Effective metastability for modified Halpern iterations in CAT(0) spaces*

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We examine convergence results for modified Halpern iterations due to Cuntavepanit and Panyanak [4]. Following Kohlenbach and Leuştean [15] we extract uniform rates of metastability. This includes extracting rates of asymptotic regularity and replacing an ineffective argument that uses Banach limits.

Keywords: Modified Halpern iteration, metastability, rate of asymptotic regularity, proof mining, CAT(0) space

1 Introduction

Recently, Kohlenbach and Leuştean developed a method for analyzing convergence proofs that make use of Banach limits (and hence – for what is known – the axiom of choice) and applied this method to obtain quantitive versions of convergence results for Halpern iterations in CAT(0) and uniformly smooth spaces (see [15, 16]). In this paper we apply this method to a recent convergence proof (again in the CAT(0)-setting) due to Cuntavepanit and Panyanak [4] for a modified scheme of Halpern iterations due to Kim and Xu [10].

Given a nonempty convex subset C in a CAT(0) space, we consider $u, x \in C$, a sequence $(\lambda_n) \subset [0,1]$ and a nonexpansive mapping $T : C \to C$ with a nonempty fixed point set. Then the *Halpern iterations* with initial point x and reference point u are given by

$$x_0 := x, \quad x_{n+1} := \lambda_n u \oplus (1-\lambda_n)Tx_n, \quad \text{for } n \in \mathbb{N}.$$

Here for $x, y \in X$ and $\lambda \in [0,1]$, $\lambda x \oplus (1-\lambda)y$ denotes the unique point $z \in X$ with $d(z,x) = (1-\lambda)d(x,y)$ and $d(z,y) = \lambda d(x,y)$.

A particularly important choice for λ_n is $\left(\frac{1}{n+2}\right)_{n\in\mathbb{N}}$. Then, if T is linear and u is chosen equal to x, one obtains the Cesàro averages of $\{T^n x\}_{n\in\mathbb{N}}$

$$x_n = \frac{1}{n+1}\sum_{i=0}^{n} T^i x.$$

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The conditions used in this paper always allow for this choice of (λ_n).

In [15], Kohlenbach and Leuştean extracted both effective rates of convergence for the asympotic regularity property

$$d(x_n, Tx_n) \to 0$$

as well as effective so-called rates of metastability (in the sense of Tao [27]) for the convergence of the sequence (x_n) of Halpern iterations applying techniques of the proof mining program (see [13] for general information) to a convergence proof due to Saejung [22]. Here, by a rate of metastability, we mean a function $\Psi : (0, \infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that

$$\forall \varepsilon > 0 \quad \forall g : \mathbb{N} \to \mathbb{N} \quad \exists N \leq \Psi(\varepsilon, g) \quad \forall m, n \in [N; N + g(N)] \quad (d(x_n, x_m) \leq \varepsilon).$$

In general, there is no computable rate of convergence for Halpern iterations $(x_n)_{n\in\mathbb{N}}$ (already for $\lambda_n := 1/(n + 2)$ and linear T) as follows from [1, Theorem 5.1]. Note, however, that the metastability property

$$\forall \varepsilon > 0 \quad \forall g : \mathbb{N} \to \mathbb{N} \quad \exists N \in \mathbb{N} \quad \forall m, n \in [N; N + g(N)] \quad (d(x_n, x_m) \leq \varepsilon)$$

(for which [15] does extract effective rates) ineffectively *is* equivalent to the usual Cauchy property.

Saejung's original proof makes substantial reference to the axiom of choice by using the existence of Banach limits. Kohlenbach and Leuştean eliminated this reference in favor of a use of a finitary functional which renders Saejung's proof admissible for the proof mining program. In this paper we apply this method to a variation of Halpern iterations, the aforementioned modified Halpern iterations due to [10]:

In the same setting as above we consider two sequences $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0, 1]$. Then the *modified Halpern iterations* with initial point x and reference point u are given by

$$x_0 := x, \quad x_{n+1} := \beta_n u \oplus (1 - \beta_n)(\alpha_n x_n \oplus (1 - \alpha_n)Tx_n), \quad \text{for } n \in \mathbb{N}.$$

So instead of using Tx_n as in the usual Halpern iteration, one takes here a so-called Krasnoselski-Mann iteration

$$\alpha_n x_n \oplus (1 - \alpha_n)Tx_n.$$

Modified Halpern iterations can be seen as generalizations of Halpern iterations by putting $\alpha_n \equiv 0$ and so our results (which allow this choice) extend the quantitative metastability and asymptotic regularity results of [15]. The main convergence result that we treat is due to Cuntavepanit and Panyanak [4].

Even for ordinary Halpern iterations, the inclusion of the case of unbounded C in our bounds is new compared to [15].

Our paper further strengthens the claim made in [15] to have developed a general method for analyzing quantitatively strong convergence proofs that use Banach limits.

2 Preliminaries

CAT(0) spaces are instances of geodesic spaces which are special metric spaces. Roughly speaking, in a geodesic space the associated metric behaves in an orderly manner, i.e. making sure there is at least one shortest path between two points. A CAT(0) space enforces further regularity in the sense that every triangle in the space is as "thin" as in Euclidean space.

The terminology of CAT(κ) spaces is due to Gromov [8]. CAT(0) spaces are uniquely geodesic. [2, Proposition II.1.4(1)]

With $(1 - \lambda)x \oplus \lambda y$ we denote the unique point z on the unique geodesic segment $[x, y]$ joining x and y such that

$$d(z, x) = \lambda d(x, y) \text{ and } d(z, y) = (1 - \lambda)d(x, y) \tag{1}$$

holds.

The following properties of CAT(0) spaces are of interest to us.

Proposition 2.1 ([5, Lemma 2.5]). *Let X be a CAT(0) space. Then the following inequality holds for all $x, y, z \in X$ and for all $t \in [0,1]$:*

$$d((1-t)x \oplus ty, z)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2. \tag{2}$$

For a uniquely geodesic space X, this property is equivalent to X being a CAT(0) space.

Proposition 2.2 ([5, Lemma 2.4]). *Let X be a CAT(0) space. If x, y, z are points in X and $t \in [0,1]$, then*

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z), \tag{3}$$

i.e. CAT(0) spaces are, in particular, convex metric spaces in the sense of Takahashi [26] by taking $W(x,y,\lambda) := (1-\lambda)x \oplus \lambda y$.

Every pre-Hilbert space is a CAT(0)-space. Another example is the open unit ball B in $\mathbb{C}$ with the Poincaré metric,

$$\rho(z,w) := 2\tanh^{-1}\left|\frac{z-w}{1-z\overline{w}}\right| \quad \text{for } z, w \in B.$$

This example is interesting for fixed point theory, since holomorphic mappings $f : B \to B$ are nonexpansive with respect to ρ (Schwarz-Pick Lemma, see [7]). $\mathbb{R}$-trees in the sense of Tits are further examples of CAT(0) spaces.
W-hyperbolic spaces are in turn generalizations of CAT(0) spaces. The following definition of W-hyperbolic spaces is due to Kohlenbach [12, Definition 2.11].

Definition 2.3. A triple (X, d, W) is called a *W-hyperbolic space* if (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is a mapping satisfying

(W1) $d(z, W(x,y,\lambda)) \le (1-\lambda)d(z,x) + \lambda d(z,y)$,

(W2) $d(W(x,y,\lambda_0), W(x,y,\lambda_1)) = |\lambda_0 - \lambda_1| d(x,y)$,

(W3) $W(x,y,\lambda) = W(y,x,1-\lambda)$,

(W4) $d(W(x,z,\lambda), W(y,w,\lambda)) \le (1-\lambda)d(x,y) + \lambda d(z,w)$.

Lemma 2.4 ([13, p.386]). *Let (X,d) be a CAT(0) space. If it is equipped with the mapping $W : X \times X \times [0,1] \to X$,*

$$W(x,y,\lambda) := (1-\lambda)x \oplus \lambda y,$$

W satisfies (W1-W4), i.e. (X,d,W) is W-hyperbolic.

We will use the following notions (here and in the following $\mathbb{N}$ is the set of natural numbers including 0 while $\mathbb{Z}_+$ denotes the set of natural numbers $n \ge 1$):

(1) A mapping $\gamma : (0,\infty) \to \mathbb{Z}_+$ is called a *Cauchy modulus* of a Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in a metric space (X,d) if

$$\forall \varepsilon > 0 \quad \forall n \in \mathbb{N} \quad (d(a_{\gamma(\varepsilon)+n}, a_{\gamma(\varepsilon)}) \le \varepsilon).$$

(2) For $(a_n)_{n\in\mathbb{N}}$ as above, a mapping $\Psi : (0,\infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{Z}_+$ is a *rate of metastability* if

$$\forall \varepsilon > 0 \quad \forall g : \mathbb{N} \to \mathbb{N} \quad \exists N \le \Psi(\varepsilon, g) \quad \forall m, n \in [N; N + g(N)] \quad (d(a_n, a_m) \le \varepsilon).$$

(3) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $\limsup_{n\to\infty} a_n \le 0$, then a mapping $\theta : (0,\infty) \to \mathbb{Z}_+$ is called an *effective rate* for $(a_n)_{n\in\mathbb{N}}$ if

$$\forall \varepsilon > 0 \quad \forall n \ge \theta(\varepsilon) \quad (a_n \le \varepsilon).$$

(4) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative reals such that $\sum_{n=1}^{\infty} a_n = \infty$. Then a function $\delta : \mathbb{Z}_+ \to \mathbb{Z}_+$ with

$$\sum_{i=1}^{\delta(n)} a_i \geq n \quad \text{forall } n \in \mathbb{Z}_+$$

is called a *rate of divergence* of $(\sum_{i=1}^{n} a_i)_{n\in\mathbb{N}}$.

The term metastability is due to Tao [27, 28]. It is an instance of the no-counterexample interpretation by Kreisel [17, 18].

3 Halpern iterations

Definition 3.1. Let C be a nonempty convex subset of a CAT(0) space X. Let $u, x \in C$, $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0,1]$ and $T : C \to C$ be nonexpansive. The *modified Halpern iterations* $x_n \in C$ with initial point x and reference point u are

$$x_0 := x, \quad x_{n+1} := \beta_n u \oplus (1-\beta_n)(\alpha_n x_n \oplus (1-\alpha_n) T x_n), \quad \text{for } n \in \mathbb{N}.$$

Combinations of the following conditions were considered for the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$.

(D1) $(a) \lim_{n\to\infty} \beta_n = 0, \quad (b) \lim_{n\to\infty} \alpha_n = 0,$

(D2) $(a) \sum_{n=0}^{\infty} \beta_n = \infty, \quad (b) \sum_{n=0}^{\infty} \alpha_n = \infty,$

(D3) $(a) \sum_{n=0}^{\infty} |\beta_n - \beta_{n+1}| < \infty, \quad (b) \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty,$

(D4) $(a) \prod_{n=0}^{\infty} (1-\beta_n) = 0, \quad (b) \prod_{n=0}^{\infty} (1-\alpha_n) = 0,$

(D5) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$

We will be concerned with (D1) to (D4).

Modified Halpern iterations are a generalization of Halpern iterations if one is permitted to set $\alpha_n := 0$ for all $n \in \mathbb{N}$. This excludes (D2.b) which, however, we will never need.

Halpern iterations were named after a paper by Halpern [9] in 1967. This is somewhat misleading since Halpern considered only an instance of Halpern iterations in which the reference point is set to 0 and hence required a closed ball around 0 to be contained the domain C of the self-mapping T.

In this paper, Halpern examines these iterations in the setting of Hilbert spaces. For the convergence of $(x_n)_{n\in\mathbb{N}}$ to a fixed point of T with the smallest norm (hence closest to $u = 0$), he showed that the conditions (D1.a) and (D2.a) were necessary. He also gave a set of sufficient conditions.

In 1977, Lions [20] improved Halpern's original result. He considers real Hilbert spaces and Halpern iterations in full generality in this article and shows the convergence of the iteration to the fixed point of T nearest to u under the following conditions: $(\lambda_n)_{n\in\mathbb{N}} \in (0,1]$, (D1.a), (D2.a) and $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$. Furthermore, he generalizes his result to a variation of Halpern iterations which deal with finite families of nonexpansive operators T_i, $0 \leq i \leq N$, with $N \in \mathbb{N}$ instead of one nonexpansive T.

Halpern's and Lion's results do not cover the choice of $(\lambda_n)_{n\in\mathbb{N}} = (\frac{1}{n+2})_{n\in\mathbb{N}}$.

In 1983, Reich [21] posed the following problem, which was referred to as Problem 6.

Let X be a Banach space. Is there a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that whenever a weakly compact convex subset C of X posses the fixed point property for nonexpansive mappings, then $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point of T for all $x \in C$ and all nonexpansive mappings $T : C \to C$?

Many partial answers were given to this problem, we will only give a brief overview. The problem in its full generality is still open.
Wittmann [29] proved a result in 1992 that finally allows for $\lambda_n = \frac{1}{n+2}$ in Hilbert spaces. While Halpern's proof relies on a limit theorem for a resolvent, Wittmann carries out a direct proof.

Theorem 3.2 (Wittmann [29, Theorem 2]). *Let C be a closed convex subset of a Hilbert space X and $T : C \to C$ a nonexpansive mapping with a fixed point. Assume $(\lambda_n)_{n\in\mathbb{N}}$ satisfies (D1.a), (D2.a) and (D3.a). Then for any $x \in C$, the Halpern iteration $(x_n)_{n\in\mathbb{N}}$ with $u = x \in C$ converges to the projection Px of x on* $\mathrm{Fix}(T)$.

Five years later, in 1997, Shioji and Takahashi [24] considered Banach spaces with uniformly Gâteaux-differentiable norm with a closed, convex subset C. They treat also the case $u \neq x$ for $u, x \in C$ for nonexpansive mappings $T : C \to C$ with a nonempty fixed point set and show the convergence of $(x_n)_{n\in\mathbb{N}}$ to a fixed point if the conditions (D1.a), (D2.a) and (D3.a) hold for $(\lambda_n)_{n\in\mathbb{N}} \subset [0, 1]$ and for $0 < t < 1$, the sequence satisfying

$$z_t := tu + (1 - t)Tz_t$$

converges strongly to $z \in \mathrm{Fix}(T)$ as $t \to 0$. The existence of this sequence follows from Banach's fixed point theorem.
In 2007, Saejung [22] considered the case of complete CAT(0) spaces, which are a generalization of Hilbert spaces – as mentioned already above – and showed that for closed and convex subsets C, $T : C \to C$ nonexpansive with a nonempty fixed point set, and $(\lambda_n)_{n\in\mathbb{N}} \subset (0, 1)$ satisfying the conditions (D1.a),(D2.a) to (D3.a) or, alternatively, (D1.a), (D2.a) and $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$, Halpern iterations converge strongly to the fixed point of T nearest to u (for the case of the Hilbert ball, which is a CAT(0) space, see already [7]). Saejung also studies Halpern iterations with finitely and countably many different nonexpansive mappings sharing a fixed point.
In 2011, Kohlenbach [14] considered Wittmann's proof and the case of $\lambda_n := \frac{1}{n+1}$ for all $n \in \mathbb{N}$ for Halpern iterations in Hilbert spaces. He extracts a rate of metastability in both bounded and unbounded domains C. Subsequently, Kohlenbach and Leuştean [15] gave an effective uniform rate of metastability for Halpern iterations in CAT(0) spaces by analyzing Saejung's proof. They now treated arbitrary $(\lambda_n)_{n\in\mathbb{N}}$ satisfying (D1.a), either (D2.a) or (D4.a) and (D3.a) and the case of bounded C. As an intermediate step, they use (improved versions of) uniform effective rates of asymptotic regularity which are due to Leuştean [19] in 2007. In [16], Kohlenbach and Leuştean also develop a new metatheorem for real Banach spaces with a norm-to-norm uniformly continuous duality selection map. This metatheorem is the applied to the convergence proof of Halpern iterations by Shioji and Takahashi [24] for the extraction of rates of metastability in the setting of the metatheorem (though only relative to a given rate of asymptotic regularity for the resolvent whose computation in this setting is still subject of ongoing research).
Kim and Xu [10] showed in 2005 for their modified Halpern iteration from Definition 3.1:
Let C be a closed convex subset of a uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping with nonempty fixed point set. Under the conditions (D1)-(D3) (a)+(b), $(x_n)_{n\in\mathbb{N}}$ converges strongly to a fixed point of T.
Independent of each other, Suzuki [25] in 2006 and Chidume and Chidume [3] in 2007 considered the following different iteration scheme:

$$x_0 := x \in C, \quad y_{n+1} := \alpha_n u \oplus +(1 - \alpha_n)((1 - \beta)x_n \oplus \beta Tx_n) \text{ for } n \in \mathbb{N}, \tag{4}$$

for $\beta \in (0, 1)$. By ruling out $\beta = 1$, they exclude original Halpern iterations in their scheme.
Let X be a Banach space with uniformly uniformly Gâteaux-differentiable norm, $C \subset X$ a closed convex subset, $T : C \to C$ nonexpansive with nonempty fixed point set, $u, x \in C$. They showed

convergence of this scheme to a fixed point of T if $(\lambda_n)_{n\in\mathbb{N}} \subset [0,1]$ satisfies (D1.a) and (D2.a) and that $(z_t)_{t\in(0,1)}$ converges strongly to some point $z \in C$ as $t \to 0$ where z_t is the unique element of C with $z_t = tu + (1-t)Tz_t$ for every $0 < t < 1$.
Note that Kim and Xu's result does not permit this iteration scheme, since a constant $\beta \in (0,1)$ does not satisfy (D1.b).
In 2011, Cuntavepanit and Panyanak [4] generalized Kim and Xu's result to CAT(0) spaces and eliminated the use of condition (D2.b). They consider C to be a nonempty closed convex subset of a complete CAT(0) space X, $x, u \in C$ and $T : C \to C$ a nonexpansive mapping with nonempty fixed point set and show strong convergence to the fixed point of T nearest to u of the modified Halpern iterations defined here under the conditions (D1.a), (D1.b), (D2.a), (D3.a) and (D3.b).
This scheme does not cover the schemes due to Suzuki and Chidume and Chidume, since the choice of (α_n) as $(1-\beta)$ is not permitted because of (D1.a). Since in [4] (D2.b) is now longer used, modified Halpern iterations can be viewed as generalizations of Halpern iterations.
Cuntavepanit and Panyanak also considered a different iteration scheme: For $(\alpha_n), (\beta_n) \subset [0,1]$, let

$$x_0 := x \in C, \quad x_{n+1} := \beta_n x_n \oplus (1-\beta_n)(\alpha_n u \oplus (1-\alpha_n)Tx_n), \text{ for } n \in \mathbb{N}, \tag{5}$$

and showed that the conditions (D1.b), (D2.b) and (D5) sufficed for strong convergence in the above setting. We will call these iterations *Secondary Modified Halpern iterations.*
This scheme excludes original Halpern iterations. Setting for all $n \in \mathbb{N}$,

$$\beta'_n := (1-\alpha_n)(1-\beta) \text{ and } \alpha'_n := \alpha_n/(1-\beta'_n)$$

this scheme includes Chidume and Chidume's and Suzuki's iteration scheme, though. The quantitative analysis of the convergence proof for (5) has to be left for future research.

4 Main results

Theorem 4.1. *Let X be a complete CAT(0) space, $C \subseteq X$ a closed subset, $x, u \in C$ and $T : C \to C$ nonexpansive with a nonempty fixed point set. Let $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subset [0,1]$ satisfy (D1.a) and (D1.b), (D2.a), (D3.a) and (D3.b). Then the modified Halpern iteration $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Furthermore, let*

$$\begin{aligned}
&\gamma_\alpha : (0,\infty) \to \mathbb{Z}_+ \text{ rate of convergence of } (\alpha_n)_{n\in\mathbb{N}} \text{ towards } 0,\\
&\gamma_\beta : (0,\infty) \to \mathbb{Z}_+ \text{ rate of convergence of } (\beta_n)_{n\in\mathbb{N}} \text{ towards } 0,\\
&\psi_\alpha : (0,\infty) \to \mathbb{Z}_+ \text{ Cauchy modulus of } \left(\sum_{i=1}^{N} |\alpha_{i+1} - \alpha_i|\right),\\
&\psi_\beta : (0,\infty) \to \mathbb{Z}_+ \text{ Cauchy modulus of } \left(\sum_{i=1}^{N} |\beta_{i+1} - \beta_i|\right),\\
&\theta_\beta : \mathbb{Z}_+ \to \mathbb{Z}_+ \text{ rate of divergence of } \left(\sum_{i=1}^{N} \beta_i\right).
\end{aligned}$$

Then for all $\varepsilon \in (0,2)$ and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leq \Sigma(\varepsilon, g, M, \gamma_\alpha, \gamma_\beta, \psi_\alpha, \psi_\beta, \theta_\beta) \quad \forall m, n \in [N; N+g(N)] \quad (d(x_n, x_m) \leq \varepsilon)$$

where

$$\Sigma(\varepsilon, g, M, \gamma_\alpha, \gamma_\beta, \psi_\alpha, \psi_\beta, \theta_\beta) := \theta_\beta^+ \left(\Gamma - 1 + \left\lceil \ln\left(\frac{12M^2}{\varepsilon^2}\right)\right\rceil\right) + 1,$$

with $M \geq 4\max\{d(u,p), d(x,p)\}$ *for some* $p \in \mathrm{Fix}(T)$,

$$\tilde{\Phi}(\varepsilon, M, \theta_\beta, \psi_\beta, \psi_\alpha) := \theta_\beta\left(\max\left\{\psi_\beta\left(\frac{\varepsilon}{8M}\right), \psi_\alpha\left(\frac{\varepsilon}{4M}\right)\right\} + \ln\left\lceil\frac{M}{\varepsilon}\right\rceil + 1\right) + 1,$$

$$\Phi(\varepsilon, M, \theta_\beta, \psi_\beta, \psi_\alpha, \gamma_\alpha, \gamma_\beta) := \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}, M, \theta_\beta, \psi_\beta, \psi_\alpha\right), \gamma_\alpha\left(\frac{\varepsilon}{4M}\right), \gamma_\beta\left(\frac{\varepsilon}{4M}\right)\right\}.$$

The other constants are

$$\varepsilon_0 := \frac{\varepsilon^2}{24(M+1)^2}, \quad \Gamma := \max\left\{\chi_k^*(\varepsilon^2/12) : \left\lceil\frac{1}{\varepsilon_0}\right\rceil \leq k \leq \tilde{f^*}^{(\lceil M^2/\varepsilon_0^2\rceil)}(0) + \left\lceil\frac{1}{\varepsilon_0}\right\rceil\right\},$$

$$\chi_k^*(\varepsilon) := \tilde{\Phi}\left(\frac{\varepsilon}{4M(\tilde{P}_k(\varepsilon/2)+1)}\right) + \tilde{P}_k(\varepsilon/2), \quad \tilde{P}_k(\varepsilon) := \left\lceil\frac{12M^2(k+1)}{\varepsilon}\Phi\left(\frac{\varepsilon}{12M(k+1)}\right)\right\rceil,$$

$$\Delta_k^*(\varepsilon, g) := \frac{\varepsilon}{3g_{\varepsilon,k}(\Theta_k(\varepsilon) - \chi_k^*(\varepsilon/3))}, \quad \Theta_k(\varepsilon) := \theta\left(\chi_k^*\left(\frac{\varepsilon}{3}\right) - 1 + \left\lceil\ln\left(\frac{3M^2}{\varepsilon}\right)\right\rceil\right) + 1,$$

$$g_{\varepsilon,k}(n) := n + g\left(n + \chi_k^*\left(\frac{\varepsilon}{3}\right)\right), \quad \theta_\beta^+(n) := \max\{\theta_\beta(i) : i \leq n\},$$

$$f(k) := \max\left\{\left\lceil\frac{M^2}{\Delta_k^*(\varepsilon^2/4, g)}\right\rceil, k\right\} - k, \quad f^*(k) = f\left(k + \left\lceil\frac{1}{\varepsilon_0}\right\rceil\right) + \left\lceil\frac{1}{\varepsilon_0}\right\rceil,$$

$$\tilde{f^*}(k) := k + f^*(k).$$

We now come to the metastability rates for the other set of conditions we consider for modified Halpern iterations.

Theorem 4.2. *In the setting of Theorem 4.1, let* $(\alpha_n)_{n\in\mathbb{N}} \subset [0,1]$ *and* $(\beta_n)_{n\in\mathbb{N}} \subset [0,1)$ *satisfy (D1.a) and (D4.a), (D3a), (D1.b) and (D3.b). Then the modified Halpern iteration* $(x_n)_{n\in\mathbb{N}}$ *is Cauchy. Furthermore, let*

$$\theta_\beta : \mathbb{Z}_0 \to \mathbb{Z}_+ \text{ be a rate of convergence of } \left\{\prod_{n=1}^{N}(1-\beta_n)\right\} \text{ towards } 0.$$

Then for all $\varepsilon \in (0,2)$ *and* $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leq \Sigma(\varepsilon, g, M, \gamma_\alpha, \gamma_\beta, \psi_\alpha, \psi_\beta, \theta_\beta) \quad \forall m, n \in [N; N+g(N)] \quad (d(x_n, x_m) \leq \varepsilon)$$

where

$$\Sigma(\varepsilon, g, M, \gamma_\alpha, \gamma_\beta, \psi_\alpha, \psi_\beta, \theta_\beta) := \max\left\{\Theta_k(\varepsilon^2/4) : \left\lceil\frac{1}{\varepsilon_0}\right\rceil \leq k \leq \widetilde{f^*}^{(\lceil M^2/\varepsilon_0^2\rceil)}(0) + \left\lceil\frac{1}{\varepsilon_0}\right\rceil\right\},$$

with $M \geq 4\max\{d(u,p), d(x,p)\}$ *for* $p \in \mathrm{Fix}(T)$

$$\tilde{\Phi}(\varepsilon, M, \theta_\beta, \psi_\beta, \psi_\alpha) := \theta_\beta\left(\frac{D\varepsilon}{M}\right) + 1$$

$$\Phi(\varepsilon, M, \theta_\beta, \psi_\beta, \psi_\alpha, \gamma_\alpha, \gamma_\beta) := \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}, M, \theta_\beta, \psi_\beta, \psi_\alpha\right), \gamma_\alpha\left(\frac{\varepsilon}{4M}\right), \gamma_\beta\left(\frac{\varepsilon}{4M}\right)\right\},$$

$$0 < D \leq \prod_{n=1}^{\gamma(\varepsilon/(2M)}(1-\beta_n),$$

$$\gamma(\varepsilon) := \max\left\{\psi_\alpha\left(\frac{\varepsilon}{2M}\right), \psi_\beta\left(\frac{\varepsilon}{4M}\right)\right\},$$
$$\Theta_k(\varepsilon) := \theta\left(\frac{D_k\varepsilon}{3M_2^2}\right) + 1,$$
$$0 < D_k \leq \prod_{n=1}^{\chi_k^*(\varepsilon/3)-1} (1-\beta_n).$$

The other functionals and constants are defined as in Theorem 4.1.

Remark 4.3.

1. The bounds in Theorems 4.1 and 4.2 only differ from the ones obtained in [15] for the usual Halpern iteration and the case of bounded C by the new functionals $\Phi, \tilde{\Phi}$ which reflect the modification in the iteration scheme and by the fact that instead of $M \geq diam(C)$ we only need $M \geq 4\max\{d(u,p), d(x,p)\}$. Making only the latter change in the bounds in [15] yields rates of metastability for the usual Halpern iterations in the unbounded case.

2. The extractability of bounds on metastability depending only on the arguments shown can be explained in terms of general logical metatheorems from [12, 6]. In particular, the fact that u, x, p and C, X, T only enter these bounds via M follows this way (note that we do not need any extra bound on $d(x, T(x))$ since $d(x,T(x)) \leq d(x,p) + d(p,T(p)) + d(T(p),T(x)) \leq 2d(x,p) \leq M$). See [23] for details.

3. Again general logical metatheorems from [12, 6] guarantee that the existence of a fixed point of T can be relaxed to the existence of arbitrarily good approximate fixed points in some fixed b-bounded neighborhood around x, where then M is taken to satisfy $M \geq 4(d(u,x)+b)+1$ (note that $d(u,p) \leq d(u,x) + d(x,p) \leq d(u,x) + b$ and that – reasoning as in the previous point – $d(x,T(x)) \leq 2b+1 \leq M$ where, in fact, '+1' can be replaced by an arbitrarily small positive number). See [23] for details.

4. The assumption of the CAT(0)-space X to be *complete* and C to be *closed* is actually not necessary as can be seen by going to the metric completion $\widehat{X}$ of X and the closure $\overline{C}$ of C in $\widehat{X}$, since T extends to a nonexpansive operator on $\overline{C}$ [15, Remark 4.5.(ii)].

5 Estimates for modified Halpern iterations

We need bounds for modified Halpern iterations. Part of the following result can be deduced from the proof of [4, Theorem 3.1].

Lemma 5.1. *For modified Halpern iterations $(x_n)_{n\in\mathbb{N}}$ as in Definition 3.1 set $y_n := \alpha_n x_n \oplus (1-\alpha_n)Tx_n$ for all $n \in \mathbb{N}$. Then the following holds for $n \in \mathbb{Z}_+$ in (1)-(3) and $n \in \mathbb{N}$ in (4)-(6):*

(1) $d(x_{n+1}, x_n) \leq (1-\beta_n)d(y_n, y_{n-1}) + |\beta_n - \beta_{n-1}|d(u, y_{n-1})$.

(2) $d(y_n, y_{n-1}) \leq \alpha_n d(x_n, x_{n-1}) + (1-\alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1})$.

(3) $d(x_{n+1}, x_n) \leq (1-\beta_n)d(x_n, x_{n-1}) + (1-\beta_n)|\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1})$
$+ |\beta_n - \beta_{n-1}|\alpha_{n-1}d(x_{n-1}, Tx_{n-1}) + |\beta_n - \beta_{n-1}|d(u, Tx_{n-1})$.

(4) $d(y_n, Tx_n) \leq \alpha_n d(x_n, Tx_n)$.

(5) $d(x_{n+1}, Tx_n) \leq \beta_n d(u, Tx_n) + (1-\beta_n)\alpha_n d(x_n, Tx_n)$.

(6) $d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + \beta_n d(u, Tx_n) + (1-\beta_n)\alpha_n d(x_n, Tx_n)$.

Proof.

(1) Let $n \in \mathbb{Z}_+$.

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(\beta_n u \oplus (1-\beta_n)y_n, \beta_{n-1}u \oplus (1-\beta_{n-1})y_{n-1}) \\
&\le d(\beta_n u \oplus (1-\beta_n)y_n, \beta_n u \oplus (1-\beta_n)y_{n-1}) \\
&\quad + d(\beta_n u \oplus (1-\beta_n)y_{n-1}, \beta_{n-1}u \oplus (1-\beta_{n-1})y_{n-1}), \\
&\text{by the triangle inequality,} \\
&\le (1-\beta_n)d(y_n, y_{n-1}) + \beta_n d(u,u) + |\beta_n - \beta_{n-1}|d(u, y_{n-1}), \\
&\text{by (W4) and (W2) from Definition 2.3 and 2.4,} \\
&= (1-\beta_n)d(y_n, y_{n-1}) + |\beta_n - \beta_{n-1}|d(u, y_{n-1}) \\
&= (1-\beta_n)d(y_n, y_{n-1}) + |\beta_n - \beta_{n-1}|d(u, \alpha_{n-1}x_{n-1} \oplus (1-\alpha_{n-1})Tx_{n-1}),
\end{aligned}$$

(2) Let $n \in \mathbb{Z}_+$.

$$\begin{aligned}
d(y_n, y_{n-1}) &= d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, \alpha_{n-1}x_{n-1} \oplus (1-\alpha_{n-1})Tx_{n-1}) \\
&\le d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, \alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_n) \\
&\quad + d(\alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_n, \alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_{n-1}) \\
&\quad + d(\alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_{n-1}, \alpha_{n-1}x_{n-1} \oplus (1-\alpha_{n-1})Tx_{n-1}), \\
&\text{by the triangle inequality,} \\
&\le (1-\alpha_n)\underbrace{d(Tx_n, Tx_n)}_{=0} + \alpha_n d(x_n, x_{n-1}) + \alpha_n \underbrace{d(x_{n-1}, x_{n-1})}_{=0} \\
&\quad + (1-\alpha_n)d(Tx_n, Tx_{n-1}) + |\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1}), \\
&\text{by (W4) and (W2) from Definition 2.3 and 2.4,} \\
&= \alpha_n d(x_n, x_{n-1}) + (1-\alpha_n)d(Tx_n, Tx_{n-1}) + |\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1}) \\
&\le \alpha_n d(x_n, x_{n-1}) + (1-\alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1}), \\
&\text{since } T \text{ is nonexpansive.}
\end{aligned}$$

(3) Let $n \in \mathbb{Z}_+$.

$$\begin{aligned}
d(x_{n+1}, x_n) &\le (1-\beta_n)d(y_n, y_{n-1}) + |\beta_n - \beta_{n-1}|d(u, \alpha_{n-1}x_{n-1} \oplus (1-\alpha_{n-1})Tx_{n-1}), \text{ by (1),} \\
&\le (1-\beta_n)[\alpha_n d(x_n, x_{n-1}) + (1-\alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1})] \\
&\quad + |\beta_n - \beta_{n-1}|d(u, \alpha_{n-1}x_{n-1} \oplus (1-\alpha_{n-1})Tx_{n-1}), \text{ by (2),} \\
&\le (1-\beta_n)d(x_n, x_{n-1}) + (1-\beta_n)|\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1}) \\
&\quad + |\beta_n - \beta_{n-1}|\alpha_{n-1}d(u, x_{n-1}) + |\beta_n - \beta_{n-1}|(1-\alpha_{n-1})d(u, Tx_{n-1}), \\
&\text{by Corollary 2.2,} \\
&\le (1-\beta_n)d(x_n, x_{n-1}) + (1-\beta_n)|\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1}) \\
&\quad + |\beta_n - \beta_{n-1}|\alpha_{n-1}[d(u, x_{n-1}) - d(u, Tx_{n-1})] \\
&\quad + |\beta_n - \beta_{n-1}|d(u, Tx_{n-1}) \\
&\le (1-\beta_n)d(x_n, x_{n-1}) + (1-\beta_n)|\alpha_n - \alpha_{n-1}|d(x_{n-1}, Tx_{n-1}) \\
&\quad + |\beta_n - \beta_{n-1}|\alpha_{n-1}d(x_{n-1}, Tx_{n-1}) + |\beta_n - \beta_{n-1}|d(u, Tx_{n-1}), \\
&\text{since } d(u, x_{n-1}) \le d(u, Tx_{n-1}) + d(Tx_{n-1}, x_{n-1}).
\end{aligned}$$

(4) Let $n \in \mathbb{N}$. By Corollary 2.2,

$$\begin{aligned}
d(y_n, Tx_n) &= d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, Tx_n) \le \alpha_n d(x_n, Tx_n) + (1-\alpha_n)d(Tx_n, Tx_n) \\
&= \alpha_n d(x_n, Tx_n).
\end{aligned}$$

(5) Let $n \in \mathbb{N}$. Again, by Corollary 2.2,

$$\begin{aligned} d(x_{n+1}, Tx_n) &= d(\beta_n u \oplus (1-\beta_n) y_n, Tx_n) \leq \beta_n d(u, Tx_n) + (1-\beta_n) d(y_n, Tx_n) \\ &\leq \beta_n d(u, Tx_n) + (1-\beta_n)\alpha_n d(x_n, Tx_n), \text{by (4)}. \end{aligned}$$

(6) Let $n \in \mathbb{N}$. By the triangle inequality,

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &\leq d(x_n, x_{n+1}) + \beta_n d(u, Tx_n) + (1-\beta_n)\alpha_n d(x_n, Tx_n), \text{by (5)}. \end{aligned}$$ □

For the general case of unbounded C, we also need some bound considerations, setting

$$M_0 := \max\{d(x,p), d(u,p)\}, \tag{6}$$
$$M_1 := 2d(p,u) \text{ and} \tag{7}$$
$$M_2 \geq 4M_0 = 4\max\{d(u,p), d(x,p)\} \geq 2M_0 + M_1 \tag{8}$$

for a fixed point p of T. Some parts of the next lemma are already implicit in the proof of [10, Theorem 1]

Lemma 5.2. *Consider the modified Halpern iterations $(x_n)_{n\in\mathbb{N}}$. We define $y_n := \alpha_n x_n \oplus (1-\alpha_n)Tx_n$ for all $n \in \mathbb{N}$. Take $p \in \mathrm{Fix}(T)$. Then the following holds for $n \in \mathbb{N}$:*

(1) $d(y_n, p) \leq d(x_n, p)$.

(2) $d(x_{n+1}, p) \leq \beta_n d(u,p) + (1-\beta_n) d(x_n, p)$.

(3) $d(x_n, p) \leq M_0$. *Hence $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are bounded.*

(4) $d(Tx_n, p) \leq d(x_n, p) \leq M_0$.

(5) $d(x_n, Tx_n) \leq 2M_0 \leq M_2$.

(6) $d(x_n, u) \leq 2M_0 \leq M_2$.

(7) $d(u, Tu) \leq 2d(u,p) = M_1 \leq M_2$.

(8) $d(Tx_n, u) \leq M_2$.

(9) $d(x_{n+1}, x_n) \leq 2M_0 \leq M_2$.

Proof.

(1) Let $n \in \mathbb{N}$.

$$\begin{aligned} d(y_n, p) &= d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, p), \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(Tx_n, p), \\ &\quad \text{by Corollary 2.2,} \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(x_n, p), \text{ since T is nonexpansive,} \\ &= d(x_n, p) \end{aligned}$$

(2) Let $n \in \mathbb{N}$.

$$\begin{aligned} d(x_{n+1}, p) &= d(\beta_n u \oplus (1-\beta_n) y_n, p) \\ &\leq \beta_n d(u,p) + (1-\beta_n) d(y_n, p), \text{ by Corollary 2.2,} \\ &\leq \beta_n d(u,p) + (1-\beta_n) d(x_n, p), \text{ by (1)}. \end{aligned}$$

(3) The induction start is trivial. For $n \in \mathbb{N}$ it holds

$$\begin{aligned} d(x_{n+1}, p) &\leq \beta_n d(u,p) + (1-\beta_n) d(x_n, p), \text{ by (2)}, \\ &\leq \beta_n d(u,p) + (1-\beta_n) M_0 \text{ by I.H.}, \\ &\leq M_0. \end{aligned}$$

(4) This follows since T is nonexpansive using (3).

(5) For $n \in \mathbb{N}$, $d(x_n, Tx_n) \leq d(x_n, p) + d(p, Tx_n) \leq 2M_0 \leq M_2$ using (3), (4).

(6) For $n \in \mathbb{N}$, $d(x_n, u) \leq d(x_n, p) + d(u, p) \leq 2M_0 \leq M_2$ using (3).

(7) $d(u, Tu) \leq d(u,p) + d(p, Tu) \leq 2d(u,p) = M_1 \leq M_2$.

(8) Let $n \in \mathbb{N}$.

$$\begin{aligned} d(Tx_n, u) &\leq d(Tx_n, Tu) + d(Tu, u) \leq d(x_n, u) + d(u, Tu) \\ &\leq 2M_0 + M_1 \leq M_2. \end{aligned}$$

(9) For $n \in \mathbb{N}$, $d(x_{n+1}, x_n) \leq d(x_{n+1}, p) + d(x_n, p) \leq 2M_0 \leq M_2$. □

6 Effective rates of asymptotic regularity

In this section we give the actual quantitative convergence results. Let in the following (X, d) be a CAT(0) space. The results hold also true if we consider a W-hyperbolic space (X, d, W). Let C be a convex subset of X. Let $T : C \to C$ be nonexpansive.
The following proposition is the quantitative version of [4, Theorem 3.1].

Proposition 6.1. *In the setting of Theorem 4.1, $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. More precisely, for all $\varepsilon \in (0, 2)$,*

$$\forall n \geq \tilde{\Phi} \quad d(x_n, x_{n+1}) \leq \varepsilon \quad \textit{and} \quad \forall n \geq \Phi \quad d(x_n, Tx_n) \leq \varepsilon,$$

where

$$\begin{aligned} \tilde{\Phi} &:= \tilde{\Phi}(\varepsilon, M_2, \theta_\beta, \psi_\beta, \psi_\alpha) := \theta_\beta\left(\max\left\{\psi_\beta\left(\frac{\varepsilon}{8M_2}\right), \psi_\alpha\left(\frac{\varepsilon}{4M_2}\right)\right\} + \ln\left\lceil\frac{M_2}{\varepsilon}\right\rceil + 1\right) + 1, \\ \Phi &:= \Phi(\varepsilon, M_2, \theta_\beta, \psi_\beta, \psi_\alpha, \gamma_\alpha, \gamma_\beta) := \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}, M_2, \theta_\beta, \psi_\beta, \psi_\alpha\right), \gamma_\alpha\left(\frac{\varepsilon}{4M_2}\right), \gamma_\beta\left(\frac{\varepsilon}{4M_2}\right)\right\}, \end{aligned}$$

where $M_2 \geq 4\max\{d(u,p), d(x,p)\}$ for a $p \in \mathrm{Fix}(T)$.

Proof. We want to apply [15, Lemma 5.5.]. For all $n \in \mathbb{Z}_+$, we know

$$\begin{aligned} d(u, Tx_{n-1}) &\leq M_2, \text{ by Lemma 5.2(8)}, && (9) \\ d(x_{n-1}, Tx_{n-1}) &\leq 2M_0 \leq M_2, \text{ by Lemma 5.2(5)}. && (10) \end{aligned}$$

By Lemma 5.1(3), we have for all $n \in \mathbb{Z}_+$

$$\begin{aligned} d(x_{n+1}, x_n) &\leq (1-\beta_n) d(x_n, x_{n-1}) + (1-\beta_n)|\alpha_n - \alpha_{n-1}| d(x_{n-1}, Tx_{n-1}) \\ &\quad + |\beta_n - \beta_{n-1}|\alpha_{n-1} d(x_{n-1}, Tx_{n-1}) + |\beta_n - \beta_{n-1}| d(u, Tx_{n-1}) \\ &\leq (1-\beta_n) d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(x_{n-1}, Tx_{n-1}) \\ &\quad + |\beta_n - \beta_{n-1}| d(x_{n-1}, Tx_{n-1}) + |\beta_n - \beta_{n-1}| d(u, Tx_{n-1}) \\ &\leq (1-\beta_n) d(x_n, x_{n-1}) + M_2[|\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|]. && (11) \end{aligned}$$

We set for all $n \in \mathbb{Z}_+$

$$\begin{aligned} s_n &:= d(x_n, x_{n-1}), \\ a_n &:= \beta_n \text{ and} \\ b_n &:= M_2[|\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|]. \end{aligned}$$

Then for all $n \in \mathbb{Z}_+$

$$\begin{aligned} s_{n+1} = d(x_{n+1}, x_n) &\leq (1-\beta_n)d(x_n, x_{n-1}) + M_2[|\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|], \\ &\text{by (11),} \\ &= (1-a_n)s_n + b_n. \end{aligned} \tag{12}$$

The sequence (s_n) is a priori bounded by $\frac{M_2}{2}$ by Lemma 5.2(9). But we know also by assumption

$$\begin{aligned} &\sum_{n=1}^{\infty} \beta_n \to \infty, \text{ with rate of divergence } \theta_\beta, \\ &\sum_{n=1}^{N} M_2(2|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|) < \infty, N \to \infty, \text{ with Cauchy modulus} \\ &\psi : (0, \infty) \to \mathbb{Z}_+, \psi(\varepsilon) = \max\left\{\psi_\beta\left(\frac{\varepsilon}{4M_2}\right), \psi_\alpha\left(\frac{\varepsilon}{2M_2}\right)\right\}. \end{aligned}$$

We have fulfilled the requirements of [15, Lemma 5.5.](1). Thus,

$$\forall \varepsilon \in (0,2) \quad \forall n \geq \tilde{\Phi} \quad s_n \leq \varepsilon,$$

where

$$\tilde{\Phi}(\varepsilon, M_2, \theta_\beta, \psi_\beta, \psi_\alpha) := \theta_\beta\left(\max\left\{\psi_\beta\left(\frac{\varepsilon}{8M_2}\right), \psi_\alpha\left(\frac{\varepsilon}{4M_2}\right)\right\} + \ln\left\lceil\frac{M_2}{\varepsilon}\right\rceil + 1\right) + 1.$$

It remains to determine Φ. By Lemma 5.1(6), (9) and (10), we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + \beta_n d(u, Tx_n) + \underbrace{(1-\beta_n)}_{\leq 1}\alpha_n d(x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + (\beta_n + \alpha_n)M_2. \end{aligned}$$

We can define a rate of convergence

$$\gamma : (0, \infty) \to \mathbb{Z}_+, \text{ with } \gamma(\varepsilon) = \max\left\{\gamma_\alpha\left(\frac{\varepsilon}{2M_2}\right), \gamma_\beta\left(\frac{\varepsilon}{2M_2}\right)\right\}$$

such that the second term on the right becomes less than ε. For our bound, we then need to consider $\gamma\left(\frac{\varepsilon}{2}\right)$ so that the term on the right side becomes less than $\frac{\varepsilon}{2}$. In total, we get for all $\varepsilon \in (0,2)$ and for all $n \geq \Phi$, we have $d(x_n, Tx_n) \leq \varepsilon$, where

$$\begin{aligned} \tilde{\Phi}(\varepsilon, M_2, \theta_\beta, \psi_\beta, \psi_\alpha) &:= \theta_\beta\left(\max\left\{\psi_\beta\left(\frac{\varepsilon}{8M_2}\right), \psi_\alpha\left(\frac{\varepsilon}{4M_2}\right)\right\} + \ln\left\lceil\frac{M_2}{\varepsilon}\right\rceil + 1\right) + 1, \\ \Phi(\varepsilon, M_2, \theta_\beta, \psi_\beta, \psi_\alpha, \gamma_\alpha, \gamma_\beta) &:= \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}, M_2, \theta_\beta, \psi_\beta, \psi_\alpha\right), \gamma_\alpha\left(\frac{\varepsilon}{4M_2}\right), \gamma_\beta\left(\frac{\varepsilon}{4M_2}\right)\right\}. \end{aligned}$$

□

We can also consider the case in which (D2.a) is replaced by (D4a). This case was not considered by Cuntavepanit and Panyanak [4].

Proposition 6.2. *In the setting of Theorem 4.2,* $(x_n)_{n\in\mathbb{N}}$ *is an approximate fixed point sequence and* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*. More precisely, for all* $\varepsilon \in (0,2)$*,*

$$\forall n \geq \tilde{\Phi} \quad d(x_n, x_{n+1}) \leq \varepsilon \quad \text{ and } \quad \forall n \geq \Phi \quad d(x_n, Tx_n) \leq \varepsilon,$$

where

$$\tilde{\Phi} := \tilde{\Phi}(\varepsilon, M_2, \theta_\beta, \psi_\beta, \psi_\alpha) := \theta_\beta \left(\frac{D\varepsilon}{M_2}\right) + 1,$$
$$\Phi := \Phi(\varepsilon, M_2, \theta_\beta, \psi_\beta, \psi_\alpha, \gamma_\alpha, \gamma_\beta) := \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}, M_2, \theta_\beta, \psi_\beta, \psi_\alpha\right), \gamma_\alpha\left(\frac{\varepsilon}{4M_2}\right), \gamma_\beta\left(\frac{\varepsilon}{4M_2}\right)\right\},$$

where $M_2 = 4\max\{d(u,p), d(x,p)\}$ *for* p *a fixed point of* T *and*

$$\gamma(\varepsilon) := \max\left\{\psi_\alpha\left(\frac{\varepsilon}{2M_2}\right), \psi_\beta\left(\frac{\varepsilon}{4M_2}\right)\right\}$$
$$D \leq \prod_{n=1}^{\gamma(\varepsilon/2)} (1-\beta_n).$$

Proof. We want to use Lemma [15, Lemma 5.5.](2). We set again for $n \in \mathbb{Z}_+$,

$$\begin{aligned} s_n &:= d(x_n, x_{n_1}), \\ a_n &:= \beta_n \text{ and} \\ b_n &:= M_2[|\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|]. \end{aligned}$$

Then the main condition of Lemma [15, Lemma 5.5.] is fulfilled by (12), since the sequences $(a_n), (s_n)$ and (b_n) were chosen the same.
The sequence (s_n) is a priori bounded by $\frac{M_2}{2}$ by Lemma 5.2(9). But also,

$$\prod_{n=1}^{\infty}(1-\beta_n) = 0, \text{ with rate of convergence } \theta_\beta,$$
$$\sum_{n=1}^{N} M_2(2|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|) < \infty, N \to \infty, \text{ with Cauchy modulus } \gamma,$$

defined as $\gamma : (0,\infty) \to \mathbb{Z}_+$,

$$\gamma(\varepsilon) := \max\left\{\psi_\alpha\left(\frac{\varepsilon}{2M_2}\right), \psi_\beta\left(\frac{\varepsilon}{4M_2}\right)\right\}.$$

Then the conditions for Lemma [15, Lemma 5.5.](2) are fulfilled and we obtain the desired rate $\widetilde{\Phi}$. The rate Φ is obtained as in Proposition 6.1. □

7 Quantitative properties of an approximate fixed point sequence

Cuntavepanit's and Panyanak's proof contains a lemma that uses the existence of Banach limits similar to the Banach limit lemma used in Saejung [22]. To make a current metatheorem applicable, this lemma has to be replaced in the proof. This can be done in the same way as carried out in [15].
Let X be a complete CAT(0) space, $C \subset X$ a closed convex subset and $T : C \to C$ a nonexpansive mapping. For $t \in (0,1)$ and $u \in C$ consider

$$T_t^u : C \to C, \quad T_t^u y = tu \oplus (1-t)Ty. \tag{13}$$

One can easily see, that T_t^u is a strict contraction with contractive constant $L = 1 - t$. Thus T_t^u has a unique fixed point $z_t^u \in C$ by Banach's fixed point theorem. Hence z_t^u solves the following equation uniquely:

$$z_t^u = tu \oplus (1-t)Tz_t^u. \tag{14}$$

The following proposition is our substitute for a use of Banach limits in convergence proofs of modified Halpern iterations.

Proposition 7.1 (see also [15, Proposition 9.1] for the bounded case)**.** *Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in C, $u \in C$, $t \in (0,1)$ and let be z as defined in (14). Define for all $n \in \mathbb{Z}_+$,*

$$\gamma_n^t := (1-t)d(u, Tz_t^u)^2 - d(y_n, u)^2. \tag{15}$$

Let $M \in \mathbb{Z}_+$ be such that $d(z_t^u, y_n), d(y_n, Ty_n), d(y_n, u) \le M$ holds for all $n \in \mathbb{Z}_+$.

(1) For all $n \in \mathbb{Z}_+$,

$$d(y_n, z_t^u)^2 \le d(y_n, u)^2 + \frac{1}{t}a_n - (1-t)d(u, Tz_t^u) = \frac{1}{t}a_n - \gamma_n^t,$$

where $a_n := d(y_n, Ty_n)^2 + 2Md(y_n, Ty_n)$.

(2) If $(y_n)_{n\in\mathbb{N}}$ is asymptotically regular with rate of asymptotic regularity φ, then for all $\varepsilon \in (0,2)$,

$$\forall p \ge P(\varepsilon, t, M, \varphi) \quad \forall m \ge 1 \quad (C_{m,p}(\gamma_n^t) \le \varepsilon),$$

where

$$P(\varepsilon, t, M, \varphi) = \left\lceil \frac{6M^2}{t\varepsilon}\varphi\left(\frac{t\varepsilon}{6M}\right)\right\rceil.$$

(3) Assume that $(y_n)_{n\in\mathbb{N}}$ is asymptotically regular and $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Then

$$\limsup_{n\to\infty} \gamma_n^t \le 0.$$

Furthermore, if φ is a rate of asymptotic regularity of $(y_n)_{n\in\mathbb{N}}$ and $\tilde{\varphi}$ is a rate of convergence of $\{d(y_n, y_{n+1})\}_{n\in\mathbb{N}}$ towards 0, then $\limsup_{n\to\infty} \gamma_n^t \le 0$ with effective rate ψ defined by

$$\psi(\varepsilon, t, M, \varphi, \tilde{\varphi}) = \tilde{\varphi}\left(\frac{\varepsilon}{2M(P(\varepsilon/2, t, M, \varphi) + 1)}\right) + P(\varepsilon/2, t, M, \varphi),$$

where P is given by the same definition as in (2).

Proof. The result follows from the proof given in [15] by collecting all the instances of $M \ge diam(C)$ used in that proof. □

Proposition 7.2 ([15, Proposition 9.3])**.** *Let $(t_k)_{n\in\mathbb{N}} \subset (0,1)$ be non increasing. For $k \in \mathbb{N}$ let $z_{t_k}^u$ be defined as in (14). Let the set C be bounded with $diam(C) \le M \in \mathbb{N}$. Then for all $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$, the following holds*

$$\exists K_0 \le K(\varepsilon, g, M) \quad \forall i, j \in [K_0, K_0 + g(K_0)] \quad \left(d(z_{t_i}^u, z_{t_j}^u) \le \varepsilon\right),$$

where

$$K(\varepsilon, g, M) := \tilde{g}^{\lceil M^2/\varepsilon^2 \rceil}(0), \tag{16}$$

and $\tilde{g}(k) := k + g(k)$.

Proof. For the case of Hilbert spaces, the bound is extracted in [14] from Halpern's proof of the convergence of (z_t^u). Since that proof extends unchanged to CAT(0) spaces as remarked by Kirk [11], the same is true for the extracted bound. □

Lemma 7.3 (Compare Saejung [22, Lemma 2.2])**.** *Let z_t^u be defined as in equation (14). If* $\mathrm{Fix}(T) \neq \emptyset$*, then*

$$d(p, z_t^u) \leq d(p, u), \qquad \textit{for } p \in \mathrm{Fix}(T).$$

In particular, $d(u, z_t^u), d(x_n, z_t^u), d(u, Tz_t^u) \leq M_2$ *and for* $n \in \mathbb{N}$.

Proof. Let $p \in \mathrm{Fix}(T)$. Then

$$\begin{aligned} d(p, z_t^u) &= d(p, tu \oplus (1-t)Tz_t^u) \leq td(p, u) + (1-t)d(p, Tz_t^u), \text{ by Corollary 2.2} \\ &\leq td(p, u) + (1-t)d(p, z_t^u). \end{aligned}$$

Hence, $d(p, z_t^u) \leq d(p, u)$.
Then, by the definition of M_0 in (6), and by Lemma 5.2(3),

$$\begin{aligned} d(z_t^u, u) &\leq d(z_t^u, p) + d(p, u) \leq 2d(p, u) \leq M_2, \\ d(z_t^u, x_n) &\leq d(z_t^u, p) + d(p, x_n) \leq d(p, u) + M_0 \leq M_2, \\ d(Tz_t^u, u) &\leq d(Tz_t^u, p) + d(u, p) \leq d(z_t^u, p) + d(u, p) \leq M_2 \end{aligned}$$

for all $n \in \mathbb{N}$. □

With this result we can generalize Proposition 7.2 to unbounded domains given a fixed point p of T.

Corollary 7.4. *In the situation of Proposition 7.2 the conclusion also holds if C is unbounded and T has a nonempty fixed point set. In this case the bound M can be replaced by M_2.*

Proof. By the logical analysis of Halpern's proof [9, Theorem 1] in [14, Theorem 4.2] one can replace the bound M on the diameter by a bound on $d(z_t^u, u)$. If we have a fixed point p of T at our disposal we can take this bound to be $M_2 \geq 4\max\{d(u, p), d(x, p)\}$ by Lemma 7.3. □

The next lemma for modified Halpern iterations interestingly is precisely of the form proved for the usual Halpern iterations (for bounded C) in [15, Lemma 9.2] though the proof is different.

Lemma 7.5. *Let $u, x \in C$ and $(x_n)_{n\in\mathbb{N}}$ be the modified Halpern iterations as in Definition 3.1. Then for all $t \in (0, 1)$ and $n \geq 0$,*

$$d(x_{n+1}, z_t^u)^2 \leq (1-\beta_n)d(x_n, z_t^u)^2 + \beta_n\left((1-t)d(u, Tz_t^u)^2 - d(x_{n+1}, u)^2\right) + M_2^2 t,$$

with $M_2 := 4\max\{d(u, p), d(x, p)\}$.

Proof. Let $n \in \mathbb{N}$ and $t \in (0, 1)$ be given.
We need the following inequalities which follow from (2).

$$d(x_{n+1}, u)^2 = d(\beta_n u \oplus (1-\beta_n)y_n, u)^2 \leq (1-\beta_n)d(y_n, u)^2 \tag{17}$$

$$d(u, z_t^u)^2 \leq d(u, tu \oplus (1-t)Tz_t^u)^2 \leq (1-t)d(Tz_t^u, u)^2. \tag{18}$$

$$\begin{aligned} d(x_{n+1}, z_t^u)^2 &= d(\beta_n u \oplus (1-\beta_n)y_n, z_t^u)^2 \\ &\leq (1-\beta_n)d(y_n, z_t^u)^2 + \beta_n d(u, z_t^u)^2 - \beta_n \underbrace{(1-\beta_n)d(u, y_n)^2}_{\geq d(x_{n+1}, u)^2}, \text{by (2)} \\ &\quad \text{in the next step, we apply (2) on } d(y_n, z_t^u)^2 \text{ and use again (17),} \\ &\leq (1-\beta_n)[\alpha_n d(x_n, z_t^u)^2 + (1-\alpha_n)d(Tx_n, z_t^u)^2 \underbrace{-\alpha_n(1-\alpha_n)d(x_n, Tx_n)^2}_{\leq 0}] \\ &\quad + \beta_n d(u, z_t^u)^2 - \beta_n d(x_{n+1}, u)^2 \\ &\leq (1-\beta_n)\alpha_n d(x_n, z_t^u)^2 + (1-\beta_n)(1-\alpha_n)d(Tx_n, z_t^u)^2 \\ &\quad + \beta_n d(u, z_t^u)^2 - \beta_n d(x_{n+1}, u)^2 \end{aligned}$$

$$
\begin{aligned}
&\le (1-\beta_n)\alpha_n d(x_n, z_t^u)^2 \\
&\quad + \underbrace{(1-\beta_n)(1-\alpha_n)}_{\le 1}[td(u,Tx_n)^2 + \underbrace{(1-t)}_{\le 1}\underbrace{d(Tz_t^u, Tx_n)^2}_{\le d(z_t^u, x_n)^2}\underbrace{-t(1-t)d(u,Tz_t^u)^2}_{\le 0}], \text{by } ((2)) \\
&\quad + \beta_n d(u, z_t^u)^2 - \beta_n d(x_{n+1}, u)^2 \\
&\le (1-\beta_n)\alpha_n d(x_n, z_t^u)^2 + (1-\beta_n)(1-\alpha_n)d(x_n, z_t^u)^2 + \beta_n \underbrace{d(u, z_t^u)^2}_{\le (1-t)d(Tz_t^u, u)^2} \\
&\quad - \beta_n d(x_{n+1}, u)^2 + td(u, Tx_n)^2, \\
&\le (1-\beta_n)d(x_n, z_t^u)^2 + \beta_n[(1-t)d(Tz_t^u, u)^2 - d(x_{n+1}, u)^2] + td(u, Tx_n)^2, \text{ by (18)} \\
&\le (1-\beta_n)d(x_n, z_t^u)^2 + \beta_n[(1-t)d(Tz_t^u, u)^2 - d(x_{n+1}, u)^2] + tM_2^2, \text{by 5.2(8).} \qquad \square
\end{aligned}
$$

Proof of Theorems 4.1 and 4.2 concluded:

The proof of Theorem 4.1 (and also of Theorem 4.2) is essentially the same as the proof of the existence of a rate of metastability for ordinary Halpern iterations in the case of bounded C by Kohlenbach and Leuştean [15, Theorem 4.2] replacing the rates of asymptotic regularity $\Phi, \tilde{\Phi}$ used in [15] by the new ones we obtained in section 6. This is due to the fact that despite of the different iteration scheme at hand, Lemma 7.5 (though by a different proof) is identical to [15, Lemma 9.2]. The only other thing we have to check is that we can use the bound $M := M_2 \geq 4\max\{d(u,p), d(x,p)\}$ in Proposition 7.1. This, however, follows from Lemma 7.3 and Lemma 5.2(5) and (6).

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