)$ is a rate of metastability for $\|Tx_n - x_n\| = \|(Tx_n - Tp) - (x_n - p)\| \to 0$. Applied to the function $g := 0$ this yields

$$\forall \varepsilon > 0 \, \exists n \leq \psi_d(\varepsilon) \; (\|Tx_n - x_n\| < \varepsilon)$$

and hence

$$\forall \varepsilon > 0 \, \forall n \geq \psi_d(\varepsilon) \; (\|Tx_n - x_n\| < \varepsilon)$$

since $(\|Tx_n - x_n\|)$ is nonincreasing.
The 2nd claim follows from the fact that for $[\Omega(d, \varphi)](\varepsilon, g) := \varphi(\omega(d, \varepsilon), g)$ (see above) we get $[\Omega(d, \varphi)](\varepsilon, 0) = \varphi(\omega(d, \varepsilon), 0) = \lceil d/\omega(d, \varepsilon) \rceil$ since $\tilde{0}(n) = n + 1$. □

Remark 2.9. *It is not hard to verify that for the claim about ω, the assumption in Theorem 2.8 that T has a fixed point can be replaced by*

$$\forall \delta > 0 \, \exists p_\delta \in S \; (\|x - p_\delta\| \leq d \wedge \|Tp_\delta - p_\delta\| < \delta).$$

In the following we give a quantitative version of the important property of strongly nonexpansive mappings being closed under composition (first proved in [9], see also [10]):

Theorem 2.10. *Let $S \subseteq X$ be a subset of a normed space X and let $T_1, T_2 : S \to S$ be SNE mappings. Consider $T := T_2 \circ T_1$.*
If T_1, T_2 have SNE-moduli ω_1, ω_2 resp., then T is an SNE mapping and

$$\omega(c, \varepsilon) := \min\{\omega_1(c, \varepsilon/2), \omega_2(c, \varepsilon/2)\}$$

is an SNE-modulus for T.
If we have n SNE mappings $T_1, \ldots, T_n : S \to S$ with respective moduli ω_i we may take

$$\omega(c, \varepsilon) := \min\{\omega_1(c, \varepsilon/n), \ldots, \omega_n(c, \varepsilon/n)\}$$

as SNE-modulus for $T := T_n \circ \ldots \circ T_1$.

Proof: Let $\varepsilon_1, \varepsilon_2 > 0$. Define $\omega(c, \varepsilon_1, \varepsilon_2) := \min\{\omega_1(c, \varepsilon_1), \omega_2(c, \varepsilon_2)\}$. Let $x, y \in S$ be with $\|x - y\| \le c$. Assume that

$$(1)\ \|x - y\| - \|Tx - Ty\| < \omega(c, \varepsilon_1, \varepsilon_2).$$

Then also

$$(2)\ \|x - y\| - \|T_1x - T_1y\| < \omega_1(c, \varepsilon_1) \text{ and } \|T_1x - T_1y\| - \|T_2(T_1x) - T_2(T_1y)\| < \omega_2(c, \varepsilon_2)$$

because of

$$\|Tx - Ty\| = \|T_2(T_1x) - T_2(T_1y)\| \le \|T_1x - T_1y\| \le \|x - y\|.$$

Since ω_i is an SNE-modulus for T_i $(i = 1, 2)$ and $\|T_1x - T_1y\| \le \|x - y\| \le c$ we get from (2) that

$$\|(x - y) - (T_1x - T_1y)\| < \varepsilon_1 \text{ and } \|(T_1x - T_1y) - (T_2(T_1x) - T_2(T_1y))\| < \varepsilon_2$$

and so

$$\|(x-y)-(Tx-Ty)\| \le \|(x-y)-(T_1x-T_1y)\|+\|(T_1x-T_1y)-(T_2(T_1x)-T_2(T_1y))\| < \varepsilon_1+\varepsilon_2.$$

From this result, we now inductively get the claim for SNE-mappings $T_1, \ldots, T_n$ with respective moduli ω_i : let $n > 1$. By induction hypothesis, $\tilde{\omega}(c, \varepsilon) := \min\{\omega_1(c, \varepsilon/(n-1)), \ldots, \omega_{n-1}(c, \varepsilon/(n-1))\}$ is an SNE-modulus for $T_{n-1} \circ \ldots \circ T_1$. Then by the result proved above, we get from assuming

$$\|x - y\| - \|Tx - Ty\| < \min\{\omega_1(c, \frac{\varepsilon}{n}), \ldots, \omega_n(c, \frac{\varepsilon}{n})\} = \min\{\tilde{\omega}(c, \frac{\varepsilon}{n}(n-1)), \omega_n(c, \frac{\varepsilon}{n})\}$$

that

$$\|(x - y) - (Tx - Ty)\| < \frac{\varepsilon}{n}(n-1) + \frac{\varepsilon}{n} = \varepsilon.$$

which yields the claim. □

Remark 2.11. *There also is a version of Theorem 2.10 in terms of moduli of metastability for T_1, T_2 : If T_1, T_2 have moduli of metastability Ω_1, Ω_2 respectively which are selfmajorizing (in the sense of the majorizability relation from [15]), then T is an SNE mapping with modulus of metastability $\Xi(\Omega_1, \Omega_2)$ (on selfmajorizing arguments) where*

$$[\Xi(\Omega_1, \Omega_2)(c, \varphi)](k, g) := \xi(k, g),$$

with

$$\begin{aligned}
&\xi(k, g) := \max\left\{\xi_1(k, \delta(k, h_{k,g}), \xi_2(k, \tilde{g}_{\xi_1(k, h_{k,g})}\right\}, \text{ where} \\
&g^*(n) := n + \max\{g(i) : i \le n\},\ \tilde{g}_l(m) := g^*(\max\{l, m\}), \\
&h_{k,g}(n) := g^*(\max\{n, \xi_2(k, \tilde{g}_n)\}),\ \xi_j(k, g) := [\Omega_j(c, \varphi)](k + 1, g) \text{ for } j = 1, 2.
\end{aligned}$$

The construction ξ is made such that it transforms two metastability rates in one which is simultaneously a rate for both. The requirement being 'selfmajorizing' is usually satisfied in practice.
Since we do not need this form of Theorem 2.10 in this paper, we do not go into further details here.

Theorem 2.10 can be used to compute approximate common fixed points of finite families $T_1, \ldots, T_k$ of SNE-mappings if they have common fixed points. For this we need a quantitative version of Lemma 2.1 from [9]

Proposition 2.12. *Let X be a Banach space and $S \subseteq X$ be a subset. Let $T_1, \cdots, T_k : S \to S$ be SNE-mappings with SNE-moduli $\omega_1, \ldots, \omega_k$, resp. Let $p \in S$ be a common fixed point of $T_1, \ldots, T_k$. Define $\rho(d, \varepsilon) := \chi_d(k-1, \varepsilon)$, where $\chi_d(0, \varepsilon) := \varepsilon/2$, $\chi_d(n+1, \varepsilon) := \min\{\omega(d, \frac{1}{2}\chi_d(n, \varepsilon)), \frac{1}{2}\chi_d(n, \varepsilon)\}$ with $\omega(d, \varepsilon) := \min\{\omega_1(d, \varepsilon), \ldots, \omega_k(d, \varepsilon)\}$. Then*

$$\forall d \in \mathbb{N}\, \forall x \in S\, \forall \varepsilon > 0\ (\|x - p\| \leq d \wedge \|T_k T_{k-1} \ldots T_1 x - x\| < \rho(d, \varepsilon) \to \bigwedge_{i=1}^{k} (\|T_i x - x\| < \varepsilon)).$$

Proof: We first observe that $\chi_d(n, \varepsilon) \leq \varepsilon/2$ for all $n \in \mathbb{N}$.
We proceed by induction on $1 \leq l \leq k$: For $l = 1$ the statement is trivial. So let $1 < l \leq k$. Assume that

$$(1)\ \|T_l T_{l-1} \ldots T_1 x - x\| < \rho(d, \varepsilon) = \min\{\omega(d, \frac{1}{2}\chi_d(l-2, \varepsilon)), \frac{1}{2}\chi_d(l-2, \varepsilon)\} \leq \frac{\varepsilon}{2}.$$

Define $y := T_{l-1} \ldots T_1 x$. Then

$$(2)\ \|x - p\| - \min\{\omega(d, \frac{1}{2}\chi_d(l-2, \varepsilon)), \frac{1}{2}\chi_d(l-2, \varepsilon)\} < \|T_l y - p\| \leq \|y - p\| \leq \|x - p\|.$$

Since ω is an SNE-modulus of T_l, (2) yields

$$(3)\ \|T_l y - y\| < \frac{1}{2}\chi_d(l-2, \varepsilon) \leq \varepsilon/4.$$

(1) also implies

$$(4)\ \|T_l y - x\| < \frac{1}{2}\chi_d(l-2, \varepsilon) \leq \varepsilon/4.$$

By (3) and (4) we get

$$(5)\ \|x - T_{l-1} \ldots T_1 x\| = \|x - y\| < \chi_d(l-2, \varepsilon)$$

as well as

$$(6)\ \|T_l x - T_l y\| \leq \|x - y\| < \frac{\varepsilon}{2}.$$

Together with (1) this gives

$$(7)\ \|x - T_l x\| \leq \|x - T_l y\| + \|T_l y - T_l x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, by (7), we have shown that x is an ε-approximate fixed point of T_l and by (5) the induction hypothesis gives the result for $T_{l-1}, \ldots, T_1$. $\square$

We can now combine Theorem 2.8, Theorem 2.10 and Proposition 2.12 to obtain the following result:

Corollary 2.13. *Let $T_1, \ldots, T_k : S \to S$ be SNE-mappings with respective moduli $\omega_1, \ldots, \omega_k$, $p \in S$ be a common fixed point of $T_1, \ldots, T_k$ and define for $x \in S$, $x_n := T^n x$ with $T := T_k \circ \ldots \circ T_1$. Let $d \in \mathbb{N}$ be such that $\|x - p\| \le d$. Then*

$$\forall \varepsilon > 0 \, \forall n \ge \zeta(d, \varepsilon) \ \left(\bigwedge_{i=1}^{k} \|T_i x_n - x_n\| < \varepsilon \right),$$

where

$$\zeta(d,\varepsilon) := \psi_d(\rho(d,\varepsilon)) \text{ with } \psi_d(\varepsilon) := \left\lceil \tfrac{d}{\omega(d,\varepsilon)} \right\rceil + 1,$$
$$\omega(d,\varepsilon) := \min\{\omega_1(d,\varepsilon/k), \ldots, \omega_k(d,\varepsilon/k)\} \text{ and } \rho(d,\varepsilon) \text{ as in Lemma 2.12.}$$

Proof: By Theorem 2.10, ω is an SNE-modulus for T. Hence, by Theorem 2.8, $\psi_d(\varepsilon)$ is a rate of asymptotic regularity of T for $x \in S$ with $\|x - p\| \le d$. Thus

$$\forall n \ge \zeta(d,\varepsilon) \ (\|Tx_n - x_n\| = \|T^{n+1}x - T^n x\| < \rho(d,\varepsilon)$$

and so, by Proposition 2.12 (since $\|x_n - p\| \le \|x - p\| \le d$),

$$\forall n \ge \zeta(d,\varepsilon) \ \left(\bigwedge_{i=1}^{k} \|T_i x_n - x_n\| < \varepsilon \right).$$

$\square$

Definition 2.14 ([7])**.** *A mapping $T : S \to X$ is λ-firmly nonexpansive for $\lambda > 0$ if*

$$\forall x, y \in S \ (\|Tx - Ty\| \le \|\lambda(x - y) + (1 - \lambda)(Tx - Ty)\|)$$

and firmly nonexpansive if this holds for all $\lambda > 0$.

In uniformly convex Banach spaces, every λ-firmly nonexpansive mapping (for $\lambda \in (0,1)$) and so, in particular, every firmly nonexpansive mapping is strongly nonexpansive (this is Proposition 2.1 in [9]). In order to apply our quantitative results obtained so far we need a quantitative version of this fact. The next lemma is essentially well-known:

Lemma 2.15. *Let $d > 0$ and $\lambda \in [0,1]$. Let X be a uniformly convex Banach space with a modulus of uniform convexity $\eta : (0,2] \to (0,1]$, i.e.*

$$(*) \ \forall \varepsilon \in (0,2] \, \forall x, y \in X \ \left(\|x\|, \|y\| \le 1 \wedge \|x - y\| \ge \varepsilon \to \left\| \frac{1}{2}(x + y) \right\| \le 1 - \eta(\varepsilon) \right).$$

Then for all $\varepsilon \in (0,2]$ and $x, y \in X$:

$$\|x\|, \|y\| \le d \wedge \|(1 - \lambda x) + \lambda y)\| > (1 - 2\lambda(1 - \lambda)\eta(\varepsilon))\, d \to \|x - y\| < \varepsilon \cdot d.$$

Remark 2.16. *Note that for $\varepsilon > 2$ one can stipulate $\eta(\varepsilon)$ to be any real > 0 as $(*)$ trivially holds. We may, therefore, without loss of generality assume that η is defined as a function $(0,\infty) \to (0,\infty)$ and we will occasionally implicitly make use of this.*

Proposition 2.17. *Let X be uniformly convex with a modulus η and let $\lambda \in (0,1)$. Let $S \subseteq X$ and $T : S \to X$ be λ-firmly nonexpansive. Then T is SNE with modulus $\omega(c,\varepsilon) := \lambda(1-\lambda)\eta(\varepsilon/c)\cdot\varepsilon$ (for $\varepsilon > 2c$ the claim is trivial and we may simply put $\omega(c,\varepsilon) := 1$).*
If η can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ such that

$$\varepsilon_1 \leq \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2), \text{ for all } \varepsilon_1, \varepsilon_2 \in (0,2],$$

then the modulus can be taken as $\omega(c,\varepsilon) := 2\lambda(1-\lambda)\tilde{\eta}(\varepsilon/c)\cdot\varepsilon$.

Proof: Since T is λ-firmly nonexpansive we have

$$(1)\ \|Tx - Ty\| \leq \|(1-\lambda)(Tx-Ty) + \lambda(x-y)\| \leq \|x-y\|.$$

Now assume that $x, y \in S$ with $\varepsilon/2 \leq \|x-y\| \leq c$ and

$$(2)\ \|x-y\| - \|Tx-Ty\| < \lambda(1-\lambda)\eta(\varepsilon/c)\cdot\varepsilon.$$

(1) and (2) yield

$$(3)\ \|(1-\lambda)(Tx-Ty) + \lambda(x-y)\| > \|x-y\| - 2\lambda(1-\lambda)\eta(\varepsilon/c)\cdot\|x-y\|.$$

Then, since $\|Tx-Ty\| \leq \|x-y\| \leq c$, Lemma 2.15 (applied to $d := \|x-y\|$) implies

$$\|(x-y)-(Tx-Ty)\| < \frac{\varepsilon}{c}\cdot\|x-y\| \leq \varepsilon.$$

If $\|x-y\| < \varepsilon/2$ (and so also $\|Tx-Ty\| < \varepsilon/2$), then trivially $\|(x-y)-(Tx-Ty)\| < \varepsilon$.
The additional claim is shown as follows: instead of (2) we now assume

$$\begin{aligned}(2)'\ \|x-y\| - \|Tx-Ty\| &< 2\lambda(1-\lambda)\cdot\varepsilon\cdot\tilde{\eta}(\varepsilon/c) \\ &\leq 2\lambda(1-\lambda)\cdot\varepsilon\cdot\tilde{\eta}(\varepsilon/\|x-y\|) \\ &= 2\lambda(1-\lambda)\cdot\eta(\varepsilon/\|x-y\|)\cdot\|x-y\|.\end{aligned}$$

(1) and $(2)'$ then yield

$$(3)'\ \|(1-\lambda)(Tx-Ty) + \lambda(x-y)\| > \|x-y\| - 2\lambda(1-\lambda)\cdot\eta(\varepsilon/\|x-y\|)\cdot\|x-y\|$$

and so by Lemma 2.15 (again applied to $d := \|x-y\|$)

$$\|(x-y)-(Tx-Ty)\| < \frac{\varepsilon}{\|x-y\|}\cdot\|x-y\| = \varepsilon.$$

$\square$

Corollary 2.18. *In Hilbert spaces we get the SNE-modulus*

$$\omega(c,\varepsilon) := \frac{\lambda(1-\lambda)}{4c}\varepsilon^2.$$

Proof: It is well-known that $\eta(\varepsilon) := \varepsilon^2/8$ is a modulus of uniform convexity in Hilbert spaces. The corollary now follows from the 2nd claim in Proposition 2.17 with $\tilde{\eta}(\varepsilon) = \varepsilon/8$. $\square$

Corollary 2.19. *Let X be a uniformly convex Banach space with modulus of uniform convexity η.*

1. *Let $\{P_i : 1 \leq i \leq k\}$ be linear norm-one projections of X onto subspaces $\{X_i : 1 \leq i \leq k\}$. Then $P_k \circ \ldots \circ P_1$ is an SNE-mapping with SNE-modulus*
$$\omega(c, \varepsilon) := \frac{\varepsilon}{4k}\eta(\varepsilon/(k \cdot c)).$$

2. *Let $C \subseteq X$ be a closed and convex subset and $P_i : C \to F_i$ be sunny nonexpansive retractions of C onto (nonempty) closed subsets $F_i \subseteq C$ for $1 \leq i \leq k$. Then $P_k \circ \ldots \circ P_1$ is an SNE-mapping with the SNE-modulus $\omega(c, \varepsilon)$ as above.*

Proof: Both items follow from Theorem 2.10 and Proposition 2.17 using that norm-one linear projections as well as sunny nonexpansive retractions are firmly nonexpansive ([9]) and so, in particular, λ-firmly nonexpansive for $\lambda := 1/2$. $\square$

For more information on (sunny) nonexpansive retracts see [19].

3 Rates of metastability for strong convergence results

As pointed out in [9], a corollary of Theorem 2.7 and a theorem proved in [4] (Theorem 1.1) is the following:

Corollary 3.1 ([9])**.** *Let X be a uniformly convex Banach space, $C \subseteq X$ be closed and convex with $C = -C$. Let $T : C \to C$ be odd and strongly nonexpansive. Then for every $x \in C$, $(T^n x)$ strongly converges to a fixed point of T.*

In [16][1] we generalized the aforementioned Theorem 1.1 from [4] to mappings which just satisfy a condition due to [30]

$$(W)\ \forall x, y \in S\, (\|Tx + Ty\| \leq \|x + y\|),$$

where now $S \subseteq X$ is any closed subset (note that any odd and nonexpansive selfmap of a symmetric set S satisfies (W)) and we gave a quantitative version of this theorem:

Theorem 3.2 ([16])**.** *Let X be a uniformly convex Banach space with a modulus of uniform convexity η, $S \subseteq X$ be any nonempty subset of X and $T : S \to S$ a selfmapping of S that satisfies Wittmann's [30] condition (W). Moreover, assume that for each $0 < d \in \mathbb{N}$ the mapping T is (uniformly on $S_d := \{x \in S : \|x\| \leq d)$ asymptotically regular with a rate $\alpha : \mathbb{N} \times \mathbb{R}^*_+ \to \mathbb{N}$, i.e.*

$$\forall \varepsilon > 0\, \forall d \in \mathbb{N}^*\, \forall x \in S_d\, \forall n \geq \alpha(d, \varepsilon)\ \big(\|T^{n+1}x - T^n x\| < \varepsilon\big).$$

[1] Typos in [16]: p.619, line 11 'Then (x_n)...', line 14 '$b^2 \cdot \lambda$' (in the definition of α) and line 16 '$\lambda\|x_n - T(x_n)\| = \|T_\lambda^{n+1}x - T_\lambda^n x\|$'.

Then $(T^n x)_{n\in\mathbb{N}}$ *converges strongly with the following rate of metastability*

$$\forall \varepsilon \in (0,2]\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall d \in \mathbb{N}^*\, \forall x \in S_d\, \exists n \leq \Phi(d,\alpha,\varepsilon,g)\\ \forall i,j \in [n, n+g(n)]\ \big(\|T^i x - T^j x\| < \varepsilon\big),$$

where

$$\begin{aligned}&\Phi(d,\alpha,\varepsilon,g) := \Psi(d, h_{d,\alpha,\varepsilon,g}, \tfrac{\delta_d(\varepsilon)}{2}) \text{ with}\\ &h_{d,\alpha,\varepsilon,g}(n) := h(n) := \max\Big\{\alpha\Big(d, \tfrac{\delta_d(\varepsilon)}{\max\{g(n),1\}}\Big) \dot{-} n, g(n)\Big\} \text{ and}\\ &\Psi(d,f,\delta) := \tilde{f}^{(\lceil d/\delta\rceil)}(0) \text{ with } \tilde{f}(n) := n + f(n) \text{ for } f : \mathbb{N} \to \mathbb{N},\\ &\delta_d(\varepsilon) := \tfrac{\varepsilon}{2}\cdot\eta(\varepsilon/d).\end{aligned}$$

If T is continuous and S closed, then the strong limit of $(T^n x)_{n\in\mathbb{N}}$ is a fixed point of T. For the metastability statement the completeness of X is not needed.

Using Theorem 2.8 together with Theorem 3.2 we get the following quantitative version of Corollary 1.2 in [9]:

Corollary 3.3. *Let X be a uniformly convex Banach space with a modulus of uniform convexity η and $S \subseteq X$ be any closed subset. Let $T : S \to S$ be a mapping which satisfies condition (W) and which is strongly nonexpansive. Assume that T has a fixed point $p \in S$ (which e.g. is the case when S is convex and bounded or when S is convex and symmetric and T is odd). Then for every $x \in S$, $(T^n x)$ strongly converges to a fixed point of T with the rate of metastability $\Phi(d,\alpha,\varepsilon,g)$ from 3.2 with $\alpha(\varepsilon) := \psi_{d+b}(\varepsilon)$ from Theorem 2.8, where $d,b \in \mathbb{N}$ with $\|x\| \leq d$ and $\|p\| \leq b$.*

Proof: Since $\|x\| \leq d$ implies that $\|x - p\| \leq d + b$, we get from Theorem 2.8 that ψ_{d+b} is a uniform rate of asymptotic regularity for all $x \in S_d$. Hence Theorem 3.2 is applicable with $\alpha(\varepsilon) := \psi_{d+b}(\varepsilon)$. ☐

The next corollary gives quantitative versions of Theorems 2.1 and 2.2 in [9]:

Corollary 3.4. *Under the conditions of Corollary 2.19.1, the sequence $\big((P_k \circ \ldots \circ P_1)^n x\big)_n$ is metastable with the rate $\Phi(d,\psi_d,\varepsilon,g)$ from Corollary 3.3, but now with*

$$\psi_d(\varepsilon) := \lceil 4kd/(\varepsilon\cdot\eta(\varepsilon/(k\cdot d)))\rceil.$$

The same is true under the conditions of Corollary 2.19.2 if we additionally assume that X is smooth and C and $F_1,\ldots,F_k$ are symmetric.

Proof: The claim follows from Corollary 3.3, Theorem 2.8 and Corollary 2.19 since norm-one linear projections as well as (under the assumptions made) sunny nonexpansive retracts are odd (by [7, 9]) and hence (as well as their compositions) satisfy (W). Note that this time 0 is a fixed point of $P_k \circ \ldots \circ P_1$ so that we can take $b := 0$. ☐

4 Generalizations to metric and uniformly convex hyperbolic spaces

In this section we show that most of the results in Section 2 hold true in the setting of general metric spaces (instead of Banach spaces) and uniformly convex hyperbolic spaces (instead of uniformly convex Banach spaces). Since it is somewhat artificial to adapt the concept of strong nonexpansivity to the metric context, we use instead a generalization of this concept, namely strong quasi-nonexpansivity and so the results in this section are also a proper generalization in the normed case (also we now allow for error terms). For the relevance of this class of mappings in the context of convex optimization even in the setting of Hilbert spaces, see e.g. [31]. In the geodesic setting, it has recently been shown that metric projections to Chebycheff sets in CAT(κ)-spaces ($\kappa > 0$) are strongly quasi-nonexpansive but not nonexpansive (while being Lipschitzian under suitable boundedness conditions, see [1]). In the setting of the Hilbert ball, strongly (quasi-)nonexpansive mappings have been defined and studied already in [25, 5].

Definition 4.1. *Let (X, d) be a metric space, $S \subseteq X$ and $T : S \to X$ be a mapping. We call T uniformly strongly quasi-nonexpansive (SQNE) if $Fix(T) \neq \emptyset$ and*

$$(+) \begin{cases} \forall c, k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall x \in S\, \forall p \in Fix(T) \\ \qquad (d(x,p) \leq c \wedge d(x,p) - d(Tx,p) < 2^{-n} \to d(x,Tx) < 2^{-k}). \end{cases}$$

If this is only claimed for **some** *fixed point $p \in Fix(T)$ we say that T is SQNE w.r.t. p.*

Remark 4.2.

1. *Note that being 'SQNE' differs from 'SQNE w.r.t. p for each $p \in Fix(T)$' as in the former case '$\exists n$' holds uniformly for all $p \in Fix(T)$.*
2. *Any mapping T that is SQNE w.r.t. some $p \in Fix(T)$, in particular, is quasi-nonexpansive w.r.t. p, i.e.*

$$\forall x \in S\ (d(Tx,p) \leq d(x,p)).$$

This definition is essentially due to [8] who considers both the nonuniform and the uniform version of being SQNE. However, we do not claim n to be uniformly for all x (as in [8]) but only provided that $d(x,p) \leq c$. It is then obvious that any SNE-mapping having a fixed point is also (uniformly) SQNE (although claimed in [8] it is not clear why this would be the case if the existence of n in SQNE would be claimed to be uniform for all x, p).
Note that the notion called also 'SQNE' in [10][2.1.38] for the Hilbert space case is much stronger in the sense that n is claimed to be (uniformly) given in a very special form there. In particular, not every SNE-mapping (having a fixed point) would be SQNE in this strong sense. More precisely, the definition given in [10] coincides with the condition (P_1) from [3] for $l = 2$ (see below).

Being SQNE w.r.t. a fixed point $p \in Fix(T)$ is equivalent to being quasi-nonexpansive w.r.t. p and satisfying the following condition (see also [25, 5]):

$$((x_n) \text{ bounded } \wedge d(x_n,p) - d(Tx_n,p) \to 0) \Rightarrow d(x_n,Tx_n) \to 0.$$

A function $\omega : \mathbb{N}^2 \to \mathbb{N}$ witnessing '$\exists n$' in $(+)$ as a function in c, k is called a (uniform) SQNE-modulus of T (resp. an SQNE-modulus w.r.t. some $p \in Fix(T)$ if T is only claimed to be SQNE w.r.t. p).
We can adopt also the concept of being a modulus Ω of metastability for mappings T :

$S \to X$ that are SQNE-mappings w.r.t. a fixed point $p \in Fix(T)$ in the following way: Ω transforms a bound c on $(d(x_n, p))$ and a rate of metastability φ for $d(x_n, p) - d(Tx_n, p) \to 0$ into a rate $\psi := \Omega(c, \varphi)$ of metastability for $d(x_n, Tx_n) \to 0$.

We next observe that being SQNE is implied (in general metric spaces) by the property (P_1) introduced in [3] and one easily can construct an SQNE-modulus:

Definition 4.3 ([3]). *Let (X, d) be a metric space and $S \subseteq X$ be a subset. A mapping $T : S \to X$ satisfies the property (P_1) if $Fix(T) \neq \emptyset$ and there exist $l, \beta > 0$ such that*

$$(P_1)\ \forall x \in S\, \forall p \in Fix(T)\ \left(d(Tx, p)^l \leq d(x, p)^l - \beta d(Tx, x)^l \right).$$

Proposition 4.4. *Every mapping T satisfying the property (P_1) is SQNE with modulus (switching again to the more convenient ε/δ-notation) $\omega(c, \varepsilon) := \alpha_c(\varepsilon^l \cdot \beta)$, where α_c is a modulus of uniform continuity for $x \mapsto x^l$ on $[0, c]$.*

Proof: Let $\alpha_c : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ be a modulus of uniform continuity of $a \mapsto a^l$ on $[0, c]$, i.e.

$$\forall \varepsilon > 0\, \forall a, b \in [0, c]\ (|a - b| < \alpha_c(\varepsilon) \to |a^l - b^l| < \varepsilon).$$

(P_1) implies that for all $x \in S, p \in Fix(T)$

$$(1)\ d(Tx, x) \leq \left(\frac{d(x, p)^l - d(Tx, p)^l}{\beta} \right)^{\frac{1}{l}}.$$

Define

$$(2)\ \omega(c, \varepsilon) := \alpha_c(\varepsilon^l \cdot \beta).$$

Now let $d(x, p) \leq c$ with $x \in S$ and $p \in Fix(T)$. (P_1), in particular, implies that $d(Tx, p) \leq d(x, p)$. Assume that

$$(3)\ d(x, p) - d(Tx, p) < \omega(c, \varepsilon).$$

Then

$$(4)\ d(x, p)^l - d(Tx, p)^l < \varepsilon^l \cdot \beta$$

and so by (1)

$$d(Tx, x) < \varepsilon.$$

□

Corollary 4.5. *Let X be a complete $CAT(\kappa)$-space ($\kappa > 0$) with $diam(X) < \pi/(2\sqrt{\kappa}) \leq d$ and $C \subseteq X$ be a nonempty closed convex subset. Then the metric projection $P_C : X \to C$ is SQNE with modulus*[2]

$$\omega(\varepsilon) := \frac{\varepsilon^2 \cdot \beta}{2d},$$

where

$$\beta = \frac{1}{2}(\pi - 2\sqrt{\kappa}\delta) \tan(\sqrt{\kappa}\delta)\ \text{with}\ 0 < \delta < \pi/(2\sqrt{\kappa}) - diam(X).$$

[2]Since X is bounded we do not need the first argument c of ω here.

Proof: By [3], metric projections onto closed convex subsets of X satisfy the property (P_1) with $l = 2$ and β as given above. Hence by Proposition 4.4 we get (using that $f(x) = x^2$ is $2d$-Lipschitz on $[0, d]$) that $\varepsilon^2 \cdot \beta/2d$ is an SQNE-modulus for P_C. $\square$

Theorem 2.10 also holds for SQNE-mappings with respective moduli in the following sense:

Theorem 4.6. *If $T_1, \ldots, T_k : S \to S$ are SQNE with moduli $\omega_1, \ldots, \omega_k$ for some common fixed point $p \in \bigcap_{i=1}^k Fix(T_i)$, then $T_k \circ \ldots \circ T_1$ is SQNE (with the modulus ω as defined in Theorem 2.10) w.r.t. p. In particular: if $T_1, \ldots, T_k$ are SQNE, then T is SQNE (again with the modulus ω as defined in Theorem 2.10) provided that $\bigcap_{i=1}^k Fix(T_i) \neq \emptyset$.*

Proof: The first part follows by inspecting the proof of Theorem 2.10. The 2nd claim follows using that if $T_1, \ldots, T_k$ are SQNE and $\bigcap_{i=1}^k Fix(T_i) \neq \emptyset$, then $Fix(T) = \bigcap_{i=1}^k Fix(T_i)$ (using that the new modulus ω is uniform if the moduli $\omega_1, \ldots, \omega_k$ were). This fact is shown in [8] (and we give a quantitative form of this result in Proposition 4.15 below; note that the continuity requirement is not necessary for the qualitative result). $\square$

Theorem 4.7. *Let (X, d) be a metric space and $S \subseteq X$ be any subset. Let $T : S \to S$ be an SQNE-map with modulus of metastability Ω w.r.t. some $p \in Fix(T)$. Then*

$$d(T^{n+1}x, T^n x) \to 0, \quad \textit{for all } x \in S,$$

and we obtain the following rate of metastability for this asymptotic regularity:

$$d(x,p) \leq d \to \forall \varepsilon > 0\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \psi_d(\varepsilon, g)\, \forall k \in [n, n + g(n)]\ \Big(d(T^{k+1}x, T^k x) < \varepsilon \Big),$$

where

$$\psi_d(\varepsilon, g) := (\Omega(d, \varphi))(\varepsilon, g) \ \textit{with}\ \varphi(\varepsilon, g) := \tilde{g}^{(\lceil d/\varepsilon \rceil)}(0),\ \ \tilde{g}(n) := n + g(n) + 1.$$

If ω is an SQNE-modulus for T w.r.t. p, then we obtain as rate of metastability for the asymptotic regularity $\psi_d(\varepsilon, g) := \varphi(\omega(d, \varepsilon), g)$.
In the case where $(d(T^{n+1}x, T^n x))$ is nonincreasing (which e.g. is the case if T additionally is nonexpansive), then $\psi_d(\varepsilon, 0)$ is a rate of asymptotic regularity (in both cases). In particular, then $\lceil d/\omega(d, \varepsilon) \rceil$ is a rate of asymptotic regularity.

Proof: For $x \in S$ define $x_n := T^n x$. Let $d \geq d(x, p)$. Since T, in particular, is quasi-nonexpansive (w.r.t. p), the sequence $(d(x_n, p))$ is nonincreasing and bounded by d. From [15][Corollary 2.28 and Remark 2.29] it follows that

$$\forall \varepsilon > 0\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \varphi(\varepsilon, g)\, \forall i, j \in [n, n + g(n) + 1]\ \ (|d(x_i, p) - d(x_j, p)| < \varepsilon)$$

and so, in particular,

$$\forall \varepsilon > 0\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \varphi(\varepsilon, g)\, \forall i \in [n, n + g(n)]\ \ (d(x_i, p) - d(x_{i+1}, p) < \varepsilon).$$

Since $d(x_n, p) - d(Tx_n, p) = d(x_n, p) - d(x_{n+1}, p)$, this means that φ is a rate of metastability for $d(x_n, p) - d(Tx_n, p) \to 0$. Hence by the assumption on T being SQNE with modulus of metastability Ω w.r.t. p we get that $\Omega(d, \varphi)$ is a rate of metastability for $d(Tx_n, x_n) \to 0$. Applied to the function $g := 0$ this yields

$$\forall \varepsilon > 0 \, \exists n \leq \psi_d(\varepsilon, 0) \ \ (d(Tx_n, x_n) < \varepsilon) \, .$$

If $(d(Tx_n, x_n))$ is, furthermore, nonincreasing we obtain

$$\forall \varepsilon > 0 \, \forall n \geq \psi_d(\varepsilon, 0) \ \ (d(Tx_n, x_n) < \varepsilon) \, .$$

The 2nd claim follows from the fact that, given an SQNE-modulus ω w.r.t. p we can just take $[\Omega(d, \varphi)](\varepsilon, g) := \varphi(\omega(d, \varepsilon), g)$ as SQNE-modulus of metastability w.r.t. p. Finally, $[\Omega(d, \varphi)](\varepsilon, 0) = \varphi(\omega(d, \varepsilon), 0) = \lceil d/\omega(d, \varepsilon) \rceil$ since $\tilde{0}(n) = n + 1$. $\square$

We now show how to introduce error terms in Theorem 4.7. We first need a lemma:

Lemma 4.8. *Let $(a_n), (\delta_n)$ be sequences of nonnegative reals with*

$$a_{n+1} \leq a_n + \delta_n,$$

where $\sum \delta_n < \infty$. Let $A, D \in \mathbb{N}$ with $A \geq a_0$ and $D \geq \sum \delta_n$. Define

$$\tilde{\varphi}_{A,D}(\varepsilon, g) := \tilde{g}^{(K)}(0), \ \text{where } K = \left\lceil \frac{4(A + 5D)}{\varepsilon} \right\rceil, \ \tilde{g}(n) := n + g(n).$$

Then $\tilde{\varphi}_{A,D}$ is a rate of metastability for (a_n).

Proof: The lemma is a special case of Lemma 3.2.3 in [17] taking there $b_n = 0$ (and consequently $B = 0$). We use '4' instead of '3' in the definition of K to get the conclusion in the metastability statement with $< \varepsilon$ (as in our paper) rather than $\leq \varepsilon$ (as in [17]). $\square$

Theorem 4.9. *Let (X, d) be a metric space and $S \subseteq X$ be any subset. Let $T : S \to S$ be an SQNE-map with modulus of metastability Ω w.r.t. some $p \in Fix(T)$. Let $x \in S$ and $d \in \mathbb{N}$ with $d \geq d(x, p)$. Let (x_n) be a sequence in S with $x_0 = x$ and $d(x_{n+1}, Tx_n) < \delta_n$ where (δ_n) is a sequence of nonnegative reals with $\sum \delta_n \leq D \in \mathbb{N}$. Then*

$$d(Tx_n, x_n) \to 0, \quad \text{for all } x \in S,$$

and we obtain the following rate of metastability for this asymptotic regularity:

$$\forall \varepsilon > 0 \, \forall g \in \mathbb{N}^{\mathbb{N}} \, \exists n \leq \psi_d(\varepsilon, g) \, \forall k \in [n, n + g(n)] \ \ (d(Tx_k, x_k) < \varepsilon) \, ,$$

where

$$\begin{aligned} &\psi_d(\varepsilon, g) := \psi_{d,D,\alpha}(\varepsilon, g) := (\Omega(d + D, \widehat{\varphi}_{d,D}))(\varepsilon, g) \\ &\text{with } \widehat{\varphi}_{d,D}(\varepsilon, g) := \tilde{\varphi}_{d,D}(\varepsilon/2, g_{\alpha(\varepsilon/2)} + 1) + \alpha(\varepsilon/2), \end{aligned}$$

where $\tilde{\varphi}_{d,D}$ is as in Lemma 4.8, $g_l(n) := g(n + l) + l$ and α is a rate of convergence for $\delta_n \to 0$, i.e. $\forall \varepsilon > 0 \, \forall n \geq \alpha(\varepsilon) \, (\delta_n < \varepsilon)$.

If ω is an SQNE-modulus for T w.r.t. p, then we obtain as rate of metastability for the asymptotic regularity $\psi_d(\varepsilon, g) := \widehat{\varphi}_{d,D}(\omega(d+D,\varepsilon), g)$.
In the case where $(d(Tx_n, x_n))$ is nonincreasing (which e.g. is the case if T additionally is nonexpansive and $\delta_n = 0$ for all n), then $\psi_d(\varepsilon, 0)$ is a rate of asymptotic regularity (in both cases).

Proof: For $x \in S$ let (x_n) be a sequence in S with $x_0 = x$ and $d(x_{n+1}, Tx_n) < \delta_n$. Let $d \geq d(x,p)$. Since T, in particular, is quasi-nonexpansive (w.r.t. p), we get that

$$d(x_{n+1}, p) \leq d(x_{n+1}, Tx_n) + d(Tx_n, p) < \delta_n + d(x_n, p)$$

and so, in particular,

$$d(x_n, p) \leq d + D.$$

By Lemma 4.8, $\tilde{\varphi}_{d,D}(\varepsilon, g)$ is a rate of metastability for the Cauchy property of $(d(x_n, p))$, i.e.

$$\forall \varepsilon > 0\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \tilde{\varphi}_{d,D}(\varepsilon, g+1)\, \forall i,j \in [n, n+g(n)+1]\ \ (|d(x_i,p) - d(x_j,p)| < \varepsilon)\,.$$

In particular, we get from this that

$$\forall \varepsilon > 0\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \tilde{\varphi}_{d,D}(\varepsilon, g+1)\, \forall i \in [n, n+g(n)]\ \ (d(x_i,p) - d(x_{i+1},p) < \varepsilon)\,.$$

Applying this to $g_{\alpha(\varepsilon)}$, where $g_l(n) := g(n+l)+l$, we get

$$\exists n \leq \tilde{\varphi}_{d,D}(\varepsilon, g_{\alpha(\varepsilon)}+1)\, \forall i \in [n, n+g(n+\alpha(\varepsilon))+\alpha(\varepsilon))]\ \ (d(x_i,p) - d(x_{i+1},p) < \varepsilon)\,.$$

Hence (adding to n from the previous line $\alpha(\varepsilon)$)

$$\exists n \in [\alpha(\varepsilon), \tilde{\varphi}_{d,D}(\varepsilon, g_{\alpha(\varepsilon)}+1) + \alpha(\varepsilon)]\, \forall i \in [n, n+g(n)]\ \ (d(x_i,p) - d(x_{i+1},p) < \varepsilon)\,.$$

Since

$$d(x_i,p) - d(Tx_i,p) < d(x_i,p) - d(x_{i+1},p) + \delta_i$$

we get that

$$\exists n \leq \tilde{\varphi}_{d,D}(\varepsilon, g_{\alpha(\varepsilon)}+1) + \alpha(\varepsilon)\ \forall i \in [n, n+g(n)]\, (d(x_i,p) - d(Tx_i,p) < 2\varepsilon)\,.$$

Hence

$$\widehat{\varphi}_{d,D}(\varepsilon, g) := \tilde{\varphi}_{d,D}(\varepsilon/2, g_{\alpha(\varepsilon/2)}+1) + \alpha(\varepsilon/2)$$

is a rate of metastability for

$$d(x_n,p) - d(Tx_n,p) \to 0.$$

Hence by the assumption on T being SQNE with modulus of metastability Ω w.r.t. p (and the fact that $(d(x_n,p))$ is bounded by $d+D$) we get that $\psi_d = \Omega(d+D, \widehat{\varphi}_{d,D})$ is a rate of metastability for $d(Tx_n, x_n) \to 0$. Applied to the function $g := 0$ this yields

$$\forall \varepsilon > 0\, \exists n \leq \psi_d(\varepsilon, 0)\ \ (d(Tx_n, x_n) < \varepsilon)\,.$$

If $(d(Tx_n, x_n))$ is, furthermore, nonincreasing this yields

$$\forall \varepsilon > 0\, \forall n \geq \psi_d(\varepsilon, 0)\ \ (d(Tx_n, x_n) < \varepsilon)\,.$$

$\square$

Corollary 4.10. *The rate of metastability ψ_d for the asymptotic regularity in Theorem 4.9, in particular, provides a so-called* $\liminf$*-bound ξ for (x_n), i.e.*

$$\forall n \in \mathbb{N}\,\forall \varepsilon > 0\,\exists m \in [n, \xi(\varepsilon, n)]\ (d(x_m, Tx_m) < \varepsilon).$$

Take as $\xi(\varepsilon, n) := \psi_d(\varepsilon, n)$, where here we identify in the notation n with the constant-n function $g(k) := n$.

Corollary 4.11. *Theorem 4.9, in particular, applies (with $S := X$) to the case where $T := P_{C_k} \circ \ldots \circ P_{C_1}$ is a composition of metric projections onto closed convex subsets C_i in a complete CAT(κ)-space X ($\kappa > 0$) with $diam(X) < \pi/(2\sqrt{\kappa}) \leq d$ provided that $\bigcap_{i=1}^{k} C_i \neq \emptyset$. In this case one can take as SQNE-modulus*

$$\omega(\varepsilon) := (\varepsilon/k)^2 \cdot \beta/2d,$$

where

$$\beta = \frac{1}{2}(\pi - 2\sqrt{\kappa}\delta)\tan(\sqrt{\kappa}\delta) \text{ with } 0 < \delta < \pi/(2\sqrt{\kappa}) - diam(X).$$

Proof: This follows from Corollary 4.5 and Theorem 4.6. □

We now show the strong convergence of (x_n) in 4.9 with a rate of metastability if S is compact and T additionally is nonexpansive. For this we need a lemma:

Lemma 4.12. *The sequence (x_n) in Theorem 4.9 (but with T being additionally nonexpansive) is uniformly quasi-Fejér-monotone w.r.t. $F = Fix(T)$ (see Definition 6.3 in [18] in the special case where G, H are the identity function and $AF_k := \{p \in S : d(p, Tp) \leq 1/(k+1)\}$ and $X := S$) with modulus $\chi(r, n, m) := m(r + 1)$, i.e.*

$$\begin{aligned}&\forall r, n, m \in \mathbb{N}\,\forall p \in S\ (d(p, Tp) < \tfrac{1}{\chi(r,n,m)+1} \rightarrow \\ &\qquad \forall l \leq m (d(x_{n+l}, p) < d(x_n, p) + \textstyle\sum_{i=n}^{n+l-1} \delta_i + \tfrac{1}{r+1}).\end{aligned}$$

Proof: Easy induction using that for all $p \in S$

$$d(x_{n+1}, p) \leq d(x_{n+1}, Tx_n) + d(Tx_n, Tp) + d(Tp, p) < \delta_n + d(x_n, p) + d(Tp, p).$$

□

Definition 4.13. *Let K be a totally bounded metric space. A function $\gamma : \mathbb{N} \to \mathbb{N}$ s.t.*

$$\forall k \in \mathbb{N}\,\forall (x_n) \subset K\,\exists i, j \leq \gamma(k)\ \left(i < j \wedge d(x_i, x_j) \leq \frac{1}{k+1}\right)$$

is called a modulus of total boundedness for K (see [18]).

Theorem 4.14. *Let X be a metric space and $S \subseteq X$ be a compact subset with a modulus of total boundedness γ. Then under the conditions of Theorem 4.9 (plus T being additionally* *nonexpansive) the sequence* (x_n) *is strongly convergent with rate of metastability* $\widehat{\Psi}$ *(for* $\varepsilon = 1/(k+1)$*) defined as* $\widehat{\Psi}(k, g, \xi', \chi, \gamma, \beta) := \widehat{\Psi}_0(P, k, g, \xi', \chi, \beta)$ *with*

$$\begin{aligned}
&\widehat{\Psi}_0(0, \ldots) := 0,\\
&\widehat{\Psi}_0(n+1, \ldots) := \xi'\left(\chi_g^M(\widehat{\Psi}_0(n, \ldots), 8k+7), \beta(8k+7)\right),\\
&\quad \chi_g^M(n, k) := \max_{i \le n}\{\chi(i, g(i), k)\},\ P := \gamma(8k+7)+1,
\end{aligned}$$

where $\xi'(n, k) := \max\{\xi(i, 1/(j+1)) : i \le n, j \le k\}$ *and* ξ *is the* $\liminf$*-bound from Corollary 4.10 and* β *is a rate of convergence for* $\sum \delta_n$ *(for* $\varepsilon = 1/(k+1)$*)*

Proof: The proof is immediate from Theorem 4.9, Lemma 4.12 and [18][Theorem 6.4] (with $AF_k = \tilde{F}_k := \{p \in S : d(p, Tp) \le 1/(k+1)\}$ and $\alpha_G := \beta_H := id$). $\square$

Further below (see Theorem 4.19) we will show that such a metastability result can also be obtained for the important case of compositions $P_k \circ \ldots \circ P_1$ of metric projections in complete CAT(κ) spaces X despite of the fact that here we no longer have the nonexpansivity available. For this we need first to generalize Proposition 2.12 to SQNE mappings in the geodesic setting:

Proposition 4.15. *Let* X *be a metric space and* $S \subseteq X$ *be a subset. Let* $T_1, \ldots, T_k : S \to S$ *be SQNE-mappings with SQNE-moduli* $\omega_1, \ldots, \omega_k$*, resp. w.r.t. some common fixed point* $p \in S$ *of* $T_1, \ldots, T_k$ *and let* $d \in \mathbb{N}$*. Assume that* $T_1, \ldots, T_k$ *are uniformly continuous on* $S_d := \{x \in S : d(x, p) \le d\}$ *with modulus of uniform continuity* $\alpha : \mathbb{R}^*_+ \to \mathbb{R}^*_+$*, i.e.*

$$\forall \varepsilon > 0\, \forall y, y' \in S_d\ (d(y, y') < \alpha(\varepsilon) \to \bigwedge_{i=1}^{k} d(T_i y, T_i y') < \varepsilon).$$

Define $\rho(d, \varepsilon) := \chi_d(k-1, \varepsilon)$*, where* $\chi_d(0, \varepsilon) := \min\{\alpha(\varepsilon/2), \varepsilon\}$*,*
$\chi_d(n+1, \varepsilon) := \min\{\omega(d, \frac{1}{2}\chi_d(n, \varepsilon)), \frac{1}{2}\chi_d(n, \varepsilon)\}$ *with* $\omega(d, \varepsilon) := \min\{\omega_1(d, \varepsilon), \ldots, \omega_k(d, \varepsilon)\}$*. Then*

$$\forall x \in S_d\, \forall \varepsilon > 0\ (d(T_k T_{k-1} \ldots T_1 x, x) < \rho(d, \varepsilon) \to \bigwedge_{i=1}^{k}(d(T_i x, x) < \varepsilon)).$$

Proof: We first observe that $\chi_d(n, \varepsilon) \le \min\{\alpha(\varepsilon/2), \varepsilon\}$ for all $n \in \mathbb{N}$. Note also that, in particular, $T_1, \ldots, T_k$ are quasi-nonexpansive w.r.t. p.
We proceed by indiction on $1 \le l \le k$: For $l = 1$ the statement is trivial. So let $1 < l \le k$ Assume that

$$(1)\ d(T_l T_{l-1} \ldots T_1 x, x) < \rho(d, \varepsilon) = \min\{\omega(d, \frac{1}{2}\chi_d(l-2, \varepsilon)), \frac{1}{2}\chi_d(l-2, \varepsilon)\} \le \frac{\varepsilon}{2}.$$

Define $y := T_{l-1} \ldots T_1 x$. Then

$$(2)\ d(x, p) - \min\{\omega(d, \frac{1}{2}\chi_d(l-2, \varepsilon)), \frac{1}{2}\chi_d(l-2, \varepsilon)\} < d(T_l y, p) \le d(y, p) \le d(x, p).$$

Since ω is an SQNE-modulus of T_l w.r.t. p, (2) yields

$$(3)\ d(T_l y, y) < \frac{1}{2}\chi_d(l-2,\varepsilon) \leq \alpha(\varepsilon/2)/2.$$

(1) also implies

$$(4)\ d(T_l y, x) < \frac{1}{2}\chi_d(l-2,\varepsilon) \leq \alpha(\varepsilon/2)/2.$$

By (3) and (4) we get

$$(5)\ d(x, T_{l-1}\ldots T_1 x) = d(x,y) < \chi_d(l-2,\varepsilon)$$

as well as

$$(6)\ d(x,y) < \alpha(\varepsilon/2)$$

and so in turn

$$(7)\ d(T_l x, T_l y) < \frac{\varepsilon}{2}.$$

Together with (1) this gives

$$(8)\ d(x, T_l x) \leq d(x, T_l y) + d(T_l y, T_l x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, by (8), we have shown that x is an ε-approximate fixed point of T_l and by (5) and the induction hypothesis we get this also for $T_{l-1}, \ldots, T_1$. $\square$

Remark 4.16. *There is an interesting logical point to mention in connection with Proposition 4.15: as briefly remarked in the introduction, behind the quantitative results in this paper, there is a general logical pattern guiding this. In fact, there are general results from logic to guarantee such quantitative enrichments to be possible. One of the crucial conditions made in these logical results is that the, otherwise considered as trivial, equality axiom*

$$x = y \rightarrow Tx = Ty$$

for functions $T : X \rightarrow X$ is not included in the formal framework but only a weaker rule: from a proof of $s = t$ one can infer that $Ts = Tt$. If this rule is not sufficient and the full equality axiom is needed, then one has to impose that T is uniformly continuous on bounded sets with a given modulus of uniform continuity as this is the correct quantitative form of the equality axiom. In the original proof of the non-quantitative version of Proposition 4.15 one does use the equality axiom in the form

$$y = x \rightarrow T_k y = T_k x$$

which is not just a use of the rule mentioned above, since the proof that $y = x$ uses the (universal) assumption '$x = (T_k \circ \ldots \circ T_1)x$' which is not permitted in this rule (note that $x = y$ is defined as $d_X(x,y) =_{\mathbb{R}} 0$ and that $=_{\mathbb{R}}$ is universal given the representation of reals as used in [15]). It is because of this use of the equality axiom that we need to assume that T_k (and in turn also $T_{k-1}, \ldots, T_1$) are uniformly continuous on bounded sets (note that *in the case of SNE-mappings in Proposition 2.12 this was trivially satisfied). See [15] for extensive discussions of this issue. In the case at hand, actually we only need the uniform continuity (as far as T_k and the argument above is concerned) on pairs of points x, y one of which is a fixed point of $T_k \circ \ldots \circ T_1$. This sometimes is already realized by certain conditions on the mappings which do not imply their full uniform continuity (see [18] for a discussion of this point).*

Combining Theorems 4.6,4.7 and Proposition 4.15 we get the following result:

Corollary 4.17. *Let $T_1, \ldots, T_k : S \to S$ be SQNE-mappings with respective moduli $\omega_1, \ldots, \omega_k$ w.r.t. some common fixed point p of $T_1, \ldots, T_k$. Assume that $T_1, \ldots, T_k$ are uniformly continuous on bounded sets. Define for $x \in S$, $x_n := T^n x$ with $T := T_k \circ \ldots \circ T_1$. Let $d \in \mathbb{N}$ and define $S_d := \{y \in S : d(y, p) \leq d\}$. Let α_d be a common modulus of uniform continuity for $T_1, \ldots, T_k$ on S_d. Then*

$$\forall x \in S_d \, \forall \varepsilon > 0 \, \forall g \in \mathbb{N}^{\mathbb{N}} \, \exists n \leq \zeta(d, \varepsilon, g) \, \forall m \in [n, n + g(n)] \, \left(\bigwedge_{i=1}^{k} d(T_i x_m, x_m) < \varepsilon \right),$$

where

$$\zeta(d, \varepsilon, g) := \psi_d(\rho(d, \varepsilon), g) \text{ with } \psi_d(\varepsilon, g) := \tilde{g}^{(\lceil d/\omega(d,\varepsilon) \rceil)}, \; \tilde{g}(n) := n + g(n) + 1,$$
$$\omega(d, \varepsilon) := \min\{\omega_1(d, \varepsilon/k), \ldots, \omega_k(d, \varepsilon/k)\} \text{ and } \rho(d, \varepsilon) \text{ as in Proposition 4.15.}$$

Proof: By Theorem 4.6, ω is an SQNE-modulus for T w.r.t. p. Hence, by Theorem 4.7, $\psi_d(\varepsilon, g)$ is a rate of metastability for the asymptotic regularity of T for $x \in S_d$. So there exists an $n \leq \zeta(d, \varepsilon, g)$ s.t.

$$\forall m \in [n, n + g(n)] \; (d(T x_m, x_m) = d(T^{m+1}x, T^m x) < \rho(d, \varepsilon)$$

and so, by Proposition 4.15 (since $d(x_m, p) \leq d(x, p) \leq d$),

$$\forall m \in [n, n + g(n)] \; \left(\bigwedge_{i=1}^{k} d(T_i x_m, x_m) < \varepsilon \right).$$

□

We now prove that (even perturbed) iterations of compositions of metric projections in complete CAT(κ) spaces are uniformly quasi-Fejér monotone provided that the projections have a common fixed point.

Lemma 4.18. *Let X be a complete CAT(κ) space with $\kappa > 0$ and $diam(X) < \pi/(2\sqrt{\kappa})$. Let $C_1, \ldots, C_k \subseteq X$ be nonempty, closed and convex subsets with $\bigcap_{i=1}^{k} C_i \neq \emptyset$. Consider $T_i := P_{C_i}$ for $i = 1, \ldots, k$ and $T := T_k \circ \ldots \circ T_1$. Let (x_n) be a sequence in X with $d(x_{n+1}, T x_n) < \delta_n$ with $\sum \delta_n < \infty$, where (δ_n) is a sequence of nonnegative reals. Then (x_n) is uniformly quasi-Fejér monotone w.r.t. $F := Fix(T)$ (in the sense of Lemma 4.12) and we can compute a modulus χ for this property (see the proof).*

Proof: By [11] we know that for all $p \in C_i$ and all $x \in X$

$$d(T_i x, p) \le d(x, p).$$

Now let $q \in X$ be s.t. $d(q, T_i q) < \delta$. Since $T_i q \in C_i$ we get

$$d(T_i x, q) \le d(T_i x, T_i q) + d(T_i q, q) \le d(x, T_i q) + d(T_i q, q) \le d(x, q) + 2d(T_i q, q) < d(x, q) + 2\delta.$$

By [1] we have that the projections T_i are Lipschitzian with Lipschitz constant

$$\lambda := \frac{M\sqrt{\kappa}}{2\arcsin(\sin(M\sqrt{\kappa}/2)\cos(M\sqrt{\kappa}))},$$

where $diam(X) \le M < \pi/(2\sqrt{\kappa})$, and so $\alpha(\varepsilon) := \varepsilon/\lambda$ is a common modulus of uniform continuity for $T_1, \ldots, T_k$. Moreover, in Corollary 4.5 we have shown that the T_i are SQNE and computed an SQNE-modulus ω for T_i which, in particular, holds for the points in $\bigcap_{i=1}^k C_i$. By Proposition 4.15 (with $S := X$) we know that any $\rho(d, \delta)$-approximate fixed point of T is a common δ-fixed point for all $T_1, \ldots, T_k$, where $d \ge \pi/2\sqrt{\kappa}$. From this and iterating the reasoning above, we get that for each $p \in X$ with $d(p, Tp) < \rho(d, \delta)$ we have that for all $x \in X$

$$d(Tx, p) < d(x, p) + 2k\delta$$

and so

$$d(x_{n+1}, p) < d(x_n, p) + 2k\delta + \delta_n.$$

Hence

$$d(x_{n+l}, p) < d(x_n, p) + 2lk\delta + \sum_{i=n}^{n+l-1} \delta_i.$$

This yields that

$$\chi(n, m, r) := \left\lceil \frac{1}{\rho(d, 1/(2mk(r+1)))} \right\rceil$$

is a modulus of (x_n) being uniformly quasi-Fejér monotone. □

Theorem 4.19. *Let $X, C_1, \ldots, C_k, P_{C_1}, \ldots, P_{C_k}, (x_n), (\delta_n), \chi$ be as in Lemma 4.18. Let X be, moreover, compact with modulus of total boundedness γ. Then the sequence (x_n) is strongly convergent with rate of metastability $\widehat{\Psi}(k, g)$ as defined in Theorem 4.14 but now with χ from Lemma 4.18.*

Proof: The proof is immediate from Corollary 4.10 with $S := X$ (which is applicable due to Corollary 4.5, Theorem 4.6), Lemma 4.18 and [18][Theorem 6.4] (with $AF_n = \tilde{F}_n := \{p \in X : d(p, Tp) \le 1/(n+1)\}$). □

Remark 4.20. *Instead of assuming X to be compact it suffices to use that C_k is compact by viewing T as a selfmap of C_k. This is obvious in the absence of error terms δ_n where $x_n \in C_k$ for $n \geq 1$. In the presence of error terms one can apply the reasoning to $y_n := T(x_n)$ noticing that as in the proof of Lemma 4.18 also (y_n) is uniformly quasi-Fejér monotone (with the same modulus χ). Using a rate of convergence for $\delta_n \to 0$, the rate of metastability for (y_n) can then be transformed into one for (x_n).*

Remark 4.21. *The metastability results in Theorems 4.14 and 4.19 can also be strengthened as follows: since T in 4.14 is nonexpansive (and Lipschitzian in 4.19) one gets moduli of explicit closedness ω_F, δ_F (for $F = Fix(T)$) in the sense of [18] which then can be used as in Theorem 5.3 in [18] to modify the rate $\widehat{\Psi}$ into a rate $\tilde{\Psi}(k,g)$ satisfying for $k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}$*

$$\exists n \leq \tilde{\Psi}(k,g)\, \forall i,j \in [n, n+g(n)] \ \left(d(x_i,x_j) \leq \frac{1}{k+1} \ \textit{and} \ \ d(x_i,Tx_i) \leq \frac{1}{k+1} \right).$$

Applying this to $k' := \lceil 1/\rho(d, 1/(k+2)) \rceil + 1 \geq k$ with ρ from Proposition 4.15 we then get (using that proposition) in the situation of Theorem 4.19

$$\exists n \leq \tilde{\Psi}(k',g)\, \forall i,j \in [n, n+g(n)] \ \left(d(x_i,x_j) \leq \frac{1}{k+1} \ \textit{and} \ \bigwedge_{m=1}^{k} (d(x_i,T_m x_i) \leq \frac{1}{k+1}) \right)$$

which constitutes a finitary quantitative version of the theorem that (x_n) converges to a common fixed point of $T_1, \ldots, T_k$.

In [2], the concept of being λ-firmly nonexpansive has been generalized (for $\lambda \in (0,1)$) to the context of W-hyperbolic spaces in the sense of [14] (in the case of the Hilbert ball this is due to [12]; W-hyperbolic spaces are closely related to the hyperbolic spaces due to [27], see [14] for a detailed discussion of the relationship): let (X,d,W) be a W-hyperbolic space, $S \subseteq X$ and $T : S \to X$. Given $\lambda \in (0,1)$, we say that T is λ-firmly nonexpansive if for all $x, y \in S$

$$d(Tx,Ty) \leq d\left((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty\right).$$

Here $(1-\lambda)x \oplus \lambda y$ is defined as $W(x,y,\lambda)$, i.e. as the unique point z in the metric segment $[x,y]$ provided by W with $d(x,z) = \lambda d(x,y)$.
We say that $T : S \to X$ with $Fix(T) \neq \emptyset$ is λ-firmly-quasi-nonexpansive if for all $x, p \in S$ with $p \in Fix(T)$

$$d(Tx,p) \leq d\left((1-\lambda)x \oplus \lambda Tx, p\right).$$

Note that if T is λ-firmly-nonexpansive and has a fixed point, then it also is λ-firmly-quasi-nonexpansive since $W(x,x,\lambda) = x$.

A hyperbolic space (X,d,W) is *uniformly convex* [22] if for any $r > 0$ and any $\varepsilon \in (0,2]$ there exists $\theta \in (0,1]$ such that for all $a, x, y \in X$,

$$\left.\begin{array}{l} d(x,a) \leq r \\ d(y,a) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\theta)r. \tag{1}$$

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing such a $\theta := \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called a *modulus of uniform convexity*.
In the sequel, (X,d,W) is a uniformly convex space and η is a modulus of uniform convexity.
As a counterpart to Lemma 2.15 we now have

Lemma 4.22 ([22]). *Let $r > 0, \varepsilon \in (0,2]$ and $a,x,y \in X$ be such that $d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq \varepsilon r$. Then for any $\lambda \in [0,1]$,*

$$d((1-\lambda)x \oplus \lambda y, a) \leq (1 - 2\lambda(1-\lambda)\eta(r,\varepsilon))r.$$

Following [23] we call a uniformly convex W-hyperbolic space with a modulus of convexity $\eta(r,\varepsilon)$ which (for fixed ε) decreases as r increases a UCW-space. We call such a modulus monotone.
We now show that T being λ-firmly(-quasi-)nonexpansive implies that T is SQNE (provided that $Fix(T) \neq \emptyset$) and we give a quantitative version of this:

Proposition 4.23. *Let (X,d,W) be a UCW-space with monotone modulus of convexity η. Let $S \subseteq X$ and $T : S \to X$ be a λ-firmly-quasi-nonexpansive mapping for some $\lambda \in (0,1)$ (resp. only w.r.t. some fixed point p of T). Then T is an SQNE-mapping (resp. SQNE w.r.t. p) with SQNE-modulus*

$$\omega(c,\varepsilon) := \lambda(1-\lambda)\eta(c,\varepsilon/c)\cdot\varepsilon$$

(for $\varepsilon/c \leq 2$ and $:= 1$, otherwise).
If η can be written as $\eta(r,\varepsilon) = \varepsilon \cdot \tilde{\eta}(r,\varepsilon)$, where $\tilde{\eta}(r,\varepsilon)$ for fixed r increases as ε increases, then ω can be improved to

$$\omega(c,\varepsilon) := 2\lambda(1-\lambda)\tilde{\eta}(c,\varepsilon/c)\cdot\varepsilon.$$

Proof: Since T is λ-firmly-quasi-nonexpansive w.r.t. p we have for all $x \in S$

$$(1)\ d(Tx,p) \leq d\left((1-\lambda)x \oplus \lambda Tx, p\right) \leq (1-\lambda)d(x,p) + \lambda d(Tx,p) \leq d(x,p).$$

Now assume that $x \in S$ with $\varepsilon/2 \leq d(x,p) \leq c$ and

$$(2)\ d(x,p) - d(Tx,p) < \lambda(1-\lambda)\eta(c,\varepsilon/c)\cdot\varepsilon.$$

By (1) and (2) we get (using the monotonicity of η)

$$(3)\ \begin{cases} d\left((1-\lambda)x \oplus \lambda Tx, p\right) > d(x,p) - 2\lambda(1-\lambda)\eta(c,\varepsilon/c)\cdot d(x,p) \\ \qquad\qquad\qquad\qquad\quad \geq d(x,p) - 2\lambda(1-\lambda)\eta(d(x,p),\varepsilon/c)\cdot d(x,p). \end{cases}$$

Since $d(Tx,p) \leq d(x,p) \leq c$, Lemma 4.22 (applied to $a := p$, $y := Tx$ and $r := d(x,p)$) implies

$$d(x,Tx) < \frac{\varepsilon}{c}d(x,p) \leq \varepsilon.$$

If $d(x,p) < \varepsilon/2$ (and hence $d(Tx,p) < \varepsilon/2$), then trivially $d(x,Tx) \leq d(x,p) + d(p,Tx) < \varepsilon$. The additional claim is shown as follows: assume instead of (2)

$$
\begin{aligned}
(2)' \; d(x,p) - d(Tx,p) &< 2\lambda(1-\lambda) \cdot \varepsilon \cdot \tilde{\eta}(c, \varepsilon/c) \\
&\leq 2\lambda(1-\lambda) \cdot \varepsilon \cdot \tilde{\eta}(c, \varepsilon/d(x,p)) \\
&= 2\lambda(1-\lambda) \cdot \eta(c, \varepsilon/d(x,p)) \cdot d(x,p).
\end{aligned}
$$

(1) and $(2)'$ then yield

$$
(3)' \left\{ \begin{aligned} d\left((1-\lambda)x \oplus \lambda Tx, p\right) &> d(x,p) - 2\lambda(1-\lambda)\eta(c, \varepsilon/d(x,p)) \cdot d(x,p) \\ &\geq d(x,p) - 2\lambda(1-\lambda)\eta(d(x,p), \varepsilon/d(x,p)) \cdot d(x,p) \end{aligned} \right.
$$

and so by Lemma 4.22

$$
d(x,Tx) < \frac{\varepsilon}{d(x,p)} d(x,p) = \varepsilon.
$$

$\square$

Remark 4.24. *1. It is well-know that for CAT(0) spaces X, one can take - as in the case of Hilbert spaces - $\eta(r,\varepsilon) := \frac{\varepsilon^2}{8}$ as a modulus of uniform convexity (see [22]) and so gets $\omega(c,\varepsilon) := \frac{\lambda(1-\lambda)}{4c}\varepsilon^2$ (for $\varepsilon \leq 2c$ and $:= 1$, otherwise) as SQNE-modulus for any λ-firmly-quasi-nonexpansive $T : S \to X$ (with $\lambda \in (0,1)$).*

2. *Theorem 4.7 (last part) and Proposition 4.23 together, in particular, yield the same bounds on the asymptotic regularity for Picard iterations of λ-firmly nonexpansive mappings as the ones obtained recently in [2](Theorem 7.1,Remark 7.2). Note that for λ-firmly-nonexpansive mappings one can (as in [2]) replace the assumption of the existence of a fixed point by that of arbitrarily good approximate fixed points in some c-ball around the starting point.*

3. *Iterations of firmly nonexpansive mappings are also asymptotically regular in general Banach spaces ([26]) and even W-hyperbolic spaces ([2]) despite of the fact that in such settings in general being firmly nonexpansive does not imply being SNE. An exponential rate of asymptotic regularity has recently been extracted using proof mining in [24].*

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