

A note on the Π_2^0 -induction rule*

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Abstract

It is well-known (due to C. Parsons) that the extension of primitive recursive arithmetic PRA by first-order predicate logic and the rule of Π_2^0 -induction Π_2^0 -IR is Π_2^0 -conservative over PRA. We show that this is no longer true in the presence of function quantifiers and quantifier-free choice for numbers $AC^{0,0}$ -qf. More precisely we show that $\mathcal{T} := PRA^2 + \Pi_2^0$ -IR + $AC^{0,0}$ -qf proves the totality of the Ackermann function, where PRA^2 is the extension of PRA by number and function quantifiers and Π_2^0 -IR may contain function parameters. This is true even for $PRA^2 + \Sigma_1^0$ -IR + Π_2^0 -IR⁻ + $AC^{0,0}$ -qf, where Π_2^0 -IR⁻ is the restriction of Π_2^0 -IR without function parameters.

1

Let (**PRA**) denote the extension of primitive recursive arithmetic obtained by adding first-order predicate logic.

By the **rule IR** of **induction** we mean

$$\mathbf{IR} : \frac{A(0) , \forall x(A(x) \rightarrow A(x'))}{\forall x A(x)},$$

where x' denotes the successor of x .

The restriction of **IR** to Σ_1^0 -formulas $\exists v A_0(x, v)$ (resp. to Π_2^0 -formulas $\forall u \exists v A_0(x, u, v)$) is denoted by Σ_1^0 -**IR** (resp. Π_2^0 -**IR**)¹.

*I am grateful to an anonymous referee whose suggestions led to an improved discussion of our results.

¹ A_0 always denotes a quantifier-free formula.

It is well-known that **(PRA)** is closed under $\Pi_2^0\text{-IR}$. This was proved first by Parsons [7],[8]. Other proofs can be found in [6] and [9].

As a consequence of this fact one has the following rule

$$(1) \left\{ \begin{array}{l} \text{If } (\mathbf{PRA}) + \Pi_2^0\text{-IR} \vdash \forall x \exists y A_0(x, y), \text{ then there} \\ \text{exists a primitive recursive function } f \text{ such that} \\ (\mathbf{PRA}) \vdash \forall x A_0(x, fx), \end{array} \right.$$

where $A_0(x, y)$ contains only x, y as free variables.

Let **(PRA²)** be the extension of **(PRA)** obtained by the addition of function variables and quantifiers for functions with the usual axioms and rules as well as the schema for explicit definition of functions

$$(ED) : \exists f \forall \underline{x} (f\underline{x} = t[\underline{x}]),$$

where $t[\underline{x}]$ is a term of **(PRA²)** and \underline{x} a tuple of number variables which may occur in t (see e.g. [9]). (For convenience we include variables and quantifiers for n -ary number-theoretic functions for each n . This theory is a definitional extension of the corresponding theory with unary function variables and quantifiers only, since coding of tuples is possible in **(PRA)**.)

Furthermore the schema **QF-IA** of quantifier-free induction

$$\mathbf{QF-IA} : A_0(0) \wedge \forall x (A_0(x) \rightarrow A_0(x')) \rightarrow \forall x A_0(x)$$

is now applicable also to formulas A_0 which **contain function variables**. Finally **(PRA²)** has a bounded μ -operator μ_b with the axioms

$$(\mu_b) : \left\{ \begin{array}{l} y \leq x \wedge fxy = 0 \rightarrow fx(\mu_b fx) = 0, \\ y < \mu_b fx \rightarrow fxy \neq 0, \\ \mu_b fx = 0 \vee (fx(\mu_b fx) = 0 \wedge \mu_b fx \leq x) \end{array} \right.$$

(These axioms express that $\mu_b fx = \min y \leq x (fxy = 0)$ if such an $y \leq x$ exists and $= 0$ otherwise).

The axiom schema **AC^{0,0}-qf** of quantifier-free choice for functions is given by

$$\mathbf{AC}^{0,0}\text{-qf} : \forall x \exists y A_0(x, y) \rightarrow \exists f \forall x A_0(x, fx).$$

In $(\mathbf{PRA}^2)+\Sigma_1^0\text{-IR}$, $(\mathbf{PRA}^2)+\Pi_2^0\text{-IR}$ the rules $\Sigma_1^0\text{-IR}$, $\Pi_2^0\text{-IR}$ are always understood w.r.t. the language of (\mathbf{PRA}^2) , i.e. instances of $\Sigma_1^0\text{-IR}$, $\Pi_2^0\text{-IR}$ may contain function variables. If these rules are restricted to instances **without** function variables we write $\Sigma_1^0\text{-IR}^-$, $\Pi_2^0\text{-IR}^-$.

In this note we show that (1) is wrong for $(\mathbf{PRA}^2)+\Pi_2^0\text{-IR}+\mathbf{AC}^{0,0}\text{-qf}$ and even for $(\mathbf{PRA}^2)+\Sigma_1^0\text{-IR}+\Pi_2^0\text{-IR}^-+\mathbf{AC}^{0,0}\text{-qf}$. In fact both theories prove the totality of the Ackermann function. This shows that the strength of $\Pi_2^0\text{-IR}$ (w.r.t. the growth of provably recursive functions) increases in the presence of the additional analytical principle $\mathbf{AC}^{0,0}\text{-qf}$ if Π_2^0 is understood w.r.t. the extended language with function variables.

Our result provides quite a limit for generalizations of (1) from $(\mathbf{PRA})+\Pi_2^0\text{-IR}$ to systems suited for the formalization of fragments of analysis.

This is in sharp contrast to the conservativity of the **axiom of Σ_1^0 -induction $\Sigma_1^0\text{-IA}$** .

The following fact is well-known (see e.g. [2]):

$$(2) \left\{ \begin{array}{l} \text{If } (\mathbf{PRA}^2)+\Sigma_1^0\text{-IA} + \mathbf{AC}^{0,0}\text{-qf} \vdash \forall x \exists y A_0(x, y), \text{ then there} \\ \text{exists a primitive recursive function } f \text{ such that} \\ (\mathbf{PRA}) \vdash \forall x A_0(x, fx), \end{array} \right.$$

where $A_0(x, y)$ contains only x, y as free variables.

This fact (which also holds for the higher type extension (\mathbf{PRA}^ω) of (\mathbf{PRA}^2) and full $\mathbf{AC}\text{-qf}$) can be proved using functional interpretation and subsequent normalization of the resulting term. Here (\mathbf{PRA}^ω) denotes the theory $\widehat{\mathbf{PA}}^\omega \upharpoonright$ from [1]. Indeed one may add also the binary König's lemma \mathbf{WKL} (For (\mathbf{PRA}^ω) this was shown first in [4] (see also [3]) with various generalizations. For the special second-order case and $\mathbf{AC}^{0,0}\text{-qf}$ instead of full $\mathbf{AC}\text{-qf}$ a proof was given already in [9]).

We now come to the main result of this note:

Proposition 1

$(\mathbf{PRA}^2)+\Sigma_1^0\text{-IR}+\Pi_2^0\text{-IR}^-+\mathbf{AC}^{0,0}\text{-qf}$ *proves the totality of the Ackermann function.*

Proof: $(\mathbf{PRA}^2) + \Sigma_1^0\text{-IR}$ proves (by induction on x with the parameters f, y_0):

$$(1) \forall x \exists y ((y)_0 = y_0 \wedge \forall i < x ((y)_{i+1} = f(i, (y)_i)))$$

(Note that the formula $\forall i < x(\dots)$ can be expressed in a quantifier-free way using μ_b).

$\mathbf{AC}^{0,0}\text{-qf}$ applied to (1) yields

$$(2) \forall f, y_0 \exists g \forall x ((gx)_0 = y_0 \wedge \forall i < x ((gx)_{i+1} = f(i, (gx)_i))).$$

For $hxi := (gx)_i$ this implies

$$(3) hx0 = y_0 \wedge \forall i < x (hxi' = f(i, hxi)).$$

By $\mathbf{QF-IA}$ applied to i with x, \tilde{x}, h as parameters, one easily proves (for h satisfying (3))

$$(4) \forall x, \tilde{x} \forall i \leq \min(x, \tilde{x}) (hxi = h\tilde{x}i).$$

Define $\tilde{h}x := hx'x$. Then (3) and (4) yield

$$(5) \tilde{h}0 = y_0 \wedge \forall x (\tilde{h}x' = f(x, \tilde{h}x)).$$

Let

$$A := \exists y_0 A_0(0, y_0) \wedge \forall x (\exists y_1 A_0(x, y_1) \rightarrow \exists y_2 A_0(x', y_2)) \rightarrow \forall x \exists y A_0(x, y)$$

be an arbitrary instance of $\Sigma_1^0\text{-IA}$.

$(\mathbf{PRA}^2) + \mathbf{AC}^{0,0}\text{-qf}$ proves that A is equivalent to

$$\exists y_0 A_0(0, y_0) \wedge \exists f \forall x, y_1 (A_0(x, y_1) \rightarrow A_0(x', fxy_1)) \rightarrow \forall x \exists y A_0(x, y).$$

Assume

$$\exists y_0 A_0(0, y_0) \wedge \exists f \forall x, y_1 (A_0(x, y_1) \rightarrow A_0(x', fxy_1)).$$

To y_0, f choose \tilde{h} such that (5) is satisfied.

Using $\mathbf{QF-IA}$ one easily shows that

$$\forall x A_0(x, \tilde{h}x)$$

and therefore

$$\forall x \exists y A_0(x, y).$$

Thus $(\mathbf{PRA}^2) + \Sigma_1^0\text{-IR} + \mathbf{AC}^{0,0}\text{-qf}$ proves every instance of $\Sigma_1^0\text{-IA}$. However it is well-known that already the first-order fragment of $(\mathbf{PRA}^2) + \Sigma_1^0\text{-IA} + \Pi_2^0\text{-IR}$ and so a fortiori $(\mathbf{PRA}^2) + \Sigma_1^0\text{-IA} + \Pi_2^0\text{-IR}^-$ proves the totality of the Ackermann function (see [7]).

Corollary 2

$(\mathbf{PRA}^2) + \Pi_2^0\text{-IR} + \mathbf{AC}^{0,0}\text{-qf}$ proves the totality of the Ackermann function and therefore is not Π_2^0 -conservative over (\mathbf{PRA}^2) .

Cor.2 taken together with the rule (2) from the introduction shows that in some sense $(\mathbf{PRA}) + \Pi_2^0\text{-IR}$ is closer to systems proving the totality of the Ackermann function than $(\mathbf{PRA}) + \Sigma_1^0\text{-IA}$ is. This is not apparent from Parson's first-order result (used in the proof of our proposition above) that $(\mathbf{PRA}) + \Pi_2^0\text{-IR} + \Sigma_1^0\text{-IA}$ proves the totality of the Ackermann function (whereas both $(\mathbf{PRA}) + \Pi_2^0\text{-IR}$ and $(\mathbf{PRA}) + \Sigma_1^0\text{-IA}$ do not), since this result is completely symmetrical w.r.t. $\Pi_2^0\text{-IR}$ and $\Sigma_1^0\text{-IA}$.

As the proof of the proposition shows, the reason for this phenomenon is that in the presence of function variables, $\mathbf{AC}^{0,0}\text{-qf}$ allows to derive $\Sigma_1^0\text{-IA}$ from $\Pi_2^0\text{-IR}$ (and even from $\Sigma_1^0\text{-IR}$) but $\Sigma_1^0\text{-IA}$ plus $\mathbf{AC}^{0,0}\text{-qf}$ does not lead to a system which is closed under $\Pi_2^0\text{-IR}$ (not even under $\Pi_2^0\text{-IR}^-$).

The derivability of $\Sigma_1^0\text{-IA}$ in $(\mathbf{PRA}^2) + \Sigma_1^0\text{-IR} + \mathbf{AC}^{0,0}\text{-qf}$ shown in the proof of proposition 1 rests on three facts

- (i) $(\mathbf{PRA}^2) + \Sigma_1^0\text{-IR}$ suffices to introduce the Π_2^0 -form of the iteration of a function f
- (ii) From (i) one derives – using $\mathbf{AC}^{0,0}\text{-qf}$ – the existence of a function \tilde{h} which is the iteration of f
- (iii) Using such an iteration function one can prove the Π_2^0 -Herbrand normal form of $\Sigma_1^0\text{-IA}$ which – again by $\mathbf{AC}^{0,0}\text{-qf}$ – implies $\Sigma_1^0\text{-IA}$.

The presence of function variables in $\Sigma_1^0\text{-IR}$ is used only to define the iteration \tilde{h} of f , i.e. $\forall x (\tilde{h}0 = y_0 \wedge \tilde{h}x' = f(x, \tilde{h}x))$, **uniformly in the function parameter f** .

Hence if we add an iteration functional Φ_{it} to (\mathbf{PRA}^2) together with the axioms

$$(\Phi_{it}) : \begin{cases} \Phi_{it}fy0 = y \\ \Phi_{it}fyx' = f(x, \Phi_{it}fyx), \end{cases}$$

we obtain

Proposition 3 $(\mathbf{PRA}^2) + (\Phi_{it}) + \Pi_2^0\text{-IR}^- + \mathbf{AC}^{0,0}\text{-qf}$ proves the totality of the Ackermann function and therefore is not Π_2^0 -conservative over (\mathbf{PRA}^2) .

As we have seen in this note the addition of $\mathbf{AC}^{0,0}\text{-qf}$ to $\Pi_2^0\text{-IR}$ in the second-order context of (\mathbf{PRA}^2) destroys the Π_2^0 -conservativity over \mathbf{PRA} .

This also happens already for $\Pi_2^0\text{-IR}^-$ if the comprehension schema

$$\Pi_1^0\text{-CA}^- : \exists f \forall x (fx = 0 \leftrightarrow A(x)),$$

where $A \in \Pi_1^0$ contains no function parameters, is added:

By applying $\mathbf{QF-IA}$ to the comprehension functions, $(\mathbf{PRA}^2) + \Pi_1^0\text{-CA}^-$ proves every function parameter free instance of $\Sigma_1^0\text{-IA}$ (i.e. $\Sigma_1^0\text{-IA}^-$) which together with $\Pi_2^0\text{-IR}^-$ yields (using Parson's first-order result mentioned above) the totality of the Ackermann function. Hence

Proposition 4 $(\mathbf{PRA}^2) + \Pi_1^0\text{-CA}^- + \Pi_2^0\text{-IR}^-$ proves the totality of the Ackermann function and therefore is not Π_2^0 -conservative over \mathbf{PRA} .

This proposition refutes a theorem stated in [6] as well as cor. 5.9 (and its generalizations thm.5.8, 5.13 and 5.14) stated in [9] where this conservativity is claimed for a certain theory $\mathbf{BT} \supset (\mathbf{PRA}^2)$ even when \mathbf{WKL} is added.

Remark: The proof of prop.3 essentially uses the fact that function parameters are allowed to occur in the schema $\mathbf{QF-IA}$ of (\mathbf{PRA}^2) (The proof does not use μ_b). However to forbid the occurrence of function parameters in this schema does not help to repair the claims in [6],[9] mentioned above: If one has a coding functional $\bar{\Phi}fx := \bar{f}x$ together with the axiom

$$\forall f, x, y (y < x \rightarrow (\bar{f}x)_y = fy),$$

then still the totality of the Ackermann function can be shown to be provable. But such a coding (used explicitly in [6],[9]) is necessary in order to deal with \mathbf{WKL} (as formulated in [6], [9]).

On the other hand one can prove by methods different from those used in [6] and [9] that the provably recursive functions of $(\mathbf{PRA}^2) + \Pi_1^0\text{-CA}^-$ are primitive recursive. Indeed one can show much more general results for extensions of (\mathbf{PRA}^2) to finite types (which however must not contain Φ_{it}) with $\Delta_2^0\text{-IA}^-$, \mathbf{WKL} , full $\mathbf{AC-qf}$ and

many other analytical principles added and for sentences involving higher types (instead of Π_2^0 -sentences). These results and a discussion of the reasons for the failure of the methods used in [6] and [9] (which – at least as they stand there – can not be used to yield our positive results) are developed in chapters 11,12 of [5] and will be published in a paper under preparation.

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