Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization*

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Abstract

We show how to extract effective bounds Φ for $\forall u^1 \forall v \leq_\gamma tu \exists w^\eta G_0$–sentences which depend on u only (i.e. $\forall u \forall v \leq_\gamma tu \exists w \leq_\eta \Phi u G_0$) from arithmetical proofs which use analytical assumptions of the form $(*) \forall x^\delta \exists y \leq_\rho sx \forall z^\tau F_0$ $(\delta, \rho, \tau$ are arbitrary finite types, $\eta \leq 2, G_0, F_0$ are quantifier–free and s, t closed terms). If $\tau \leq 2$, $(*)$ can be weakened to
$\forall x^\delta, z^\tau \exists y \leq_\rho sx \forall \tilde{z} \leq_\tau zF_0$. This is used to establish new conservation results about weak Knig's lemma WKL. Applications to proofs in classical analysis, especially uniqueness proofs in approximation theory, will be given in subsequent papers.

1 Introduction and basic notions

Various theorems in classical analysis have the form

$$A \equiv \forall x \in X \exists y \in K_x \subseteq Y \; A_1(x, y),$$

where X, Y are complete separable metric spaces, K_x is compact in Y and A_1 is purely universal. If an analytical sentence B is proved in using besides lemmata A only arithmetical constructions and induction, then $A \to B$ is provable in classical arithmetic $\mathcal{A}$ (formulated in the language of all finite types). This is the case for an important class of uniqueness theorems (e.g. in approximation theory). Here B is essentially of the form

$$\forall u \in U, v \in V_u \exists k \in \mathbb{N} \; B_1(u, v, k),$$

where U, V are complete separable metric spaces, V_u is compact in V and $B_1 \in \Sigma^0_1$.

Using a suitable standard representation of such spaces, A and B can be expressed in $\mathcal{A}$ as $\tilde{A} \equiv \forall x^1 \exists y \leq_1 sx \forall z^{0/1} A_0$ and $\tilde{B} \equiv \forall u^1 \forall v \leq_1 tu \exists k^0 B_0$ $(A_0, B_0$ are quantifier–free and s, t closed terms).

In this paper we establish results which are (in their simplest form) of the following type:

¿From a proof of $(*)$ $\tilde{A} \to \tilde{B}$ in $\mathcal{A}$ one can extract an effective bound Φ for $\exists k$ in $\tilde{B}$ which depends on u only, i.e.

$$\tilde{A} \to \forall u^1 \forall v \leq_1 tu \exists k \leq_0 \Phi u \; B_0.$$

This also holds if x, y, z, v have arbitrary types and k has a type $\eta \leq 2$.

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*The results of this paper (except 2.9, 2.12, 2.15, 3.8 and 4.17) form the main part of chapter 7 of my dissertation Kohlenbach (1990). Some of them were presented at the 1989 European Summer Meeting of the ASL in Berlin.
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A similar extraction yields a bound Ψ for z in $\tilde{A}$:

$$\forall u^1 \Big(\forall x^\delta \exists y \leq_\rho sx \forall z \leq_\tau \Psi u \; A_0 \to \forall v \leq_\gamma tu \exists k \leq_\eta \Phi u \; B_0\Big) \;\; (\tau, \eta \leq 2);$$

thus a stronger conclusion follows from a weaker assumption. A proof of $\tilde{A}$ is not needed for the extraction of Φ, Ψ. The correctness of Φ follows from the truth of (the weakened) $\tilde{A}$.

We extract Φ and Ψ using a combination of functional interpretation and pointwise majorizability of primitive recursive functionals of finite type. If only elementary instances of induction occur in the proof of $(*)$ then Φ and Ψ are simple constructions in the numerically relevant terms and datas of the proof.

As logical applications of these results we obtain new conservation results for weak Knig's lemma WKL, e.g. conservation w.r.t. $\forall u^1 \forall v \leq_\gamma tu \exists w^\eta B_0$–sentences.

Mathematical applications to uniqueness proofs in approximation theory yielding new numerical estimates for rates of convergency will be given in subsequent papers.

Let $E-PA^\omega$ be the classical extensional arithmetic in all finite types (i.e. $(E-HA^\omega)^c$ in the notation of Troelstra (1973)), where the set $\tilde{T}$ of all finite types is given by the clauses

$0 \in \tilde{T}$ and $\rho, \tau \in \tilde{T} \Rightarrow \rho(\tau) \in \tilde{T}$ (each functional of type $\rho(\tau)$ maps objects of type τ to objects of type ρ; we often omit brackets which are uniquely determined and write e.g. $0(00)$ for $0(0(0))$). If the axiom of extensionality for each type is replaced by a quantifier–free rule of extensionality

$$ER\text{–qf} \; \frac{A_0 \to s =_\rho t}{A_0 \to r[s] =_\tau r[t]} \; ,$$

where A_0 is quantifier–free, then one obtains the system $WE-PA^\omega$. For the corresponding theories with intuitionistic logic only, we write $E-HA^\omega$ and $WE-HA^\omega$. All these systems $\mathcal{T}$ have the same quantifier–free part qf–$\mathcal{T}$ (in the sense of Troelstra (1973),1.6.13) which we call T. T is an extensional version of the Hilbert (1926)/Gdel (1958) calculus of primitive recursive functionals of finite type.

For functionals x_1^ρ, x_2^ρ we have the following natural inequality relation:

$$\begin{cases} x_1 \leq_0 x_2 :\equiv x_1 \leq x_2 \;\; (\text{where"} \leq \text{" is primitive recursively defined as usual}) \\ x_1 \leq_{\tau\delta} x_2 :\equiv \forall y^\delta (x_1 y \leq_\tau x_2 y). \end{cases}$$

The axiom schema of full choice is defined by $AC := \bigcup_{\rho,\tau \in \tilde{T}} \{(AC)^{\rho,\tau}\}$, where

$(AC)^{\rho,\tau} : \; \forall x^\rho \exists y^\tau A(x,y) \to \exists Y^{\tau\rho} \forall x^\rho A(x, Yx).$

Quantifier–free choice AC–qf is AC restricted to quantifier–free formulas.

We begin our investigations in this paper with the following observation:

Let $A \in \mathcal{L}(WE-PA^\omega)$ be a closed formula (i.e. a sentence) having the form

$$(1)\ \forall x^1 \forall \tilde{x} \leq_\rho sx \exists y^\tau A_0(x, \tilde{x}, y)$$

with A_0 quantifier–free and $s^{\rho 1} \in T$ closed, where $\rho \in \tilde{T}$ is arbitrary and $\tau \leq 2$ (i.e. $\tau = 0, 00$ or $0(00)$). If A is provable in $WE-PA^\omega + AC$–qf then by a combination of functional interpretation and a pointwise version of hereditary majorization of functionals from T, one can extract a closed term $\Phi^{\tau 1} \in T$ from any given proof such that

$$(2)\ WE-HA^\omega \vdash \forall x \forall \tilde{x} \leq sx \exists y \leq_\tau \Phi x A_0(x, \tilde{x}, y)$$

(see Kohlenbach (1992),3.4 and the proof of 2.3 below). This allows the extraction of uniform bounds $\in T$ for sentences of the form (1) which are proved in $WE-PA^\omega+AC$–qf from assumptions having the form (3) $\forall x^\delta \exists y \leq_\rho sx \forall z^\eta A_0$:

$$(4)\ WE-PA^\omega + AC\text{–qf} \vdash \forall x^\delta \exists y \leq_\rho sx \forall z^\eta A_0 \rightarrow \forall u^1 \forall v \leq_\gamma tu \exists w^\tau B_0 \quad \text{implies}$$

$$WE-PA^\omega + AC\text{–qf} \vdash \forall Y \leq_{\rho\delta} s \forall u \forall v \leq tu \exists x, z, w \big(A_0(x, Yx, z) \rightarrow B_0\big).$$

Using the extraction of the bound for "$\exists w$" above, one can construct a closed term $\Phi \in T$ such that

$$WE-HA^\omega \vdash \exists Y \leq s \forall x, z A_0(x, Yx, z) \rightarrow \forall u \forall v \leq tu \exists w \leq_\tau \Phi u\ B_0$$

and therefore

$$(5)\ WE-HA^\omega + AC \vdash \forall x \exists y \leq sx \forall z A_0 \rightarrow \forall u \forall v \leq tu \exists w \leq_\tau \Phi u\ B_0\ (2.3).$$

For $\tau = 0$ this yields an algorithm $\tilde{\Phi} \in T$ for w:

$$\tilde{\Phi} uv = \begin{cases} \min w \leq_0 \Phi u [B_0(u, v, w)] & \text{if such a } w \text{ exists,} \\ 0^0 \ \text{ otherwise (2.5).} \end{cases}$$

Using a more complicated extraction (2.9) one can prove (5) within $E-HA^\omega+AC$–qf, thus avoiding the axiom of choice for formulas containing quantifiers.

The result (5) is of **mathematical** interest since various important (non-constructive) theorems of classical analysis have the logical form

(6) $\forall x^1 \exists y \leq_1 sx \forall z^{0/1} A_0$ (modulo a suitable standard representation of complete separable metric spaces and compact metric spaces) and are therefore admissible premises for our extraction (5). In

the case where only premises of the form (6) are used, (5) can be proved even in $WE-HA^\omega$ (thus choice can be avoided altogether) and if furthermore $\gamma \le 1$, then $WE-PA^\omega$ may be replaced by $E-PA^\omega$ in (4) (see 3 below).

Examples of theorems of analysis having the logical form (6) are:

a) $\forall f \in \mathcal{C}[0,1] \exists x_0 \in [0,1]\big(f(x_0) = \sup\limits_{x\in[0,1]} f(x)\big)$,

b) The intermediate value theorem for $f \in \mathcal{C}[0,1]$ and the mean value theorem for the Riemann integral.

A more specific example is

c) the existence of a best approximation and an extremal alternant for the best Chebycheff approximation of $f \in \mathcal{C}[0,1]$ by algebraic polynomials of degree $\le n$ (P_n):

$$(7)\ \ \forall f \in \mathcal{C}[0,1] \exists p_b \in P_n, (x_1,...,x_{n+2}) \in [0,1]^{n+2}, j \in \{0,1\}\Big(\|f-p_b\|_\infty = \mathrm{dist}(f,P_n)$$
$$\wedge \bigwedge_{i=1}^{n+1}(x_{i+1}-x_i \ge 0)^1 \wedge \bigwedge_{i=1}^{n+2}(-1)^{i+j}\big(p_b(x_i)-f(x_i)\big) = \mathrm{dist}(f,P_n)\Big),$$

where $\|.\|_\infty$ denotes the sup norm and $\mathrm{dist}(f,P_n) := \inf\limits_{p\in P_n} \|f-p\|_\infty$.

(Since $\|p_b - f\|_\infty = \mathrm{dist}(f,P_n)\Big(\le \|f\|_\infty\Big) \Rightarrow \|p_b\|_\infty \le 2\|f\|_\infty, P_n$ can be replaced by

$K_{f,n} := \{p \in P_n \ :\ \|p\|_\infty \le 2\|f\|_\infty\}$ which is compact and therefore has a bounded standard representation).

Such sentences are in fact used in proofs of $\forall u^1 \forall v \le_1 tu \exists w^0 B_0$–theorems in classical analysis. Examples are uniqueness proofs, e.g. proofs of the uniqueness of the best Chebycheff approximation. Uniqueness in this case means

$$(8) \forall f \in \mathcal{C}[0,1], n \in \mathbb{N}, p_1, p_2 \in K_{f,n}\Big(\|p_1 - f\|_\infty = \mathrm{dist}\big(f,P_n\big) = \|p_2 - f\|_\infty \to \|p_1 - P_2\|_\infty = 0\Big).$$

If we now write "$\forall l^0\Big(\|p_i - f\|_\infty - \mathrm{dist}\big(f,P_n\big) \le \frac{1}{l+1}\Big)$" for "$\|p_i - f\|_\infty = \mathrm{dist}\big(f,P_n\big)$" and "$\forall k^0\Big(\|p_1 - p_2\|_\infty < \frac{1}{k+1}\Big)$" for "$\|p_1 - p_2\|_\infty = 0$" then (8) transforms (modulo standard representa-

[1] If $\mathrm{dist}(f,P_n) > 0$ $(i.e. f \notin P_n)$ then the last conjunction implies $x_{i+1} - x_i \ge 0 \to x_{i+1} > x_i$.

tions of $\mathcal{C}[0,1]$ and $K_{f,n}$) into

$$(9)\forall f^1, n^0 \forall p_1, p_2 \leq_1 s(f,n) \forall k^0 \exists l^0 \Big(\underbrace{\|p_{1/2} - f\|_\infty - \mathrm{dist}\big(f, P_n\big) \leq \frac{1}{l+1} \to \|p_1 - p_2\|_\infty < \frac{1}{k+1}}_{F:\equiv} \Big).$$

Since $\mathrm{dist}\big(f, P_n\big)$ and $\|.\|_\infty$ are primitive recursively computable (we consider $f \in \mathcal{C}[0,1]$ always endowed with a modulus of uniform continuity) the premise part of F is (equivalent to) a $\forall^0$–formula, while the conclusion is a $\exists^0$–formula. Therefore, by prenexing, F has the form $\exists^0 F_0$. Hence (9) is of the form (1). Furthermore (9) can be proved from assumptions of the form (6), for example a) and c), relative to $WE-PA^\omega + AC^{0,0}$–qf. Therefore our logical analysis of such uniqueness proofs yields a realization Φfnk of l which does **not depend on** p_1, p_2:

$$(10)\forall f, n; p_1, p_2 \leq s(f,n), k\Big(\|p_{1/2} - f\|_\infty - \mathrm{dist}\big(f, P_n\big) \leq \frac{1}{\Phi fnk + 1} \to \|p_1 - p_2\|_\infty < \frac{1}{k+1}\Big).$$

For a best approximant $p_b^{f,n} \in P_n$ of f this yields

$$(11)\forall f, n; p \leq s(f,n), k\Big(\|p - f\|_\infty - \mathrm{dist}\big(f, P_n\big) \leq \frac{1}{\Phi fnk + 1} \to \|p - p_b^{f,n}\|_\infty < \frac{1}{k+1}\Big).$$

A classical result in approximation theory says (non-constructively) that a best approximant $p_b^{f,n}$ always exists. This can be constructivized to

$$\forall f \in \mathcal{C}[0,1], n, l \bigvee_{\text{eff.}} p_l \in K_{f,n}\Big(\|p_l - f\|_\infty - \mathrm{dist}\big(f, P_n\big) < \tfrac{1}{l+1}\Big).$$

Now let $\Psi(f,n,l)$ denote an algorithm for p_l then (11) implies

$$(12)\forall f \in \mathcal{C}[0,1], n, k\Big(\|\Psi(f,n,\Phi fnk) - p_b^{f,n}\|_\infty < \frac{1}{k+1} \wedge \Psi(...) \in K_{f,n}\Big), \text{i.e.}$$

$\lambda k.\Phi fnk$ is a modulus of convergence for the sequence $\big(\Psi fnl\big)_{l \in \mathbb{N}} \overset{l\to\infty}{\longrightarrow} p_b^{f,n}$ (w.r.t. the norm $\|.\|_\infty$). One easily shows that $\lambda k.2\Phi fnk + 1$ is also a modulus of (pointwise) continuity of the Chebycheff projection

$$\mathcal{P}: \ \mathcal{C}[0,1] \times \mathbb{N} \to \bigcup_n P_n$$
$$(f,n) \mapsto \ p_b^{f,n}.$$

All this will be elaborated in subsequent papers where we analyse, in particular, various classical proofs for the uniqueness of best Chebycheff approximation and extract the corresponding moduli Φ with all numerical details. This yields new a priori moduli of uniqueness and continuity and estimates for strong unicity and Lipschitz continuity (which improve results of D. Bridges (1980,1982) by an n (=degree) in the exponent).

In this paper we develope the underlying proof–theoretic method and apply it to obtain conservation results for weak Knig's lemma WKL (4.1).

By using an argument analogous to the one for (5) one can construct a bound $\Psi \in T$ for "$\forall z^\eta$" if $\eta \le 2$ such that

$$(13)\ WE-HA^\omega + AC \vdash \forall u\big(\forall x \exists y \le sx \forall z \le_\eta \Psi u A_0 \to \forall v \le tu \exists w \le_\tau \Phi u B_0\big)\ (2.13).$$

For $\eta = 0$, this reduces the logical complexity of the implicative assumption

$$\forall x \exists y \le sx \forall z^0 A_0$$

to

$$\forall x, z \exists y \le sx \bigwedge_{k=0}^{z} A_0(x,y,k),$$

where $\bigwedge\limits_{k=0}^{z} A_0(x,y,k)$ can be expressed in a quantifier–free way in $WE-HA^\omega$.

In particular, proofs using sentences as e.g.

$$\forall f \in \mathcal{C}[0,1] \exists x_0 \in [0,1]\big(fx_0 = \sup_{x\in[0,1]} fx\big)$$

can be transformed into proofs which use only their "ε–versions"

$$\forall f \in \mathcal{C}[0,1], k \in \mathbb{N} \exists x_0 \in [0,1]\Big(|fx_0 - \sup_{x\in[0,1]} fx| \le \frac{1}{k+1}\Big).$$

These ε–versions are usually provable in $WE-HA^\omega$ (Thus e.g. the results for best approximation, mentioned above, can be verified within $WE-HA^\omega$).

Furthermore (13) can be generalized classically to formulas

$$\forall u^1 \exists a^\delta \forall b \le_\rho rua \exists w^\tau B_0 \quad (\delta, \tau \le 2)$$

– instead of $\forall u^1 \forall v \le tu \exists w B_0$ – yielding bounds (depending on u only) for w **and** a (2.12).

All the above results also hold for the restricted (in the sense of Feferman (1977)) systems $(W)\widehat{E-PA}^{\omega}\restriction, (W)\widehat{E-H}$ and $\widehat{PR}$ instead of $(W)E-PA^\omega, (W)E-HA^\omega$ and T with quantifier–free induction and elementary recursor constants only.

Finally we show that WKL is an admissible premise (6) since

$$\widehat{WE-HA}^{\omega}\restriction \vdash WKL \leftrightarrow \forall x^1 \exists y \le_1 \lambda k.1 \forall z^0 A_0^K,$$

for a suitable quantifier–free formula $A_0^K \in \mathcal{L}(\widehat{WE-HA}^{\omega}\restriction)$ and

$$\widehat{WE-HA}^{\omega}\restriction \vdash \forall x, z^0 \exists y \le \lambda k.1 \bigwedge_{i=0}^{z} A_0^K(x,y,i) \ \ (4.7).$$

As a corollary we see that $WE-PA^\omega$ plus AC–qf and WKL is a conservative extension of $WE-HA^\omega$ $(WE-PA^\omega+AC$–qf) w.r.t. sentences of the form
$\forall u^1\forall v\le_\gamma tu\exists w^\tau A_0$ $(\forall u^1\forall v\le_\gamma tu\exists w^0\forall z^1 A_0)$ where $\gamma,\tau\in\tilde{T}$ are arbitrary.

By constructing a counterexample we show that WKL is not conservative for $\forall x^2\exists y^1 A_0(x,y)$–sentences (4.11).

The above conservation results are also valid for the systems with restricted induction $\widehat{WE-PA}^\omega\restriction, \widehat{WE-HA}^\omega\restriction$.

If $\gamma,\tau\le 1$ then $WE-PA^\omega$ $(\widehat{WE-PA}^\omega\restriction)$ can be replaced by $E-PA^\omega$ $(\widehat{E-PA}^\omega\restriction)$.

The following conservation results can be found in the literature:

Sieg (1985) showed proof–theoretically that for the restriction RCA_0 of $\widehat{WE-PA}^\omega\restriction+AC^{0,0}$–qf to objects of type 0,1 only RCA_0+WKL is conservative over primitive recursive arithmetic PRA for Π^0_2–sentences. He used normalization of proofs and majorization of primitive recursive terms $t^0[f]$ with function parameters (primitive recursive in the sense of Kleene (1952)) to establish his results. H. Friedman previously proved the same result using modeltheoretic methods due to Kirby/Paris (1977). Feferman (1988) states the result also for $(WE-)\widehat{PA}^\omega\restriction+AC$–qf and refers to unpublished work of Sieg and himself for a (proof–theoretical) proof.

In Sieg (1987),(1991) a proof that RCA_0+WKL is Π^1_1–conservative over RCA_0 is formulated as well as for the corresponding theory with full induction and various intermediate systems (with $\exists_n$–induction in Sieg's terminology): The proof relies on normalization of infinitary proofs using infinite terms (in the sense of Tait (1965)), but makes incorrect use of Herbrand normal forms in order to generalize $\forall f^1\exists n^0 A_0$–conservation ($A_0$ quantifier–free) to arbitrary Π^1_1–sentences and thus establishes conservation only for the former case which is a special form of our $\forall u^1\forall v\le_\gamma tu\exists w^\tau A_0$–sentences (For details see Kohlenbach (1992 A) where a counterexample to Sieg's use of Herbrand normal form is given). There is mention in the literature, e.g. Sieg (1985), of an unpublished model–theoretic proof by Harrington that RCA_0+WKL is Π^1_1–conservative over RCA_0 (more precisely for a variant of RCA_0 with set variables instead of function variables). This is generalized in Clote/Hajek/Paris (1990) to systems with Σ^0_n–induction (instead of Σ^0_1–induction, which is provable in RCA_0 and $\widehat{WE-PA}^\omega\restriction+AC^{0,0}$–qf).

Finally there is the classical result (due to Kreisel (1963),(1966), Scott (1962) and Troelstra (1974)) that $WE-PA^\omega+WKL+AC$–qf is conservative over PA.

Furthermore we prove that, relative to $WE-PA^\omega+WKL$, each
$\forall x^1\exists y\le_1 sx\forall z^{0/1}A_0$–sentence is equivalent to a $\forall n^0 B_0$–sentence, where $A_0,B_0\in\mathcal{L}(WE-PA^\omega)$ are quantifier–free. An analogous result holds for $\widehat{WE-PA}^\omega\restriction$ instead of $WE-PA^\omega$ (4.15).

We conclude this paper by showing how one can extract bounds from proofs which use assumptions of the form $(*)\ \forall x^1(\forall w^0 A_0 \to \exists y \le_1 sx \forall z^0 B_0(x,y,z))$ (4.17). In this case, a proof of the ε–version of $(*)$ is needed (This contrasts to the assumptions $\forall x^\delta \exists y \le_\rho sx \forall z^\tau A_0$ considered above).

1.1 Notation

The theories $E-PA^\omega, E-HA^\omega, WE-PA^\omega, WE-HA^\omega$ and T all contain recursor constants R_ρ with the defining axioms

$$\begin{cases} R_\rho 0yz =_\rho y \\ R_\rho(Sx^0)yz =_\rho z(R_\rho xyz)x, \end{cases}$$

where y and z are of type ρ and $\rho 0 \rho$.
$s =_\rho t$ (for $\rho = 0\rho_k...\rho_1$) is used as an abbreviation for $\forall y_1^{\rho_1},...,y_k^{\rho_k}(sy_1...y_k =_0 ty_1...y_k)$ (resp. for $sy_1...y_k =_0 ty_1...y_k$ in the quantifier–free calculus T) with different variables $y_1,...,y_k$ not occuring in s and t.
We often denote finite tuples $y_1,...,y_k$ of functionals by $\underline{y}$.

If the constants R_ρ are replaced by elementary recursor constants $\widehat{R}_\rho$ characterized by

$$\begin{cases} \widehat{R}_\rho 0yz\underline{v} =_0 y\underline{v} \\ \widehat{R}_\rho(Sx^0)yz\underline{v} =_0 z(\widehat{R}_\rho xyz\underline{v})x\underline{v}, \end{cases}$$

where $\underline{v} = v_1^{\rho_1}...v_k^{\rho_k}$ such that $y\underline{v}$ is of type 0, and if the schema of full induction is replaced by the axiom of quantifier–free induction

$$(IA)\text{–qf} : \forall f^1\Big(f0 = 0 \wedge \forall x\big(fx = 0 \to f(Sx) = 0\big) \longrightarrow \forall x(fx = 0)\Big),$$

then one obtains the restricted systems $\widehat{E-PA}^\omega\!\restriction$,... ect. with quantifier–free part $\widehat{PR}$ due to Feferman (1977) (The functionals of $\widehat{PR}$ are essentially the primitive recursive functionals in the sense of Kleene (1959)).

All the systems above allow the definition of a term $\lambda x^\tau.t^\rho[x] \in T\ (\widehat{PR})$ for each term $t^\rho \in T\ (\widehat{PR})$ such that $\big(\lambda x.t[x]\big)\big(t'^{\tau}\big) =_\rho t[t']$ (see Troelstra (1973),1.6.8,1.8.4). For a theory $\mathcal{T}$ the language of $\mathcal{T}$ is denoted by $\mathcal{L}(\mathcal{T})$.

We usually use "$\forall x \le_\rho yA$", "$\exists x \le_\rho yA$", "$\bigwedge\limits_{i=0}^{k} A(i)$", "$\bigvee\limits_{i=0}^{k} A(i)$" as abbrevations for "$\forall x(x \le_\rho y \to A)$", "$\exists x(x \le_\rho y \wedge A)$", "$\forall i \le kA(i)$", "$\exists i \le kA(i)$". Furthermore "$\forall x; \tilde{x} \le_\rho sxA$" stands for "$\forall x \forall \tilde{x} \le_\rho sxA$", but "$\forall x, \tilde{x} \le_\rho yA$" stands for "$\forall x \le_\rho y \forall \tilde{x} \le_\rho yA$".

A_0, B_0, C_0,... denote quantifier–free formulas.

By the principle of **bounded choice** we mean the schema

$$(b-AC)^{\rho,\tau}: \forall Z^{\tau\rho}\Big(\forall x^\rho \exists y \leq_\tau Zx\ A(x,y,Z) \to \exists Y \leq_{\tau\rho} Z \forall x A(x, Yx, Z).$$

$(b-AC)^{\rho,\tau}$–qf $\Big((b-AC)^{\rho,\tau}-\forall,\ (b-AC)^{\rho,\tau}-\forall^b\Big)$ is $(b-AC)^{\rho,\tau}$ restricted to quantifier–free formulas (formulas having the form $\forall u^\delta A_0$ resp. $\forall u \leq_\delta v A_0$). $b-AC := \bigcup\limits_{\rho,\tau\in\tilde{T}} \{(b-AC)^{\rho,\tau}\}$.

$(C)^\rho:\ \exists y^{0\rho}\forall x^\rho\big(yx =_0 0 \leftrightarrow A(x)\big)$ (comprehension),

$(MP)^\omega:\ \forall \underline{x}\big(A(\underline{x}) \vee \neg A(\underline{x})\big) \wedge \neg\neg\exists\underline{x}A(\underline{x}) \to \exists\underline{x}A(\underline{x})$ (Markov principle),

$\Sigma^0_1 - AC^{0,0}:\ \forall x^0 \exists y^0, z^0 A_0(x,y,z) \to \exists f \forall x \exists z A_0(x, fx, z)$,

$\Sigma^0_1 - IA:\ \forall f^1\Big(\exists y^0(f0y=0) \wedge \forall x^0\big(\exists y(fxy=0) \to \exists y(fx'y=0)\big) \to \forall x \exists y(fxy=0)\Big)$,[2]

$\Delta^0_1 - CA:\ \forall x^0\big(\exists y^0 A_0(x,y) \leftrightarrow \forall y^0 B_0(x,y)\big) \to \exists f^1 \forall x\big(fx = 0 \leftrightarrow \exists y A_0(x,y)\big)$.

For a set Γ of sentences $\in \mathcal{L}(WE-PA^\omega)$, $WE-PA^\omega \oplus \Gamma$ means that the sentences from Γ are added as new axioms to $WE-PA^\omega$ but that application of the extensionality rule is allowed only when $A_0 \to s = t$ is proved in $WE-PA^\omega$ (i.e. without using the axioms Γ). $WE-PA^\omega$ satisfies the deduction theorem w.r.t. $\oplus$ but not w.r.t. $+$.

1.2 Definition

$min_\rho(x_1^\rho, x_2^\rho)$ is defined by induction on the type ρ:

$$\begin{cases} min_0(x_1,x_2) := min(x_1,x_2) \\ min_{\tau\rho}(x_1,x_2) := \lambda y^\rho.min_\tau(x_1 y, x_2 y). \end{cases}.$$

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Clearly $min_\rho \in \widehat{PR} \subset T$.

1.3 Proposition

1) $\widehat{WE-PA}^\omega\!\restriction + AC^{0,0}$–qf $\vdash \Sigma^0_1 - AC^{0,0},\ \Sigma^0_1 - IA,\ \Delta^0_1 - CA$.

2) $\widehat{E-HA}^\omega\!\restriction + AC$–qf $\vdash b-AC$–qf (Analogous for $E-HA^\omega$).

Proof:

1) is standard.

2) follows from $\widehat{E-HA}^\omega\!\restriction\ \vdash \exists y \leq_\rho x A_0(x,y) \leftrightarrow \exists y A_0(x, min_\rho(x,y))$.

[2] x' stands for Sx where S is the successor function.

2 Extraction of uniform bounds in higher types

2.1 Notational conventions

1) In the following r, s, t and Φ, Ψ always denote closed terms of Gdel's calculus T or Kleene/Feferman's calculus $\widehat{PR}$ of primitive recursive functionals of finite type as defined in 1.

2) ¿From now on (up to the end of this paper) all free variables of formulas are indicated, i.e. expressions of the form "$\forall u; v \leq tu\Big(\forall x \exists y \leq suvx \forall z A_0(u,v,x,y,z) \rightarrow \exists w^2 B_0(u,v,w)\Big)$ always stay for closed formulas (which are called sentences as usual). Sometimes we abbreviate e.g. $\forall x \exists y \forall z A(x,y,z)$ by $\forall x \exists y \forall z A$.

¿From Kohlenbach (1992) we recall the following definition of pointwise strong majorization, which is a variant of notions due to W.A. Howard (1973) and M. Bezem (1985).

2.2 Definition

For x^*, x of type ρ, $x^* \, maj_\rho \, x$ is given by

$$\begin{cases} x^* \, maj_0 \, x \; :\equiv \; x^* \; \geq \; x, \\ x^* \, maj_{\rho 0} \, x \; :\equiv \; \forall n^0 \, (x^* n \, maj_\rho \, xn), \\ x^* \, maj_{\rho\tau} \, x \; :\equiv \; \forall y^*, y \, (y^* \, maj_\tau \, y \; \rightarrow \; x^* y^* \, maj_\rho \, x^* y, xy) \\ (\tau \neq 0) \end{cases}$$

Here "$\geq$" denotes the usual primitive recursively defined inequality relation for objects of type 0. A discussion of the basic properties of maj_ρ can be found in Kohlenbach (1992).
The following theorem shows how one can extract bounds for existence quantifiers of type $\leq$ 2(not depending on bounded parameters), which are proved from premises of the form $\forall x^\delta \exists y \leq_\rho sx \forall z^\tau A_0(x,y,z)$:

2.3 Theorem

1) $WE-PA^\omega + AC\text{–qf} \vdash \forall u^1; v \leq_\gamma tu\Big(\forall x^\delta \exists y \leq_\rho suvx \forall z^\tau A_0(u,v,x,y,z) \rightarrow \exists w^2 B_0(u,v,w)\Big)$

   $\Rightarrow \exists \Phi^{21} \in T$ such that

   $$WE-HA^\omega + b-AC^{\delta,\rho} - \forall \vdash \forall u; v \leq_\gamma tu\Big(\forall x^\delta \exists y \leq_\rho suvx \forall z^\tau A_0(u,v,x,y,z) \\ \rightarrow \exists w \leq_2 \Phi u B_0(u,v,w)\Big).$$

   Φ can be extracted from any given proof of the assumption by functional interpretation combined with majorization.

2) The systems $WE-PA^\omega, T$ and $WE-HA^\omega$ in 1) can be replaced by $\widehat{WE-PA^\omega}\restriction, \widehat{PR}$ and $\widehat{WE-HA^\omega}\restriction$.

Proof:

1) $WE-PA^\omega + AC\text{–qf} \vdash \forall u^1; v \leq_\gamma tu\big(\forall x^\delta \exists y \leq_\rho suvx \forall z^\tau A_0 \to \exists w^2 B_0\big) \ \Rightarrow$
$WE-PA^\omega + AC\text{–qf} \vdash \forall u^1; v \leq_\gamma tu\Big(\exists Y \leq_{\rho\delta} suv \forall x^\delta, z^\tau A_0(u,v,x,Yx,z) \to \exists w B_0\Big) \ \Rightarrow$
$WE-PA^\omega + AC\text{–qf} \vdash \forall u^1; v \leq_\gamma tu; Y \leq_{\rho\delta} suv \exists x,z,w(A_0 \to B_0)$.

By functional interpretation (see Kohlenbach (1992),3.3) one extracts a closed term $\Phi_0 \in T$ such
that

$$(+)\, WE-HA^\omega \vdash \forall u^1; v \leq_\gamma tu; Y \leq suv\Big(\forall x,z A_0 \to B_0(u,v,\Phi_0 uvY)\Big).$$

By Kohlenbach (1992) 2.15, one can construct closed terms $\Phi_0^*, s^*, t^* \in T$ with

$$WE-HA^\omega \vdash \Phi_0^* \, maj \, \Phi_0 \wedge s^* \, maj \, s \wedge t^* \, maj \, t.$$

Define $\Phi := \lambda u^1.\Phi_0^* u(t^*u)\big(s^*u(t^*u)\big) \in T$ if $\gamma > 0$ and $\Phi := \lambda u^1.\big(\Phi_0^* u\big)^M(t^*u)\big((s^*u)^M(t^*u)\big)$ if $\gamma = 0$, where $(x^{\rho 0})^M = \lambda n.max_\rho(x0,...,xn)$ (see Kohlenbach (1992) 2.11). As in the proof of 3.1.1 in Kohlenbach (1992) one shows that

$$WE-HA^\omega \vdash \forall u; v \leq tu; Y \leq suv(\Phi u \, maj_2 \, \Phi_0 uvY)$$

which implies

$$(++)\, WE-HA^\omega \vdash \forall u; v \leq tu; Y \leq suv(\Phi u \geq_2 \Phi_0 uvY) \text{ (Kohlenbach (1992),2.5.2)}$$

(+) and (++) imply that

$$WE-HA^\omega \vdash \forall u; v \leq tu; Y \leq suv\Big(\forall x,z A_0 \to \exists w \leq_2 \Phi u B_0(u,v,w)\Big)$$

Hence

$$WE-HA^\omega \vdash \forall u; v \leq tu\Big(\exists Y \leq suv \forall x,z A_0(u,v,x,Yx,z) \to \exists w \leq_2 \Phi u \ B_0\Big).$$

Using $b-AC^{\delta,\rho}-\forall$ one concludes

$$WE-HA^\omega + b-AC^{\delta,\rho}-\forall \vdash \forall u; v \leq tu\big(\forall x \exists y \leq suvx \forall z A_0 \to \exists w \leq_2 \Phi u \ B_0\big).$$

2) is proved similar using the analogous result for $\widehat{PR}$ proved in Kohlenbach (1992).

2.4 Corollary to the proof of 2.3

1) If the quantifier "$\forall x$" does not occur then the conclusion can be proved without $b-AC$:

$$\begin{cases} WE-PA^{\omega}+AC\text{–qf}\vdash \forall u^1; v\leq_{\gamma} tu\big(\exists y\leq_{\rho} suv\forall z^{\tau}A_0 \ \to\ \exists w^2 B_0\big) \\ \Rightarrow \exists\Phi^{21}\in T \text{ such that} \\ WE-HA^{\omega}\vdash \forall u^1; v\leq_{\gamma} tu\big(\exists y\leq_{\rho} suv\forall z A_0 \ \to\ \exists w\leq_2 \Phi u B_0\big). \end{cases}$$

2) For variables w^{λ} of an arbitrary type λ, 2.3 holds with "$\exists w\big(\Phi u\, maj_{\lambda}\, w\wedge B_0(u,v,w)\big)$" instead of "$\exists w\leq_{\lambda}\Phi u B_0(u,v,w)$". If one has "$\exists w^2,\tilde{w}^{\lambda}B_0(u,v,w,\tilde{w})$" instead of "$\exists w^2 B_0$" then it is still possible to bound "$\exists w^2$" by "$\exists w\leq_2 \Phi u\exists\tilde{w}B_0$".

3) 2.3 holds also if $grad(type/w)\leq 2$, where $grad(\rho)$ is defined by

   $grad(0):=0,\ grad(\rho\tau):=max\big(grad(\rho),grad(\tau)+1\big)$.

   Furthermore, 2.3. generalizes to tuples $\underline{u},\underline{v},\underline{z},\underline{w}$ of variables $(grad(u_i)\leq 1, grad(w_i)\leq 2)$ and (with a corresponding modification of $b-AC-\forall$) also for tuples $\underline{x},\underline{y}$ instead of the single variables u,v,z,w,x,y.

4) The theorem generalizes immediately to the situation where one has a finite conjunction of premises having the form $\forall x^{\delta}\exists y\leq_{\rho} suvx\forall z^{\tau}A_0$.

2.5 Corollary to 2.3

1) $WE-PA^{\omega}+AC\text{–qf}\vdash \forall u^1; v\leq_{\gamma} tu\Big(\forall x^{\delta}\exists y\leq_{\rho} suvx\forall z^{\tau}A_0(u,v,x,y,z)$

$$\to \exists w^0 B_0(u,v,w)\Big)$$

$\Longrightarrow \exists\tilde{\Phi}^{0\gamma 1}\in T$ such that

$WE-HA^{\omega}+b-AC^{\delta,\rho}-\forall\vdash \forall u; v\leq_{\gamma} tu\Big(\forall x^{\delta}\exists y\leq_{\rho} suvx\forall z^{\tau}A_0\to B_0(u,v,\tilde{\Phi}uv)\Big)$.

In particular, if Γ is a set of sentences having the form

$\forall x^{\delta}\exists y\leq_{\rho} sx\forall z^{\tau}A_0(x,y,z)$ then the following rule holds

$$\begin{cases} WE-PA^{\omega}\oplus AC\text{–qf}\oplus\Gamma\vdash \forall u^1\exists w^0 B_0(u,w)\Rightarrow \exists\Phi^{01}\in T: \\ WE-HA^{\omega}\oplus\Gamma\oplus b-AC-\forall\vdash \forall u B_0(u,\Phi u). \end{cases}$$

2) Analogous for $\widehat{WE-PA}^{\omega}\!\restriction, \widehat{PR}$ and $\widehat{WE-HA}^{\omega}\!\restriction$ instead of $WE-PA^{\omega}, T$ and $WE-HA^{\omega}$.

Proof:

1) Using 2.3 one gets a bound $\Phi\in T$ such that $\exists w\leq \Phi u B_0(u,v,w)$. Since B_0 is quantifier–free, there exists a closed term $\chi_{B_0}\in T$ with

   $WE-HA^{\omega}\vdash \forall u,v,w\big(\chi_{B_0}uvw=_0 0\leftrightarrow B_0(u,v,w)\big)$

(see e.g. Luckhardt (1973) or Troelstra (1973). Define $\tilde{\Phi} \in T$ such that

$$\tilde{\Phi}uv = \begin{cases} \min w \leq_0 \Phi u[\chi_{B_0}uvw =_0 0] \text{ if such a } w \text{ exists,} \\ 0^0 \qquad\qquad \text{otherwise.} \end{cases}$$

Since $\Phi, \chi_{B_0} \in T$ it follows that such a $\tilde{\Phi} \in T$ exists. $\tilde{\Phi}$ fulfils the claim.

Now assume $WE-PA^\omega \oplus AC\text{–qf} \oplus \Gamma \vdash \forall u^1 \exists w^0 B_0(u,v)$. There exist finitely many sentences $A_1, ..., A_n \in \Gamma$ such that

$$WE-PA^\omega \oplus AC\text{–qf} \oplus A_1 \oplus ... \oplus A_n \vdash \forall u^1 \exists w^0 B_0.$$

Hence

$$WE-PA^\omega \oplus AC\text{–qf} \vdash \bigwedge_{i=1}^{n} A_i \to \forall u \exists w B_0.$$

The corollary now follows by the reasoning above together with 2.4.4.

2) follows analogously since $\widehat{PR}$ is also closed under bounded search.

2.6 Remarks

1) The bound Φ in 2.3 is extracted by functional interpretation of the proof of a sentence having the form $\forall a^1; b \leq_\delta sa \forall c \leq_\rho rab \exists d^\tau F_0(a,b,c,d)$ and majorizing the resulting primitive recursive term $\Phi_0 \in T$. As the proof of 2.15 in Kohlenbach (1992) shows, such a majorizing functional can be constructed in a quite simple manner and uses only the operation $x^{\rho 0} \mapsto x^M$ where $x^M := \lambda n^0.max_\rho(x0, ..., xn)$ besides Φ_0. In applications to concrete mathematical examples, this construction will be done in the mathematically most natural way and not follow in detail the general procedure from the proof in Kohlenbach (1992). Also in mathematical applications the terms s,t are usually majorizable in a straightforeward way.

2) The proof of theorem 2.3 uses essentially the majorizability of primitive recursive functionals of **higher** types (the raising of types reduces the logical complexity of the original formula!): Even for the special case

$$(*)\ \forall x^1 \exists y \leq_1 sx \forall z^0 A_0 \to \forall u^0 \exists w^0 B_0.$$

one has to majorize a functional Φ_0 of type 3 in order to obtain a bound for Φ_0 on arguments $Y \leq_{1(1)} s$. While majorizability for type–2–objects follows also from the uniform continuity of primitive recursive functionals of type 2 (on bounded domains), this hereditary boundedness for types ≥ 3 is an important property of the mathematical structure of the T–definable functionals which no longer holds for (proof–theoretic inessential enlargements as) $T + \mu_1$ (where

$\mu_1 x^{0(00)} y^{00} := \min n[x(\overline{y,n}) =_0 xy]$, see Kohlenbach (1992), or type structures as HEO or ECF (see Troelstra (1973),Kohlenbach (1990)).

3) The use of bounded search in the definition of the algorithm $\tilde{\Phi}$ in 2.5 may be replaced by more simple operations using additional information in concrete applications, e.g. if B_0 is monotonic w.r.t w ,i.e

$$\forall u; v \leq tu; w_1, w_2\Big(B_0(u,v,w_1) \wedge w_2 \geq w_1 \rightarrow B_0(u,v,w_2)\Big),$$

then $\tilde{\Phi}$ can be identified with Φ. This is the case for an important class of examples (namely uniqueness sentences in classical analysis), where Φ is of mathematical interest since it does not depend on v (this will be discussed in detail in a subsequent paper).

2.7 Proposition

If in 2.3 $grad(\rho), grad(\gamma) \leq 1$ then $WE-PA^\omega + AC$–qf can be replaced by $E-PA^\omega + AC^{\alpha,\beta}$–qf where ($\alpha = 0 \wedge \beta$ arbitrary) or ($\alpha = 1 \wedge \beta = 0$). In 2.5 $WE-PA^\omega \oplus AC$–qf$\oplus\Gamma$ can be replaced by $E-PA^\omega + AC^{\alpha,\beta}$–qf $+\Gamma$ if Γ consists of sentences of the form $\forall x^\delta \exists y \leq_\rho sx \forall z^\tau A_0$ where ρ, γ fulfil the above restriction. This also holds for the corresponding restricted systems.

Proof:
Assume for simplicity $\rho = \gamma = 1$ and

$$E-PA^\omega + AC^{\alpha,\beta}\text{–qf} \vdash \forall u^1; v \leq_1 tu\big(\forall x^\delta \exists y \leq_1 suvx \forall z^\tau A_0 \rightarrow \exists w^2 B_0\big).$$

By elimination of extensionality (see Luckhardt (1973)) this sentence can be proved also without the axiom of extensionality, in particular it can be proved within $WE-PA^\omega + AC^{\alpha,\beta}$–qf. One easily verifies that the elimination procedure can also be applied to the restricted system $\widehat{E-PA}^\omega\!\restriction$.

In the following we show that, using a more complicated extraction of the bound Φ, one can prove the conclusion of 2.3 within $WE-HA^\omega + b-AC$–qf and $E-HA^\omega + AC$–qf which avoids the need of **higher** bounded choice.
Firstly, we need the following

2.8 Lemma

1) Let $A_0 \in \mathcal{L}(WE-PA^\omega)$ be a quantifier–free formula (possible containing further variables than x, y, z). Then the following holds:

$$WE-PA^\omega + AC\text{–qf} \vdash \forall z^\rho\Big(\forall x \leq_\rho z \exists y^\tau A_0(x,y,z) \rightarrow \exists Y^{\tau\rho} \forall x \leq_\rho z A_0(x, Yx, z)\Big).$$

2) Analogous for $\widehat{WE-PA}^\omega\!\restriction$ instead of $WE-PA^\omega$.

Proof:
Assume $\rho = 0\rho_k...\rho_1$. Then

$$\forall x \leq_\rho z \exists y^\tau A_0(x,y,z) \rightarrow \forall x\Big(\forall v_1^{\rho_1},...,v_k^{\rho_k}(x\underline{v} \leq_0 z\underline{v}) \rightarrow \exists y^\tau A_0(x,y,z)\Big)$$
$$\rightarrow \forall x \exists v_1,...,v_k, y\big(x\underline{v} \leq_0 z\underline{v} \rightarrow A_0(x,y,z)\big).$$

The lemma now follows from the fact that $\widehat{WE-PA^{\omega}}\restriction$ allows the coding of tuples $v_1, ..., v_k, y$ of functionals into a single functional (of suitable type, which depends on $\rho_1, ..., \rho_k, \tau$ only).

2.9 Theorem

1) $WE-PA^{\omega}+AC\text{–qf}\vdash \forall u^1; v\leq_{\gamma} tu\Big(\forall x^{\delta}\exists y\leq_{\rho} suvx\forall z^{\tau}A_0(u,v,x,y,z)$

$$\to \exists w^2 B_0(u,v,w)\Big)$$

$\Rightarrow \exists \Phi^{21}\in T$ such that

$$WE-HA^{\omega}+b-AC\text{–qf}\vdash \forall u; v\leq_{\gamma} tu\Big(\forall x^{\delta}\exists y\leq_{\rho} suvx\forall zA_0(u,v,x,y,z)$$

$$\to \exists w\leq_2 \Phi u B_0(u,v,w)\Big).$$

The conclusion can also be proved within $E-HA^{\omega}+AC$–qf.
Φ can be extracted by functional interpretation and majorization.

2) Analogous for $\widehat{WE-PA^{\omega}}\restriction, \widehat{WE-HA^{\omega}}\restriction, \widehat{E-HA^{\omega}}\restriction, \widehat{PR}$.

Proof:

1) Let $\mathcal{T}$ denote $WE-PA^{\omega}+AC$–qf. The assumption implies

$$\mathcal{T}\vdash \forall u; v\leq tu\big(\forall x\neg\forall y\leq_{\rho} suvx\exists z^{\tau}\neg A_0\to \exists w^2B_0\big)\Rightarrow (2.8)$$

$$\mathcal{T}\vdash \forall u; v\leq tu\Big(\forall x\neg\exists Z^{\tau\rho}\forall y\leq_{\rho} suvx\neg A_0(u,v,x,y,Zy)\to \exists w^2B_0\Big)\ \Rightarrow$$

$$\mathcal{T}\vdash \forall u; v\leq tu\Big(\forall x,Z^{\tau\rho}\exists y\leq_{\rho} suvxA_0(u,v,x,y,Zy)\to \exists w^2B_0\Big)\ \Rightarrow$$

$$\mathcal{T}\vdash \forall u; v\leq tu\Big(\exists Y\leq \lambda\tilde{Z}^{\tau\rho}.suv\forall x,Z^{\tau\rho}A_0(u,v,x,YZx,Z(YZx))\to \exists w^2B_0\Big).$$

By 2.4.1 one can extract a closed term $\Phi^{21}\in T$ such that

$$WE-HA^{\omega}\vdash \forall u; v\leq tu\Big(\exists Y\leq \lambda\tilde{Z}^{\tau\rho}.suv\forall x,ZA_0(u,v,x,YZx,Z(YZx))\to \exists w\leq_2 \Phi uB_0\Big).$$

Since $\forall x, Z$ can be replaced by a single $\forall$–quantifier via coding, this implies

$$WE-HA^{\omega}+b-AC\text{–qf}\vdash \forall u; v\leq tu\big(\forall x,Z\exists y\leq suvxA_0(...,Zy)\to \exists w\leq_2 \Phi uB_0\big)$$

$\Rightarrow$

$$WE-HA^{\omega}+b-AC\text{–qf}\vdash \forall u; v\leq tu\big(\forall x\exists y\leq suvx\forall zA_0(...,z)\to \exists w\leq_2 \Phi uB_0\big)$$

$\Rightarrow$ (1.3.2)

$$E-HA^{\omega}+AC\text{–qf}\vdash \forall u; v\leq tu\big(\forall x\exists y\leq suvx\forall zA_0\to \exists w\leq_2 \Phi u\ B_0(u,v,w)\big).$$

2) is proved analogously.

2.10 Remark

Using (a suitable) negative translation (e.g. the translation * from Luckhardt (1973)), the assumption of 2.9 implies

$$(*) WE-HA^{\omega}+AC\text{–qf}+(MP)^{\omega}\vdash \forall u; v\leq tu\big(\forall x^{\delta}\neg\neg\exists y\leq suvx\forall zA_0\to\exists wB_0\big).$$

If one treats the bounded quantifiers for the moment as usual quantifiers in the definition of functional interpretation,then functional interpretation applied to $(*)$ yields

$$WE-HA^{\omega}\vdash \forall u; v\leq tu; Y\leq\lambda\tilde{Z}.suv\exists x,Z,w\Big(A_0\big(u,v,x,YZx,Z(YZx)\big)\to B_0\Big),$$

which corresponds to the reasoning in the proof of 2.9.
On the other hand, if $(*)$ is weakened by deleting "$\neg\neg$" in the premise, i.e.

$$(**) WE-HA^{\omega}+AC\text{–qf}+(MP)^{\omega}\vdash \forall u; v\leq tu\big(\forall x\exists y\leq suvx\forall zA_0\to\exists wB_0\big),$$

then functional interpretation applied directly to $(**)$ gives

$$WE-HA^{\omega}\vdash \forall u; v\leq tu; Y\leq suv\exists x,z,w\big(A_0(u,v,x,Yx,z)\to B_0\big)$$

as in the proof of 2.3. Therefore the difference in the extraction of Φ in 2.3 and 2.9 is due to a different use of $\neg\neg$–translation. Since the extraction in 2.3 is much easier (compared with 2.9), and since in the (most interesting) analytical case $b-AC$ can be eliminated altogether from the proof of the conclusion in 2.3 (see 3.8), this extraction seems to be more useful for applications.

Next, we generalize 2.3 from "$\exists wB_0$" to formulae having the form "$\exists a^2\forall b\leq_{\tau} ruva\exists w^2B_0$" and show how one can extract primitive recursive bounds for w **and** a. The proof uses the following

2.11 Lemma

1) Let $A_0\in\mathcal{L}(WE-HA^{\omega})$ be a quantifier–free formula (possibly containing other free variables in addition to $\tilde{x},\tilde{Y},\tilde{z},x,y,z$). Then

$$E-HA^{\omega}+b-AC-\forall\vdash\forall\tilde{x},\tilde{Y},\tilde{z}\Big(\forall x\leq_{\rho}\tilde{x}\exists y\leq_{\tau}\tilde{Y}x\forall z\leq_{\delta}\tilde{z}A_0(\tilde{x},\tilde{Y},\tilde{z},x,y,z)$$
$$\to\exists Y\leq_{\tau\rho}\tilde{Y}\forall x\leq_{\rho}\tilde{x}; z\leq\tilde{z}A_0(\tilde{x},\tilde{Y},\tilde{z},x,Yx,z)\Big)$$

2) An analogous result holds for $\widehat{E-HA}^{\omega}\restriction$.

Proof:

1) Provable within $E-HA^{\omega}$ one has

$$\forall x\leq_{\rho}\tilde{x}\exists y\leq_{\tau}\tilde{Y}x\forall z\leq_{\delta}\tilde{z}A_0(\tilde{x},\tilde{Y},\tilde{z},x,y,z)$$

$$\to\forall x\exists y\leq_{\tau}\tilde{Y}\big(min_{\rho}(x,\tilde{x})\big)\forall zA_0(\tilde{x},\tilde{Y},\tilde{z},min_{\rho}(x,\tilde{x}),y,min_{\delta}(z,\tilde{z})).$$

Applying $b-AC-\forall$ to the conclusion one gets

$$\exists \widehat{Y} \leq \lambda x.\tilde{Y}\big(min(x,\tilde{x})\big)\forall x \leq \tilde{x}; z \leq \tilde{z} A_0(\tilde{x},\tilde{Y},\tilde{z},min(x,\tilde{x}),Yx,min(z,\tilde{z})).$$

Using extensionality the assertion in the lemma follows by putting $Y := min(\widehat{Y},\tilde{Y})$.

2) The proof is analogous.

2.12 Theorem

1) $WE-PA^{\omega}+AC\text{–qf} \vdash \forall u^1; v \leq_{\gamma} tu\big(\forall x^{\delta}\exists y \leq_{\rho} suvx\forall z^{\eta}A_0 \rightarrow$

$$\exists a^2 \forall b \leq_{\tau} ruva\exists w^2 B_0(u,v,a,b,w)\big)$$

$\Rightarrow \exists \Phi^{21}, \tilde{\Phi}^{21} \in T$ such that

$E-PA^{\omega}+b-AC-\forall \vdash \forall u^1; v \leq_{\gamma} tu\big(\forall x\exists y \leq suvx\forall z A_0 \rightarrow$

$$\exists a \leq_2 \tilde{\Phi}u\forall b \leq_{\tau} ruva\exists w \leq_2 \Phi u B_0(u,v,a,b,w)\big)$$

$\Phi, \tilde{\Phi}$ can be extracted by functional interpretation and majorization.

2) 1) holds analogously for $\widehat{WE-PA}^{\omega}\upharpoonright, \widehat{PR}$ and $\widehat{E-PA}^{\omega}\upharpoonright$.

Proof:

1) The following implications hold by logic:

$\exists a^2 \forall b \leq_{\tau} ruva\exists w^2 B_0(u,v,a,b,w) \rightarrow$

$\neg\forall a^2 \exists b \leq_{\tau} ruva\forall w^2 \neg B_0(u,v,a,b,w) \rightarrow$

$\neg\exists B \leq_{\tau 2} ruv\forall a^2, w^2 \neg B_0(u,v,a,Ba,w) \rightarrow$

$\forall B \leq_{\tau 2} ruv\exists a^2, w^2 B_0(u,v,a,Ba,w).$

Therefore the assumption of the theorem implies

$WE-PA^{\omega}+AC\text{–qf} \vdash \forall u^1; v \leq_{\gamma} tu\big(\forall x^{\delta}\exists y \leq_{\rho} suvx\forall z^{\eta}A_0 \rightarrow$

$$\forall B \leq_{\tau 2} ruv\exists a^2, w^2 B_0(u,v,a,Ba,w)\big).$$

By 2.3 one can extract closed terms $\tilde{\Phi}, \Phi \in T$ such that

$$WE-HA^\omega+b-AC-\forall \vdash \forall u^1; v \leq_\gamma tu\big(\forall x \exists y \leq suvx \forall z A_0 \rightarrow$$

$$\forall B \leq_{\tau 2} ruv \exists a \leq_2 \tilde{\Phi} u \exists w \leq_2 \Phi u B_0(u,v,a,Ba,w)\big).$$

Since

$$\forall B \leq_{\tau 2} ruv \exists a \leq_2 \tilde{\Phi} u \exists w \leq_2 \Phi u B_0(u,v,a,Ba,w) \rightarrow \text{(by logic)}$$

$$\neg \exists B \leq_{\tau 2} ruv \forall a \leq_2 \tilde{\Phi} u \forall w \leq_2 \Phi u \neg B_0(u,v,a,Ba,w) \rightarrow (\text{in } E-HA^\omega+b-AC-\forall, 2.11)$$

$$\neg \forall a \leq_2 \tilde{\Phi} u \exists b \leq_\tau ruva \forall w \leq_2 \Phi u \neg B_0(u,v,a,b,w) \rightarrow \text{(by logic)}$$

$$\exists a \leq_2 \tilde{\Phi} u \forall b \leq_\tau ruva \exists w \leq_2 \Phi u B_0,$$

the theorem follows.

2) is proved analogously.

In the proof of 2.3 we reduced the original situation to

$$WE-PA^\omega+AC\text{–qf} \vdash \forall u^1; v \leq_\gamma tu; Y \leq_{\rho\delta} suv \exists x,z,w\big(A_0 \rightarrow B_0\big)$$

and constructed a bound Φu for w. If the types of x and z are ≤ 2 then it is also possible to bound "$\exists x$" and "$\exists z$". Thus $\exists w B_0$ can be proved from the weakened assumption $\forall x \leq \chi u \exists y \leq suvx \forall z \leq \psi u A_0$ for suitable $\chi, \psi \in T$. We formulate this only for "$\exists z$" since the possibility of bounding x is not used in this paper:

2.13 Theorem

1) $$WE-PA^\omega+AC\text{–qf} \vdash \forall u^1; v \leq_\gamma tu\Big(\forall x^\delta \exists y \leq_\rho suvx \forall z^2 A_0(u,v,x,y,z)$$

$$\rightarrow \exists w^\tau B_0(u,v,w)\Big)$$

$\Rightarrow \exists \Psi^{21} \in T:$

$$WE-HA^\omega+b-AC^{\delta,\rho}-\forall^b \vdash \forall u; v \leq tu\Big(\forall x \exists y \leq suvx \forall z \leq_2 \Psi u A_0 \rightarrow \exists w B_0\Big).$$

2) $$WE-PA^\omega+AC\text{–qf} \vdash \forall u^1; v \leq_\gamma tu\Big(\forall x^\delta \exists y \leq_\rho suvx \forall z^2 A_0 \rightarrow \exists w^2 B_0\Big)$$

$\Rightarrow \exists \Phi^{21}, \Psi^{21} \in T:$

$$WE-HA^\omega+b-AC^{\delta,\rho}-\forall^b \vdash \forall u; v \leq tu\Big(\forall x \exists y \leq suvx \forall z \leq_2 \Psi u A_0 \rightarrow \exists w \leq_2 \Phi u B_0\Big).$$

If $type/w = 0$ one can compute an algorithm $\tilde{\Phi}$ for w which depends on u and v(as in 2.5). If $type/z = 0$, then $b-AC^{\delta,\rho}-\forall^b$ can be weakened to $b-AC^{\delta,\rho}$–qf.

3) $WE-PA^\omega+AC\text{–qf}\vdash \forall u^1; v\leq_\gamma tu\Big(\forall x^\delta\exists y\leq_\rho suvx\forall z^2A_0\to\exists a^2\forall b\leq_\eta ruva\exists w^2B_0\Big)$

$\Rightarrow \exists\Phi^{21},\tilde{\Phi}^{21},\Psi^{21}\in T:$

$$E-PA^\omega+b-AC-\forall^b\vdash\forall u;v\leq_\gamma tu\Big(\forall x\exists y\leq suvx\forall z\leq_2\Psi uA_0$$

$$\to\exists a\leq_2\tilde{\Phi}u\forall b\leq_\eta ruva\exists w\leq_2\Phi uB_0\Big).$$

1),2) and 3) are also valid for $\widehat{WE-PA}^\omega\restriction,\widehat{PR},\ \widehat{WE-HA}^\omega\restriction$ and $\widehat{E-PA}^\omega\restriction$.
Proof:

1) As in the proof of 2.3 it follows that

$$WE-PA^\omega+AC\text{–qf}\vdash\forall u;v\leq tu;Y\leq suv\exists x,z,w\big(A_0(u,v,x,Yx,z)\to B_0\big).$$

Using functional interpretation one extracts a closed term $\Psi_0\in T$ such that

$$WE-HA^\omega\vdash\forall u;v\leq tu,Y\leq suv\big(\forall xA_0(u,v,x,Yx,\Psi_0uvY)\to\exists wB_0\big).$$

By a construction analogous to the one used in the proof of 2.3 one obtains a closed term $\Psi\in T$ such that

$$WE-HA^\omega\vdash\forall u;v\leq tu;Y\leq suv\big(\Psi u\geq_2\Psi_0uvY\big).$$

Hence

$$WE-HA^\omega\vdash\forall u;v\leq tu\big(\exists Y\leq suv\forall x;z\leq_2\Psi uA_0\to\exists wB_0\big).$$

The theorem now follows by applying $b-AC^{\delta,\rho}-\forall^b$ to $\forall x^\delta\exists y\leq suvx\forall z\leq\Psi uA_0$.

2) follows from the proofs of 1) and 2.3,2.5.

3) follows from the proofs of 1) and 2.12.

2.14 Corollary to the proof of 2.13

1) If "$\forall z^2$" and "$\forall w^2$" in 2.13 are replaced by "$\forall z^2,\tilde{z}^\tau$" and "$\forall w^2,\tilde{w}^\eta$" where $\tau,\eta\in\tilde{T}$ are arbitrary, then it is still possible to extract primitive recursive bounds Ψ and Φ for z^2 and w^2 (which depend on u only).

2) A remark analogous to 2.4 holds for theorems 2.12,2.13.

2.15 Corollary

$$WE-PA^{\omega}+AC\text{–qf}\vdash \forall u^1;v\leq_{\gamma} tu\big(\forall x^{\delta}\exists y\leq_{\rho} suvx\forall z^2 A_0\to \exists w^0\forall f^1 B_0(u,v,w,f)\big)\quad\Rightarrow$$

$$WE-PA^{\omega}+AC^{0,1}\text{–qf}+b-AC^{\delta,\rho}-\forall^b\vdash \forall u;v\leq tu\big(\forall x^{\delta},\tilde{z}^2\exists y\leq_{\rho} suvx\forall z\leq_2 \tilde{z}A_0$$
$$\to \exists w^0\forall f^1 B_0(u,v,w,f)\big).$$

If $type/z=0$, then $b-AC^{\delta,\rho}-\forall^b$ can be
weakened to $b-AC^{\delta,\rho}$–qf.

$WE-PA^{\omega}$ can be replaced by $\widehat{WE-PA}^{\omega}\restriction$ (for $A_0,B_0\in\mathcal{L}(\widehat{WE-PA}^{\omega}\restriction)$).

Proof:
The assumption implies that

$$WE-PA^{\omega}+AC\text{–qf}\vdash \forall u^1;v\leq_{\gamma} tu\big(\forall x^{\delta}\exists y\leq_{\rho} suvx\forall z^2 A_0\to \forall F^{1(0)}\exists w^0 B_0(u,v,w,Fw)\big).$$

By 2.13.1 it follows that $\exists\Psi\in T$ (since $grad(1(0))=1$) such that

$$WE-HA^\omega+b-AC^{\delta,\rho}-\forall^b\vdash\forall u;v\le tu,F^{1(0)}\big(\forall x\exists y\le suvx\forall z\le_2\Psi uF\ A_0$$

$$\to\exists w^0B_0(u,v,w,Fw)\big).$$

$$\Rightarrow WE-HA^\omega+b-AC^{\delta,\rho}-\forall^b\vdash\forall u;v\le tu,F\big(\forall x,\tilde{z}^2\exists y\le suvx\forall z\le_2\tilde{z}A_0$$

$$\to\exists wB_0(u,v,w,Fw)\big).$$

Since $WE-PA^\omega+AC^{0,1}$–qf $\vdash\forall F^{1(0)}\exists w^0B_0(u,v,w,Fw)\to\exists w^0\forall f^1B_0(u,v,w,f)$

the corollary follows.

3 The analytical case

In this paragraph we show that the conclusion of 2.13.3 can be proved in $WE-HA^\omega$ (so in particular without any choice!) if all variables have types≤ 1.

3.1 Primitive recursive coding and some notations

We use the following primitive recursive coding of finite sequences of objects of type 0:

$j(x,y):=2^x(2y+1)-1,\ j_1z:=min\ x\le z[\exists y\le z(2^x(2y+1)=Sz)]$,
$j_2z:=min\ y\le z[\exists x\le z(2^x(2y+1)=Sz)]$.
$\nu_1(x):=x,\ \nu_{n+1}(x_0,x_1,...,x_n):=j\big(x_0,\nu_n(x_1,...,x_n)\big)$,

$$j_1^1(x):=x,\ j_i^n(x):=\begin{cases} j_1\circ(j_2)^{i-1}(x)\text{ if }1\le i<n\\ (j_2)^{n-1}(x)\text{ if }1<i=n\end{cases}\quad(\text{if }n>1).$$

It follows that $j_i^n\big(\nu_n(x_1,...,x_n)\big)=x_i\ \ (1\le i\le n),\ \nu_n(j_1^n(x),...,j_n^n(x))=x$.
$<>:=0,<x_0,...,x_n>:=S\big(\nu_2\big(n,\nu_{n+1}(x_0,...,x_n)\big)\big)$.

As an abbreviation we use $\widehat{x}:=<x>$. One can construct primitive recursive functions $*,\ lth,\ \Pi$ such that

$<x_0,...,x_n>*<y_0,...,y_m>=<x_0,...,x_n,y_0,...,y_m>,\ lth(<x_0,...,x_n>)=n+1$.

$$\Pi(n,y)=\begin{cases} x_y\text{ if }y\le m,\\ 0^0\text{ otherwise,}\quad\text{ for }n=<x_0,...,x_m>.\end{cases}$$

We usually use the notation $(n)_y$ for $\Pi(n,y)$. For functions (i.e. functionals of type 1) a^1, we define

$\overline{a}0:=<>,\ \overline{a}(Sx):=\overline{a}x*<ax>$. Thus for $x\ne 0$ one has $\overline{a}x=<a0,...,a(x-1)>$. $\overline{a}x$ is primitive recursive in a.

$$(\overline{a}x*v^1)(y):=\begin{cases} ay\text{ if }y<x\\ v(y-x)\text{ otherwise.}\end{cases}$$

$0^1:=\lambda x^0.0^0,\ \ 1^1:=\lambda x^0.1^0$, where $1^0:=S0^0$. $\overline{a,x}:=\overline{a}x*0^1$.

3.2 Lemma

1) Let $A_0(x^1, \underline{y}) \in \mathcal{L}(WE-HA^\omega)$ be a quantifier–free formula whose free variables are x^1, $\underline{y} = y_1, ..., y_n$ and $type/y_i \leq 1$ $(1 \leq i \leq n)$.

   Then the following holds:

$$WE-HA^\omega \vdash \underset{\exists}{\bigwedge} x^1 A_0(x, \underline{y}) \leftrightarrow \underset{\exists}{\bigwedge} k^0 A_0(\lambda m.(k)_m, \underline{y}).$$

2) An analogous result holds for $\widehat{WE-HA}^\omega\upharpoonright$ if $A_0 \in \mathcal{L}(\widehat{WE-HA}^\omega\upharpoonright)$.

Proof:

1) There exists a closed term $t \in T$ such that

   1. $WE-HA^\omega \vdash \forall x, \underline{y}\big(tx\underline{y} =_0 0 \leftrightarrow A_0(x, \underline{y})\big)$ (see e.g. Troelstra (1973),1.6.14).

   By Troelstra (1973),2.7.8 there exists, furthermore, a primitive recursive modulus of continuity $\tilde{t} \in T$ for t (w.r.t. the variable x):

   2. $WE-HA^\omega \vdash \forall x, \underline{y}, v^1\big(tx\underline{y} =_0 t(\overline{x}(\tilde{t}x\underline{y}) * v)\underline{y}\big)$:

   1. and 2. imply

   3. $WE-HA^\omega \vdash \forall x, \underline{y}\big(A_0(x, \underline{y}) \leftrightarrow A_0(\overline{x}(\tilde{t}x\underline{y}) * 0^1, \underline{y})\big)$.

   Since $\lambda m.\big(\overline{x}(\tilde{t}x\underline{y})\big)_m = \overline{x}(\tilde{t}x\underline{y}) * 0^1$, the lemma follows.

2) is proved analogously using the fact that each $t^2 \in \widehat{PR}$ possesses a modulus $\tilde{t} \in \widehat{PR}$ of pointwise continuity (provable in $\widehat{WE-HA}^\omega\upharpoonright$), which can be shown by using an adaptation of Troelstra's proof for T (see Kohlenbach (1990) for details).

3.3 Remark

In the proof of 3.2 we could also have used a modulus $\widehat{t} \in T$ $(\widehat{PR})$ of **uniform** continuity w.r.t. x for $t \in T$ $(\widehat{PR})$, i.e.

2.* $WE-HA^\omega \vdash \forall \tilde{x}; x, v \leq_1 \tilde{x}; \underline{y}\big(\overline{v}(\widehat{t}\tilde{x}\underline{y}) =_0 \overline{x}(\widehat{t}\tilde{x}\underline{y}) \rightarrow tx\underline{y} =_0 tv\underline{y}\big)$.

$(\widehat{WE-HA}^\omega\upharpoonright)$.

Then 3. in the proof of 3.2 holds also with $\widehat{t}x\underline{y}$ instead of $\tilde{t}x\underline{y}$. Such a modulus of uniform continuity can be extracted from extensionality proofs of t using functional interpretation and pointwise majorization (see Kohlenbach (1992),3.6).

3.4 Corollary

1) Let $A_0(x^0, y^1) \in \mathcal{L}(WE-PA^\omega)$ be a quantifier–free formula, whose free variables are of type≤ 1. Then

$$(i)\ \ WE-PA^\omega + AC^{0,0}\text{–qf} \vdash \forall x^0 \exists f^1 A_0(x,f) \rightarrow \exists F^{1(0)} \forall x^0 A_0(x, Fx),$$

$$(ii)\ \ WE-PA^\omega + AC^{0,0}\text{–qf} \vdash \exists f^1 A_0(0,f) \wedge \forall x^0 \big(\exists f A_0(x,f) \rightarrow \exists g A_0(x',g)\big) \rightarrow \forall x \exists f A_0(x,f),$$

i.e. $WE-PA^\omega + AC^{0,0}$–qf implies $AC^{0,1}$–qf and $\exists f^1 A_0$–induction for formulas A_0 having only free variables of type≤ 1.

2) Analogous for $\widehat{WE-PA}^\omega\!\restriction$ instead of $WE-PA^\omega$.

Proof:
1)(i),2)(i) follow immediately from 3.2, and 1)(ii),2)(ii) are proved using 1.3.1 and again 3.2.

3.5 Lemma

1) Let $A_0(x, \tilde{x}^1, \underline{y}) \in \mathcal{L}(WE-HA^\omega)$ be a quantifier–free formula whose free variables are $x, \tilde{x}, \underline{y} = y_1, ..., y_n$ where $type/x, \underline{y} \leq 1$. Assume that $s^{1\delta} \in T$ is closed and $\delta = type/x (\leq 1)$. Then there are (effectively) quantifier–free formulas $B_0(x, \underline{y})$ and $C_0(x, \underline{y})$ (containing only $x, \underline{y}$ free) such that

   1. $WE-HA^\omega \vdash \forall \tilde{x} \leq_1 sx A_0(x, \tilde{x}, \underline{y}) \leftrightarrow B_0(x, \underline{y})$,

   2. $WE-HA^\omega \vdash \exists \tilde{x} \leq_1 sx A_0(x, \tilde{x}, \underline{y}) \leftrightarrow C_0(x, \underline{y})$.

2) 1) holds also for $\widehat{WE-HA}^\omega\!\restriction$ instead of $WE-HA^\omega$.

Proof:

1) As in the proof of 3.2 there exists a closed term $t \in T$ such that

$$WE-HA^\omega \vdash \forall x; \tilde{x} \leq_1 sx; \underline{y}\big(tx\tilde{x}\underline{y} =_0 0 \leftrightarrow A_0(x, \tilde{x}, \underline{y})\big).$$

By Kohlenbach (1992) (3.5,3.6) one can compute a modulus $\widehat{t} \in T$ of uniform continuity for t on $\{\tilde{x}^1 | \tilde{x} \leq_1 sx\}$, so in particular

$$WE-HA^\omega \vdash \forall x; \tilde{x} \leq_1 sx; \underline{y}\Big(tx\tilde{x}\underline{y} =_0 tx\big(\overline{\tilde{x}}(\widehat{t}x\underline{y}) * 0^1\big)\underline{y}\Big).$$

It follows that

$$WE-HA^\omega \vdash \underset{\exists}{\bigwedge} \tilde{x} \leq sx A_0 \leftrightarrow \underset{\exists}{\bigwedge} k^0 \leq \Phi x \underline{y} \begin{cases} \forall i < lth\, k\big((k)_i \leq sxi\big) \rightarrow A_0(x, \lambda m.(k)_m, \underline{y}) \\ \forall i < lth\, k\big((k)_i \leq sxi\big) \wedge A_0(x, \lambda m.(k)_m, \underline{y}), \end{cases}$$

where $\Phi \in T$ such that

$$\Phi x\underline{y} = \max\left\{ < \tilde{x}_0, ..., \tilde{x}_{\widehat{t}x\underline{y}-1} > \mid \bigwedge_{i=0}^{\widehat{t}x\underline{y}-1} \tilde{x}_i \leq_0 sxi \right\}.$$

There are closed terms $t_1, t_2 \in T$ such that
$WE-HA^\omega \vdash \forall x, \underline{y}\Big[\Big(t_1 x\underline{y} =_0 0 \leftrightarrow \forall k \leq \Phi x\underline{y}\big(\forall i < lth\, k\big((k)_i \leq sxi\big) \to A_0(x, \lambda m.(k)_m, \underline{y})\big)\Big)$

$$\wedge\Big(t_2 x\underline{y} =_0 0 \leftrightarrow \exists k \leq \Phi x\underline{y}\big(\forall i < lth\, k\big((k)_i \leq sxi\big) \wedge A_0(x, \lambda m.(k)_m, \underline{y})\big)\Big)\Big].$$

$B_0(x, \underline{y}) :\equiv (t_1 x\underline{y} =_0 0)$ and $C_0(x, \underline{y}) :\equiv (t_2 x\underline{y} =_0 0)$ fulfil the lemma.

2) can be proved analogously since by Kohlenbach (1992) (3.5,3.6) a modulus $\widehat{t} \in \widehat{PR}$ of uniform continuity for $t \in \widehat{PR}$ can be constructed (provable in $\widehat{WE-HA}^\omega\!\restriction$).

3.6 Corollary to the proof of 3.5

1) For each sentence of the form $\forall \underline{x}^1 \exists y \leq_1 s\underline{x} A_0(\underline{x}, y) \in \mathcal{L}(WE-HA^\omega)$ one can construct a closed term $\chi \in T$ such that

$$WE-HA^\omega \vdash \forall \underline{x}\big(\exists y \leq s\underline{x} A_0(\underline{x}, y) \leftrightarrow A_0(\underline{x}, \chi\underline{x}) \wedge \chi\underline{x} \leq_1 s\underline{x}\big).$$

2) 1. holds analogous for $\widehat{WE-HA}^\omega\!\restriction, \widehat{PR}$ instead of $WE-HA^\omega, T$.

Proof:

1) The proof of 3.5 yields the construction of a closed $\Phi \in T$ such that

$$WE-HA^\omega \vdash \forall \underline{x}\Big(\exists y \leq sxA_0(\underline{x}, y) \leftrightarrow \exists k \leq_0 \Phi\underline{x}\big(A_0(\underline{x}, \lambda m.(k)_m) \wedge \forall i < lth\, k\big((k)_i \leq sxi\big)\big)\Big).$$

Define $\chi_0, \chi \in T$ such that

$$\chi_0\underline{x} = \begin{cases} \min k \leq \Phi\underline{x}\left[A_0(\underline{x}, \lambda m.(k)_m) \wedge \forall i < lth\, k\big((k)_i \leq sxi\big)\right] \text{ if existent,} \\ 0^0 \text{ otherwise,} \end{cases}$$

and $\chi\underline{x} = \lambda m.(\chi_0\underline{x})_m$. χ fulfils the claim.

2) An analogous assertion holds for $\widehat{WE-HA}^\omega\!\restriction, \widehat{PR}$ instead of $WE-HA^\omega, T$.

3.2 permits the construction of an algorithm for w in 2.5 even when *type*$/w = 1$ instead of $= 0$:

3.7 Proposition

1) $$\begin{cases} WE-PA^{\omega}+AC\text{–qf}\vdash \forall u^1;v\le_1 tu\big(\forall x^{\delta}\exists y\le_{\rho} suvx\forall z^{\tau}A_0\to\exists w^1B_0(u,v,w)\big)\\ \Rightarrow \exists\tilde{\Phi}^{011}\in T \text{ such that}\\ WE-HA^{\omega}+b-AC^{\delta,\rho}-\forall\vdash \forall u^1;v\le_1 tu\big(\forall x^{\delta}\exists y\le_{\rho} suvx\forall z^{\tau}A_0\to B_0(u,v,\lambda m.(\tilde{\Phi}uv)_m)\big).\end{cases}$$

2) 1) holds also for $\widehat{WE-PA}^{\omega}\restriction,\widehat{PR}$ and $\widehat{WE-HA}^{\omega}\restriction$.

Proof:
By 3.2 one can replace "$\exists w^1B_0(u,v,w)$" by "$\exists k^0B_0(u,v,\lambda m.(k)_m)$". The conclusion now follows from 2.5.

3.8 Theorem

Assume that $(\alpha=0\wedge\beta$ arbitrary) or $(\alpha=1\wedge\beta=0)$.

1) $E-PA^{\omega}+AC^{\alpha,\beta}\text{–qf}\vdash\forall u^1;v\le_1 tu\big(\forall x^1\exists y\le_1 suvx\forall z^1A_0\to$

$$\exists a^1\forall b\le_1 ruva\exists w^1B_0(u,v,a,b,w)\big)$$

$\Rightarrow\exists\Phi,\tilde{\Phi},\Psi\in T$ such that

$$WE-HA^{\omega}\vdash\forall u^1;v\le_1 tu\Big(\forall x^1\exists y\le_1 suvx\bigwedge_{i=0}^{\Psi u}A_0(u,v,x,y,\lambda m.(i)_m)\to$$

$$\bigvee_{j=0}^{\tilde{\Phi}u}\forall b\le_1 ruv(\lambda m.(j)_m)\bigvee_{k=0}^{\Phi u}B_0(u,v,\lambda m.(j)_m,b,\lambda m.(k)_m)\Big).$$

2) Analogous for $\widehat{E-PA}^{\omega}\restriction,\widehat{PR}$ and $\widehat{WE-HA}^{\omega}\restriction$ instead of $E-PA^{\omega},T$ and $WE-HA^{\omega}$.

$\Phi,\tilde{\Phi},\Psi$ can be extracted by functional interpretation combined with majorization.

Proof:

1) By the elimination of extensionality (see Luckhardt (1973)) and 3.2 the assumption implies

$$WE-PA^{\omega}+AC^{\alpha,\beta}\text{–qf}\vdash\forall u^1;v\le_1 tu\big(\forall x^1\exists y\le_1 suvx\forall iA_0(u,v,x,y,\lambda m.(i)_m)\to$$

$$\exists a^1\forall b\le_1 ruva\exists k^0B_0(u,v,a,b,\lambda m.(k)_m)\big).$$

As in the proof of 2.12.1 one shows

$$WE-PA^{\omega}+AC^{\alpha,\beta}\text{–qf}\vdash\forall u;v\le tu\big(\forall x^1\exists y\le_1 suvx\forall iA_0\to$$

$$\forall B\le_{11} ruv\exists a^1,k^0B_0(u,v,a,Ba,\lambda m.(k)_m)\big).$$

By the proofs of 2.3 and 2.13 one can extract closed terms $\Phi, \widehat{\Phi}, \Psi \in T$ such that

$$(*)\ WE-HA^\omega \vdash \forall u; v \leq tu\Big(\exists Y \leq_{11} suv \forall x^1; i \leq_0 \Psi u A_0(u, v, x, Yx, \lambda m.(i)_m) \to$$

$$\forall B \leq_{11} ruv \exists a \leq_1 \widehat{\Phi} u; k \leq_0 \Phi u B_0(u, v, a, Ba, \lambda m.(k)_m)\Big).$$

Claim:

$$(i)\ WE-HA^\omega \vdash \forall x^1 \exists y \leq_1 suvx \forall i \leq_0 \Psi u A_0(u, v, x, y, \lambda m.(i)_m) \to$$

$$\exists Y \leq_{11} suv \forall x^1; i \leq_0 \Psi u A_0(u, v, x, Yx, \lambda m.(i)_m).$$

$$(ii)\ WE-HA^\omega \vdash \forall u; v \leq tu\Big(\forall B \leq_{11} ruv \exists a \leq_1 \widehat{\Phi} u; k \leq_0 \Phi u B_0(u, v, a, Ba, \lambda m.(k)_m) \to$$

$$\exists j \leq_0 \tilde{\Phi} u \forall b \leq ruv(\lambda m.(j)_m) \exists k \leq_0 \Phi u B_0(u, v, \lambda m.(j)_m, b, \lambda m.(k)_m)\Big),$$

for some closed $\tilde{\Phi} \in T$ which can be extracted from the given data.

Proof of the claim:

(i) There exists a closed $t_{A_0} \in T$ such that

$$WE-HA^\omega \vdash t_{A_0} uvxy =_0 0 \leftrightarrow \forall i \leq_0 \Psi u A_0(u, v, x, y, \lambda m.(i)_m).$$

Now applying 3.6 to "$t_{A_0} uvxy = 0$" we find a $\chi \in T$ such that

$$WE-HA^\omega \vdash \exists y \leq_1 suvx\big(t_{A_0} uvxy =_0 0\big) \to t_{A_0} uvx(\chi uvx) =_0 0 \land \chi uvx \leq_1 suvx \ \Rightarrow$$

$$WE-HA^\omega \vdash \forall x \exists y \leq_1 suvx\big(t_{A_0} uvxy =_0 0\big) \to \forall x\big(t_{A_0} uvx(\chi uvx) =_0 0 \land \chi uvx \leq suvx\big) \ \Rightarrow$$

$$WE-HA^\omega \vdash \forall x \exists y \leq_1 suvx\big(t_{A_0} uvxy = 0\big) \to \exists Y^{11}\big(\forall x(Yx \leq_1 suvx)$$

$$\land \forall x\big(t_{A_0} uvx(Yx) = 0)\big)$$

$\Rightarrow$ (i).

(ii) $WE-PA^\omega \vdash \forall B \leq_{11} ruv \exists a \leq_1 \widehat{\Phi} u; k \leq_0 \Phi u B_0(u, v, a, Ba, \lambda m.(k)_m) \to$

$$\neg \exists B \leq_{11} ruv \forall a \leq_1 \widehat{\Phi} u; k \leq_0 \Phi u \neg B_0(u, v, a, Ba, \lambda m(k)_m) \overset{(!)}{\to}$$

$$\neg \forall a \leq_1 \widehat{\Phi} u \exists b \leq_1 ruva \forall k \leq_0 \Phi u \neg B_0(u, v, a, b, \lambda m.(k)_m) \to$$

$$(**)\ \exists a \leq_1 \widehat{\Phi} u \forall b \leq_1 ruva \exists k \leq_0 \Phi u B_0(u, v, a, b, \lambda m.(k)_m).$$

Ad !: The implication follows analogously to the proof of claim (i).

Let $t_{B_0} \in T$ be such that

$$WE - HA^\omega \vdash t_{B_0} uvab =_0 0 \leftrightarrow \exists k \leq_0 \Phi u B_0(u, v, a, b, \lambda m.(k)_m).$$

3.5 applied to $\forall b \leq ruva\big(t_{B_0} uvab =_0 0\big)$ yields a closed term $\tilde{t}_{B_0} \in T$ such that

$$WE - HA^\omega \vdash \forall b \leq ruva\big(t_{B_0} uvab =_0 0\big) \leftrightarrow \tilde{t}_{B_0} uva =_0 0.$$

Let $\widehat{\tilde{t}}u \in T$ be a modulus of uniform continuity of $\tilde{t}_{B_0}$ on $\{a | a \leq_1 \widehat{\Phi}u\}$ and $\{v | v \leq_1 tu\}$ (see Kohlenbach (1992),3.5,3.6). Then

$$WE - HA^\omega \vdash \forall v \leq_1 tu\Big(\exists a \leq_1 \widehat{\Phi}u\big(\tilde{t}_{B_0} uva =_0 0\big) \to \exists j^0(lth\, j \leq \widehat{\tilde{t}}u \wedge$$

$$\forall m < lth\, j\big((j)_m \leq \widehat{\Phi}um\big) \wedge \tilde{t}_{B_0}(u, v, \lambda m.(j)_m) =_0 0)\Big).$$

Since $lth\, j \leq \widehat{\tilde{t}}u \wedge \forall m < lth\, j\big((j)_m \leq \widehat{\Phi}um\big)$ implies $j \leq_0 < \widehat{\Phi}u0, ..., \widehat{\Phi}u(\widehat{\tilde{t}}u - 1) >=: \tilde{\Phi}u$, it follows that

$$WE - HA^\omega \vdash \forall v \leq tu\big(\exists a \leq_1 \widehat{\Phi}u\big(\tilde{t}_{B_0}(u, v, a) = 0\big) \to \exists j \leq_0 \tilde{\Phi}u\big(\tilde{t}_{B_0}(u, v, \lambda m.(j)_m) = 0\big)\big).$$

Combining this with $(**)$, we have

$$(***) WE - PA^\omega \vdash \forall u; v \leq tu\Big(\forall B \leq ruv \exists a \leq \widehat{\Phi}u; k \leq_0 \Phi u B_0 \to \exists j \leq_0 \tilde{\Phi}u\big(\tilde{t}_{B_0}(u, v, \lambda m.(j)_m) = 0\big)\Big),$$

where

$$WE - HA^\omega \vdash \tilde{t}_{B_0}(u, v, \lambda m.(j)_m) = 0 \leftrightarrow$$

$$\forall b \leq_1 ruv(\lambda m.(j)_m) \exists k \leq_0 \Phi u B_0(u, v, \lambda m.(j)_m, b, \lambda m.(k)_m).$$

Applying negative translation we conclude that $(***)$ is provable in $WE - HA^\omega$, which implies (ii).

End of the proof of the claim.

The theorem follows immediately from $(*)$ and the claim.

2) is proved analogously.

The proof of 3.8 easily generalizes to tuples of variables (with types ≤ 1) instead of the single variables u, v, x, y, z, a, b, w.

4 Weak Knig's lemma

4.1 Definition (WKL, Troelstra (1974))

The weak Knig's lemma is defined to be

$$WKL: \quad \forall f^1\Big(Tf \wedge \forall x^0 \exists n^0 (lth\ n = x \wedge fn = 0) \to \exists b \leq_1 \lambda k.1 \forall x^0 \big(f(\bar{b}x) = 0\big)\Big),$$

where $Tf :\equiv \forall n, m\big(f(n * m) = 0 \to fn = 0\big) \wedge \forall n, x\big(f(n* < x >) = 0 \to x \leq 1\big)$.

Tf asserts that f represents a 0,1–tree.

(The designation "weak Knig's lemma (WKL)" is due to H. Friedman).

WKL is equivalent (relative to $\widehat{WE-PA^{\omega}}\!\restriction$) to the variant WKL^*, where the knots of the tree are bounded by an arbitrary function h instead of $\lambda k.1$ (see 4.12 and 4.13 below).

On the other hand, the "full" Knig's lemma KL is the strengthening of WKL, obtained when only the number of branchings (i.e. the number of successor knots to each knot) in the tree is bounded (The knots themselves, which are represented by natural numbers, are not bounded). It is known that KL is equivalent to the schema of arithmetical comprehension ACA (relative to $\widehat{WE-PA^{\omega}}\!\restriction + AC$–qf; see Friedman (1975)). This equivalence also holds if KL is restricted to trees with at most two branchings). It follows that $WE-PA^{\omega}+AC$-qf+KL is proof-theoretically stronger then PA (see Feferman (1977),5.5.2), but $\widehat{WE-PA^{\omega}}\!\restriction + AC$–qf+$KL$ has the same strenght as PA (Shoenfield (1954), Feferman (1977),5.5.1).

If arbitrary (logically complex) formulas for the definition of 0,1–trees are allowed in WKL, then the resulting strengthening of WKL also implies ACA (see Troelstra (1974)).

These results contrast with the conservation results for WKL given below.

A detailed discussion of WKL and KL can be found in Kreisel/Mints/Simpson (1975).

WKL is equivalent to

$$(+)\ \forall f, g\Big(Tf \wedge \forall x\big(lth(gx) = x \wedge f(gx) = 0\big) \to \exists b \leq_1 \lambda k.1 \forall x^0 \big(f(\bar{b}x) = 0\big)\Big).$$

Since $\forall x^0 \exists n^0 \big(lth\ n = x \wedge fn = 0\big) \overset{\mathrm{T(f)}}{\to} \forall x \exists n \leq \overline{1^{00}}x\big(lth\ n = x \wedge fn = 0\big)$ the proof of this equivalence needs no AC–qf.

(+) is a sentence having the logical form

$$(*)\ \forall x^1 \big(\forall n^0 A_0(n, x) \to \exists y \leq_1 sx \forall z^0 B_0(x, y, z)\big),$$

where A_0 and B_0 are quantifier–free formulas.

One could try now to generalize the results above for premises of the form $\forall x \exists y \leq sx \forall z A_0(x,y,z)$ directly to assumptions having the form $(*)$, in particular one can ask:

$$(**)\begin{cases} WE-PA^\omega + AC\text{–qf} \vdash \forall x\big(\forall n A_0(n,x) \to \exists y \leq_1 sx \forall z^0 B_0\big) \to \forall k^0 \exists l^0 C_0 \overset{?}{\Rightarrow} \exists \Phi \in T: \\ WE-HA^\omega \vdash \forall x\big(\forall n A_0(n,x) \to \exists y \leq_1 sx \forall z^0 B_0\big) \to \forall k \exists l \leq \Phi k C_0. \end{cases}$$

This is false, however, as the following example shows (but see theorem 4.17):

Let $\forall m \exists n T_K(e,m,n)$ define a total recursive function, which is not definable in T (here T_K denotes Kleene's T–predicate). Classical logic yields

$$WE-PA^\omega \vdash \forall m\big(\forall n \neg T_K(\overline{e},m,n) \to 0=1\big) \longrightarrow \forall k \exists l T_K(\overline{e},k,l).$$

Modulo "dummy" quantifiers, $\forall m\big(\forall n \neg T_K(\overline{e},m,n) \to 0=1\big)$ has the logical form $\forall m\big(\forall n \neg T_K(\overline{e},m,n) \to \exists y \leq_1 \lambda k.1 \forall z B_0(y,z)\big)$. By $(**)$ we could extract a term $\Phi \in T$ such that $\forall k \exists l \leq \Phi k \; T_K(\overline{e},k,l)$. But this implies that the recursive function defined by $\forall k \exists l \; T_K(\overline{e},k,l)$ is definable in T, which is a contradiction.

Nevertheless the results proved so far can be applied to proofs which use WKL, since WKL is equivalent (provable within $\widehat{WE-HA}^\omega\!\upharpoonright$) to a sentence of the form $\forall \underline{x}^1 \exists y \leq_1 \lambda k.1 \forall z^0 A_0^K(\underline{x},y,z)$ and $\widehat{WE-HA}^\omega\!\upharpoonright \vdash \forall \underline{x}, z \exists y \leq_1 \lambda k.1 \bigwedge_{k=0}^{z} A_0^K(\underline{x},y,k)$:

4.2 Construction

1) $$\widehat{f}n := \begin{cases} fn \ \text{ if } fn \neq 0 \vee \Big(\forall k,l(k*l=n \to fk=0) \wedge \forall i < lth\, n\big((n)_i \leq 1\big)\Big), \\ 1^0 \ \text{ otherwise.} \end{cases}$$

2) $$f_g n := \begin{cases} fn \ \text{ if } f\big(g(lth\ n)\big)=0 \wedge lth\big(g(lth\ n)\big) = lth\ n, \\ 0^0 \ \text{ otherwise.} \end{cases}$$

4.3 Remark

$\widehat{f}$ (f_g) is primitive recursive in f (f and g) in the sense of Kleene (1952) and therefore also in the sense of $\widehat{PR}$ and T).

The operation   modifies f in such a way that the resulting function represents a 0,1–tree, i.e. $T(\widehat{f})$. If f satisfies already Tf, then   doesn't change f:

4.4 Lemma

1) $\widehat{WE-HA}^\omega\!\upharpoonright \vdash \forall f\big(T(\widehat{f})\big)$,

2) $\widehat{WE-HA}^\omega\!\upharpoonright \vdash \forall f\big(T(f) \to f =_1 \widehat{f}\big)$.

Proof:

1) $\widehat{f}(n*m)=0 \to \widehat{f}(n*m)=f(n*m)=0 \to$

$\forall k,l\big(k*l=n*m \to fk=0\big) \wedge \forall i<lth(n*m)\big((n*m)_i \le 1\big) \to$

$\forall k,l\big(k*l=n \to fk=0\big) \wedge \forall i<lth\ n\big((n)_i \le 1\big) \to \widehat{f}n=fn=0.$

$\widehat{f}(n*<x>)=0 \to \widehat{f}(n*<x>)=f(n*<x>)=0 \to$

$\forall i<lth(n*<x>)\big((n*<x>)_i \le 1\big) \to x \le 1.$

2) Assume Tf. Then

$$fn=0 \to \forall k,l\big(k*l=n \to fk=0\big) \wedge \forall i<lth\ n\big((n)_i \le 1\big).$$

Therefore $\widehat{f}n=fn$ for all $n \in \omega$.

f_g always satisfies the foundation condition $\forall x \exists n\big(lth\ n=x \wedge f_g n=0\big)$. If already $\forall x\big(lth(gx)=x \wedge f(gx)=0\big)$, then $f_g =_1 f$:

4.5 Lemma

1) $\widehat{WE-HA}^{\omega}{\restriction} \vdash \forall f,g \forall x \exists n\big(lth\ n=x \wedge f_g n=0\big)$,

2) $\widehat{WE-HA}^{\omega}{\restriction} \vdash \forall f,g\Big(\forall x\big(lth(gx)=x \wedge f(gx)=0\big) \to f_g =_1 f\Big)$.

Proof:

1) Define $n_1 := <\underbrace{0^0,...,0^0}_{x\text{–times}}>$. Then $lth\ n_1=x$.

Case (i): $f\big(g(lth\ n_1)\big)=0 \wedge lth\big(g(lth\ n_1)\big)=lth\ n_1$: Put $n:=g(lth\ n_1)$. Then $f(n)=0$ and therefore also $f_g(n)=0$.

Case (ii): $f\big(g(lth\ n_1)\big)\neq 0 \vee lth\big(g(lth\ n_1)\big)\neq lth\ n_1$: By f_g–definition $f_g(n_1)=0$. Therefore $n:=n_1$ fulfils the lemma in this case.

2) $\forall x\big(lth(gx)=x \wedge f(gx)=0\big) \to \forall n\Big(f\big(g(lth\ n)\big)=0 \wedge lth\big(g(lth\ n)\big)=lth\ n\Big)$

$\to \forall n(f_g n=fn)$.

4.6 Definition

$WKL' :\ \forall f^1,g^1 \exists b \le_1 \lambda k.1 \forall x^0\Big(\big(\widehat{f}\big)_g(\bar{b}x) =_0 0\Big).$

4.7 Proposition

1) $\widehat{WE-HA}^{\omega}\restriction \vdash \forall f, g, x \exists b \leq_1 \lambda k.1 \bigwedge_{y=0}^{x} \Big(\widehat{(\widehat{f})}_g(\bar{b}y) =_0 0\Big).$

2) $\widehat{WE-HA}^{\omega}\restriction \vdash WKL \leftrightarrow WKL'.$

Proof:

1) We show by induction on x

$$(*)\ \forall x \exists n \Big(lth\ n = x \wedge \forall i < x\big((n)_i \leq 1\big) \wedge \widehat{(\widehat{f})}_g(n) = 0\Big)$$

(Since the quantifier "$\exists n$" can be bounded by $\overline{1^1}x$, this induction is an application of (IA)–qf only).

$(*)$ implies 1): Define $b := n * 0^1$. By 4.4.1 $\widehat{(\widehat{f})}_g(\bar{b}x) = 0$ implies $\bigwedge_{y=0}^{x} \widehat{(\widehat{f})}_g(\bar{b}y) = 0$.

$\underline{x = 0}$: $lth\ n = x \leftrightarrow n = <> = 0$.

Case (i): $\widehat{f}(g0) = 0 \wedge lth(g0) = 0$: $g0 = 0 \wedge \widehat{f}(0) = 0$.

$\widehat{f}(0) = 0$ implies $(\widehat{f})_g(0) = 0$ and furthermore $\widehat{(\widehat{f})}_g(0) = 0$.

Case (ii): $\widehat{f}(g0) \neq 0 \vee lth(g0) \neq 0$: $(\widehat{f})_g(0) = 0$ and therefore $\widehat{(\widehat{f})}_g(0) = 0$.

$\underline{x \to x+1}$: By the induction hypothesis there exists a n_0 such that

$lth\ n_0 = x$, $\widehat{(\widehat{f})}_g(n_0) = 0$ and $\forall i < lth\ n_0\big((n_0)_i \leq 1\big)$.

Define $n_1 := n_0 * < 0 >$.

Case (i): $\widehat{f}\big(g(lth\ n_1)\big) = 0 \wedge lth\big(g(lth\ n_1)\big) = lth\ n_1$:

Then $n := g(lth\ n_1)$ fulfils the claim:

$$lth\ n = lth\big(g(lth\ n_1)\big) = lth\ n_1 = x + 1. \text{ Furthermore :}$$

$$\widehat{f}n = 0 \overset{4.4.1}{\to} \forall k, l\big(n = k * l \to \widehat{f}k = 0\big) \wedge \forall i < lth\big((n)_i \leq 1\big)$$

$$\overset{4.2.2}{\to} \forall k, l\big(n = k * l \to (\widehat{f})_g(k) = 0\big) \wedge \forall i < lth\big((n)_i \leq 1\big)$$

$\stackrel{4.2.1}{\to} \widehat{(f)}_g(n) = 0 \wedge \forall i < lth\ n\big((n)_i \leq 1\big).$

Case (ii): $\widehat{f}\big(g(lth\ n_1)\big) \neq 0 \vee lth\big(g(lth\ n_1)\big) \neq lth\ n_1$: Then $n := n_1$ fulfils the claim:

The case implies $(+)$ $\big(\widehat{f}\big)_g(n_1) = 0$ by 4.2.2. Since $\widehat{(f)}_g(n_0) = 0$, it follows by 4.2.1 that $\forall k, l\big(n_0 = k * l \to \big(\widehat{f}\big)_g(k) = 0\big)$.

Together with

$(+)$ and $n_1 = n_0 * < 0 >$ this implies $\widehat{(f)}_g(n_1) = 0$.

2) "$\to$": By 4.4.1, $T\big(\widehat{(f)}_g\big)$ holds for all f, g.

Using 1), WKL' now follows by WKL.

"$\leftarrow$": Assume $T(f)$ and $\forall x \exists n\big(lth\ n = x \wedge fn = 0\big)$. Then

$$(++)\ \forall x \exists n \leq \overline{1^1}x\big(lth\ n = x \wedge fn = 0\big).$$

Define

$$gx := \begin{cases} min\ n \leq \overline{1^1}x[lth\ n = x \wedge fn = 0] & \text{if such an } n \text{ exists,} \\ 0^0 \ \ \text{otherwise.} \end{cases}$$

g is primitive recursive in f and $(++)$ implies $\forall x\big(lth(gx) = x \wedge f(gx) = 0\big)$. 4.5.2 yields $f_g =_1 f$. Since $f =_1 \widehat{f}$ (4.4.2), this proves that $\widehat{(f)}_g =_1 f$. Using WKL' one derives $\exists b \leq_1 \lambda k.1 \forall x^0\big(f(\bar{b}x) = 0\big)$.

We are now able to conclude the following

4.8 Theorem

The results 2.3,2.4,2.5,2.7,2.9,2.12–2.15,3.7,3.8 also hold if $WE - PA^\omega + AC$–qf (resp. $E - PA^\omega + AC^{\alpha,\beta}$–qf) is replaced by $WE - PA^\omega \oplus WKL \oplus AC$–qf (resp. $E - PA^\omega + WKL + AC^{\alpha,\beta}$–qf). In particular

$$1) \quad \begin{cases} WE - PA^\omega \oplus AC\text{–qf} \oplus WKL \vdash \forall u^1; v \leq_\gamma tu \exists a^2 \forall b \leq_\eta ruva \exists w^2 A_0(u, v, a, b, w) \\ \Rightarrow \exists \Phi, \tilde{\Phi} \in T, \text{ closed such that} \\ E - PA^\omega + b - AC - \forall^b \vdash \forall u; v \leq_\gamma tu \exists a \leq_2 \tilde{\Phi} u \forall b \leq_\eta ruva \exists w \leq_2 \Phi u A_0. \end{cases}$$

$$2) \quad \begin{cases} WE - PA^\omega \oplus AC\text{–qf} \oplus WKL \vdash \forall u^1; v \leq_\gamma tu \exists w^2 A_0(u, v, w) \\ \Rightarrow \exists \Phi \in T, \text{ closed such that} \\ WE - HA^\omega \vdash \forall u; v \leq_\gamma tu \exists w \leq_2 \Phi u A_0. \end{cases}$$

$$
3) \qquad \begin{cases} E-PA^\omega + AC^{\alpha,\beta}\text{–qf} + WKL \vdash \forall u^1; v \leq_1 tu \exists a^1 \forall b \leq_1 ruva \exists w^1 A_0(u,v,a,b,w) \\ \Rightarrow \exists \Phi, \tilde{\Phi} \in T, \text{ closed such that} \\ WE-HA^\omega \vdash \forall u; v \leq tu \bigvee_{j=0}^{\tilde{\Phi}u} \forall b \leq_1 ruv\big(\lambda m.(j)_m\big) \bigvee_{k=0}^{\Phi u} B_0(u,v,\lambda m.(j)_m, b, \lambda m.(k)_m). \end{cases}
$$

Variables of type 1 (resp. 2) can be replaced by variables of type δ with $grad(\delta) \leq 1$ (≤ 2).
All the results above also hold for the corresponding restricted systems $\widehat{WE-PA}^\omega\restriction$, $\widehat{WE-HA}^\omega\restriction$, $\widehat{E-PA}^\omega\restriction$ and $\widehat{PR}$.

Proof:
We prove one case of the theorem, namely that in 2.3 $WE-PA^\omega + AC$–qf can be replaced by $WE-PA^\omega \oplus AC$–qf $\oplus WKL$.

$$
WE-PA^\omega \oplus AC\text{–qf} \oplus WKL \vdash \forall u^1; v \leq_\gamma tu\big(\forall x^\delta \exists y \leq_\rho suvx \forall z^\tau A_0(u,v,x,y,z) \\ \to \exists w^2 B_0(u,v,w)\big) \implies
$$

$$
WE-PA^\omega \oplus AC\text{–qf} \vdash WKL \to \forall u^1; v \leq_\gamma tu\big(\forall x^\delta \exists y \leq_\rho suvx \forall z A_0 \to \exists w^2 B_0\big) \overset{4.7.2}{\implies}
$$

$$
WE-PA^\omega \oplus AC\text{–qf} \vdash \forall u^1; v \leq_\gamma tu\Big(\forall f, g \exists b \leq_1 \lambda k.1 \forall n^0 \Big(\big(\widehat{f}\big)_g(\bar{b}n) =_0 0\Big) \\ \wedge \forall x^\delta \exists y \leq_\rho suvx \forall z A_0 \to \exists w^2 B_0\Big)
$$

$\overset{\text{Proofof2.13,2.4}}{\implies} \exists \Psi, \Phi \in T$:

$$
WE-HA^\omega + b-AC^{\delta,\rho} - \forall \vdash \forall u; v \leq_\gamma tu\Big(\exists B \leq \lambda f, g, k.1 \forall f, g \forall i \leq \Psi u\Big(\big(\widehat{f}\big)_g\Big(\big(\overline{Bfg}\big)i\Big) =_0 0\Big) \\ \wedge \forall x^\delta \exists y \leq_\rho suvx \forall z A_0 \to \exists w \leq_2 \Phi u B_0\Big) \overset{3.6}{\implies}
$$

$$
WE-HA^\omega + b-AC^{\delta,\rho} - \forall \vdash \forall u; v \leq_\gamma tu\Big(\forall f, g \exists b \leq_1 \lambda k.1 \forall i \leq \Psi u\Big(\big(\widehat{f}\big)_g(\bar{b}i) =_0 0\Big) \\ \wedge \forall x \exists y \leq suvx \forall z A_0 \to \exists w \leq_2 \Phi u B_0\Big) \implies
$$

$$
WE-HA^\omega + b-AC^{\delta,\rho} - \forall \vdash \forall u; v \leq_\gamma tu\Big(\forall f, g, n \exists b \leq_1 \lambda k.1 \forall i \leq n\Big(\big(\widehat{f}\big)_g(\bar{b}i) =_0 0\Big) \\ \wedge \forall x \exists y \leq suvx \forall z A_0 \to \exists w \leq_2 \Phi u B_0\Big) \overset{4.7.1}{\implies}
$$

$$
WE-HA^\omega + b-AC^{\delta,\rho} - \forall \vdash \forall u; v \leq_\gamma tu\big(\forall x \exists y \leq suvx \forall z A_0 \to \exists w \leq_2 \Phi u B_0\big).
$$

The other assertions in the theorem can be proved in a similar way.

Furthermore we obtain the following conservation results concerning WKL:

4.9 Theorem

1) $WE-PA^\omega \oplus AC$–qf $\oplus WKL$ is conservative over $WE-PA^\omega \oplus AC^{0,1}$–qf $(WE-HA^\omega)$ w.r.t. sentences of the form $\forall u^1; v \leq_\gamma tu\exists w^0\forall z^1 A_0$ $(\forall u^1; v \leq_\gamma tu\exists w^\tau A_0)$ $(\gamma, \tau \in \tilde{T}$ arbitrary).

2) $E-PA^\omega + AC^{\alpha,\beta}$–qf+$WKL$ $(\alpha = 0 \wedge \beta$ arbitrary) or $(\alpha = 1 \wedge \beta = 0)$ is conservative over $WE-PA^\omega + AC^{0,0}$–qf $(WE-HA^\omega)$ w.r.t. sentences of the form $\forall u^1; v \leq_1 tu\exists w^0\forall z^1 A_0$ $(\forall u^1; v \leq_1 tu\exists a^1\forall b \leq_1 ruva\exists w^1 A_0)$

(A_0 is quantifier–free and $t, r \in T$ are closed).

Variables of type 1 can be replaced by (tuples of) variables of type δ with $grad(\delta) \leq 1$.

1) and 2) also hold for the restricted systems $\widehat{WE-PA}^\omega\!\restriction$, $\widehat{WE-HA}^\omega\!\restriction$ and $\widehat{E-PA}^\omega\!\restriction$ instead of $WE-PA^\omega$, $WE-HA^\omega$, $E-PA^\omega$ (Then, of course, $t, r \in \widehat{PR}$).

Proof:

1) follows from 2.15 (2.13) and 4.7 (By 3.6 the proof of the conclusion does not need $b-AC-\forall^b$).

2) Assume $E-PA^\omega + AC^{\alpha,\beta}$–qf+$WKL \vdash \forall u^1; v \leq_1 tu\exists w^0\forall z^1 A_0$. Using elimination of extensionality and 3.2,4.7 one concludes

$$WE-PA^\omega + AC^{\alpha,\beta}\text{–qf} \vdash WKL' \to \forall u^1; v \leq_1 tu\exists w^0\forall k^0 A_0(u, v, w, \lambda m.(k)_m).$$

As in 1), 2.15 and 4.7 now imply

$$WE-PA^\omega + AC^{0,0}\text{–qf} \vdash \forall u^1; v \leq_1 tu\exists w^0\forall k^0 A_0.$$

(Since $type/k = 0$, only $AC^{0,0}$–qf but not $AC^{0,1}$–qf is needed; the use of $b-AC-\forall^b$ can be avoided by 3.6).

The second assertion of 2) follows immediately from 3.8 and 4.7.

4.10 Remark

Theorem 4.9.1 remains valid if $WE-PA^\omega \oplus AC$–qf$\oplus WKL$ and $WE-PA^\omega \oplus AC^{0,1}$–qf $(WE-HA^\omega)$ are replaced by $WE-PA^\omega \oplus AC$–qf$\oplus WKL \oplus \Gamma$ and $WE-PA^\omega \oplus AC^{0,1}$–qf$\oplus\Gamma$ $(WE-HA^\omega \oplus \Gamma)$, where Γ is an arbitrary set of sentences of the form $\exists x^1\forall y^\rho F_0(x, y) \in \mathcal{L}(WE-PA^\omega)$ $(F_0(x, y)$ is quantifier–free and contains no further free variables than $x, y)$. An analogous generalization holds for 4.9.2 (with + instead of $\oplus$) if Γ is a set of sentences of the form $\exists x^1\forall y^{0/1} F_0(x, y)$.

This remark is also correct for the corresponding restricted systems.

Proof:
Assume

$$WE-PA^\omega \oplus AC\text{–qf} \oplus WKL \oplus \Gamma \vdash \forall u^1; v \leq_\gamma tu\exists w^0\forall z^1 A_0.$$

Then there are finitely many sentences $F_1, ..., F_n \in \Gamma$ such that

$$WE-PA^\omega \oplus AC\text{–qf} \oplus WKL \vdash \bigwedge_{i=1}^{n} F_i \to \forall u^1; v \leq_\gamma tu \exists w^0 \forall z^1 A_0.$$

For notational simplicity, we assume that $n=1$. Let $F :\equiv F_1 :\equiv \exists x^1 \forall y^\rho F_0(x,y)$. Then

$$WE-PA^\omega \oplus AC\text{–qf} \oplus WKL \vdash F \to \forall u^1; v \leq_\gamma tu; Z^{1(0)} \exists w^0 A_0(u,v,w,Zw) \;\Rightarrow$$

$$WE-PA^\omega \oplus AC\text{–qf} \oplus WKL \vdash \forall x^1, Z^{1(0)}; v \leq_\gamma tu \exists y^\rho, w\big(F_0(x,y) \to A_0(u,v,w,Zw)\big).$$

By 4.9.1 we conclude that

$WE-HA^\omega \vdash \forall x^1, u, Z^{1(0)}; v \leq_\gamma tu \exists y, w(F_0 \to A_0)$ and therefore

$WE-HA^\omega \vdash \exists x \forall y F_0(x,y) \to \forall u; v \leq_\gamma tu; Z^{1(0)} \exists w^0 A_0(u,v,w,Zw)$, which implies

$WE-PA^\omega \oplus AC^{0,1}\text{–qf} \vdash \exists x \forall y F_0(x,y) \to \forall u; v \leq_\gamma tu \exists w^0 \forall z^1 A_0.$

The other assertions of 4.10 can be proved in a similar manner.

4.11 Remark

WKL is not conservative w.r.t. sentences of the form $(i)\ \exists y^1 \forall x^0 A_0(y,x)$ or $(ii)\ \forall x^2 \exists y^1 A_0(x,y)$:

(i) Each instance of WKL' with $f,g \in T$ has the form $\exists y \leq_1 \lambda k.1 \forall x^0 A_0(y,x)$ and is provable in $WE-PA^\omega \oplus AC$–qf$\oplus WKL$ but in general not in $WE-PA^\omega + AC$–qf:

Let $\exists y \leq_1 \lambda k.1 \forall x^0 A_0^{rec}$ be the application of WKL' to the primitive recursive Kleene–tree (here $f,g \in T$). Assume: $\mathcal{T} := WE-PA^\omega + AC$–qf$\vdash \exists y \leq_1 \lambda k.1 \forall x^0 A_0^{rec}$. Then $\mathcal{T} \vdash \exists y^1(y$ is not recursive) since no recursive y realizes $\exists y \leq \lambda k.1 \forall x^0 A_0^{rec}$ (provable within $\mathcal{T}$). But this contradicts the fact that $HEO \models \mathcal{T} + \forall y^1(y$ is recursive) (see Troelstra (1973),2.6.20,2.6.21). Analogous for $\exists y^1 \forall x^0 A_0'$ with $A_0'(y,x) :\equiv A_0^{rec}(min_1(y, \lambda k.1), x)$.

(ii) Assume

$$\Big(WE-PA^\omega \oplus WKL \oplus AC\text{–qf} \vdash \forall x^2 \exists y^1 A_0 \Rightarrow WE-PA^\omega + AC\text{–qf} \vdash \forall x^2 \exists y^1 A_0\Big)$$

for arbitrary quantifier–free A_0. Then

$$\Big(WE-PA^\omega \oplus WKL \oplus AC\text{–qf} \vdash \exists y^1 \forall x^0 A_0(y,x) \Longrightarrow$$

$$WE-PA^\omega \oplus WKL \oplus AC\text{–qf} \vdash \forall X^2 \exists y^1 A_0(y, Xy) \overset{\text{assumption}}{\Longrightarrow}$$

$$WE-PA^\omega + AC\text{–qf} \vdash \forall X^2 \exists y^1 A_0(y, Xy) \Longrightarrow$$

$$WE-PA^\omega + AC\text{–qf} \vdash \exists y^1 \forall x^0 A_0(y,x)\Big),$$

but this contradicts (i).

As mentioned above, our majorant–construction allows the extraction of primitive recursive bounds (resp. algorithms) for $\forall u^1; v \leq_\gamma tu \exists w^0 B_0$–sentences from proofs, which use **arbitrary** assumptions of the form $\forall x^\delta \exists y \leq_\rho sx \forall z^\tau A_0$. Furthermore, weak Knig's lemma WKL is equivalent to a **special** $\forall x^1 \exists y \leq_1 sx \forall z^0 A_0$–sentence (namely WKL'). Mathematical experience indicates that $\forall x^1 \exists y \leq_1 sx \forall z^0 A_0$–sentences are quite often provable from their "ε–versions" using WKL (e.g. the theorem that each $f \in \mathcal{C}[0,1]$ assumes its maximum on [0,1]). We want to prove now this universal character of WKL (for the case $\delta, \rho, \tau \leq 1$):

Each sentence having the form $\forall x^1 \exists y \leq_1 sx \forall z^1 A_0 \in \mathcal{L}(WE-PA^\omega)$ follows provable in $WE-PA^\omega + WKL$ from $(*)\ \forall x, k^0 \exists y \leq sx \bigwedge\limits_{i=0}^{k} A_0(x, y, \lambda m.(i)_m)$. By 3.5 $(*)$ is equivalent to a $\forall x^1 A_0$–sentence and therefore (by 3.2) to a $\forall x^0 A_0$–sentence. Hence each true $\forall x^1 \exists y \leq_1 sx \forall z^{0/1} A_0$–sentence is provable within $WE-PA^\omega + WKL$+a true $\forall x^0 A_0'$–sentence.

This result is mainly of theoretical interest. For concrete extractions of bounds from given proofs it is much easier to apply our method directly to mathematical assumptions having the form $\forall x \exists y \leq sx \forall z A_0$ instead of first reducing them to WKL+true universal sentences and then eliminating WKL. Furthermore, for higher types $\delta, \rho, \tau \geq 1$, it is in general not possible to reduce $\forall x^\delta \exists y \leq_\rho sx \forall z^\tau A_0$–sentences to WKL plus universal sentences (see 4.16) but we still can apply our method to extract bounds from proofs which use such assumptions (as was shown in 2.3).

4.12 Definition

$$WKL^*:\ \forall f^1, h^1 \Big(T\big(h, f\big) \wedge \forall x \exists n \big(lth\ n = x \wedge fn = 0\big) \to \exists b \leq_1 h \forall x^0 f(\bar{b}x) = 0 \Big),$$

where

$$T\big(f, h\big) :\equiv \forall n, m \big(f(n*m) = 0 \to fn = 0\big) \wedge \forall n, x \big(f(n* < x >) = 0 \to x \leq h\big(lth(n)\big)\big).$$

One easily proves the following lemma (using Troelstra (1973),1.9.24)

4.13 Lemma

$$\widehat{WE-PA}^\omega\!\upharpoonright \vdash WKL \leftrightarrow WKL^*.$$

4.14 Proposition

1) Let $\forall x^1 \exists y \leq_1 sx \forall z^1 A_0(x, y, z)$ be a sentence in $\mathcal{L}(WE-PA^\omega)$, where $s^{11} \in T$ is a closed term. Then

$$WE-PA^\omega + WKL \vdash \forall x^1, k^0 \exists y \leq_1 sx \bigwedge_{i=0}^{k} A_0(x, y, \lambda m.(i)_m) \leftrightarrow \forall x^1 \exists y \leq_1 sx \forall z^1 A_0.$$

2) 1) holds also for $\widehat{WE-PA}^\omega\!\upharpoonright$ instead of $WE-PA^\omega$.

Proof:

1) By 3.2 it suffices to consider $\forall x \exists y \leq sx \forall k^0 A_0(x,y,k)$.

$\forall x, k \exists y \leq sx \bigwedge\limits_{i=0}^{k} A_0(x,y,i)$ implies

$$(*)\ \forall x,k \exists n \underbrace{\Big(lth\ n = k \wedge \forall j < k\big((n)_j \leq sxj\big) \wedge \exists y \leq sx \bigwedge_{i=0}^{k} A_0(x, n * \lambda m.y(m+k), i) \Big)}_{\overline{A_0}(x,k,n):\equiv}.$$

By 3.5 $\overline{A_0}$ is quantifier–free definable in $WE-PA^\omega$. Therefore we can define in $WE-PA^\omega$ a function f_x such that

$$f_x n := \begin{cases} 0 \text{ if } \overline{A_0}(x, lth\ n, n), \\ 1 \text{ otherwise.} \end{cases}$$

For all x, $T(f_x, sx)$ holds. Furthermore $(*)$ implies $\forall x,k \exists n (lth\ n = k \wedge f_x n = 0)$. Therefore WKL^* applied to f_x, sx yields

$$(**)\ \forall x \exists y_0 \leq sx \forall k \Big(\exists y \leq sx \bigwedge_{i=0}^{k} A_0(x, \overline{y}_0 k * \lambda m.y(m+k), i) \Big).$$

It remains to show that $\forall k A_0(x, y_0, k)$: Assume there exists a $k \in \omega$ such that $\neg A_0(x, y_0, k_0)$. Since A_0 is quantifier–free, there exists a closed term $t \in T$ such that

$WE-PA^\omega \vdash \forall x,y,k\big(txyk =_0 0 \leftrightarrow \neg A_0(x,y,k)\big)$. By Troelstra (1973),2.7.8 there exists a modulus of pointwise continuity $\tilde{t} \in T$ for t w.r.t. y:

$$(***)\ WE-PA^\omega \vdash \forall x, y, \tilde{y}, k\big(tx(\overline{y}(\tilde{t}xyk) * \tilde{y})k =_0 txyk\big).$$

Define $n_0 := \tilde{t}xy_0k_0$. Since $txy_0k_0 = 0$, $(***)$ implies $\forall m \geq n_0, y\big(\neg A_0(x, \overline{y}_0 m * y, k_0)\big)$. Define $m := max(n_0, k_0)$, then $(**)$ yields $\exists y \leq sx A_0(x, \overline{y}_0 m * \lambda k.y(m+k), k_0)$, which is a contradiction.

2) is proved analogously using the fact that each $t^2 \in \widehat{PR}$ possesses a modulus of pointwise continuity $\tilde{t} \in \widehat{PR}$ provable within $\widehat{WE-PA}^\omega\!\upharpoonright$ (see Kohlenbach (1990)).

4.15 Corollary

1) For each sentence of the form $\forall x^1 \exists y \leq_1 sx \forall z^{0/1} A_0(x,y,z) \in \mathcal{L}(WE-PA^\omega)$ one can construct a corresponding Π^0_1–sentence $\forall n^0 B_0(n) \in \mathcal{L}(WE-PA^\omega)$ such that

$$WE-PA^\omega + WKL \vdash \forall x^1 \exists y \leq_1 sx \forall z^{0/1} A_0(x,y,z) \leftrightarrow \forall n^0 B_0(n).$$

2) An analogous result holds for $\widehat{WE-PA^{\omega}}\!\restriction$ instead of $WE-PA^{\omega}$.

Proof:
The corollary follows from 4.14 together with 3.5 and 3.2.

For types >1 the above corollary no longer holds:

4.16 Proposition

There exist true sentences $A \equiv \exists x \leq_2 1^2 \forall y^1 A_0(x,y) \in \mathcal{L}(E-PA^{\omega})$ such that $E-PA^{\omega}+WKL+\Gamma \nvdash A$, where Γ is the set of all true sentences having the form $\forall x^{\rho} F_0(x) \in \mathcal{L}(E-PA^{\omega})$ with $grad(\rho) \leq 2$ and F_0 is quantifier–free ("True" here means valid in the full type structure of all set–theoretical functionals of finite type).

Proof:
$b-AC$ applied to WKL' yields $A :\equiv \exists \Phi \leq 1^{1(1)(1)} \forall f,g,x \Big(\big(\widehat{f}\big)_g \big((\overline{\Phi f g})x \big) = 0 \Big)$. One easily shows that Φ is not continuous w.r.t. f. Hence $ECF \not\models A$ (see Troelstra (1973),2.6.5 for the definition of $ECF := ECF(\omega^{\omega})$).

On the other hand $ECF \models E-PA^{\omega}+WKL+\Gamma$: Since ECF_{00} contains all functions ω^{ω}, $ECF \models WKL$. Since the continuous x^{ρ} are a subset of all set-theoretical x^{ρ} $(grad(\rho) \leq 2!)$, the truth of Γ implies the truth of $[\Gamma]_{ECF}$. $ECF \models E-PA^{\omega}$ follows from Troelstra (1973),2.6.5.

Our last theorem states that it is possible to extract bounds for (1) $\forall u^1; v \leq_1 tu \exists k^0 D_0$–sentences which are proved from assumptions having the form (2) $\forall x^0(\forall w^0 B_0 \to \exists y \leq_1 sx \forall z^0 C_0)$ by analysing both the proof of $(2) \to (1)$ and a proof of the ε–version of (2). Thus, in contrast to assumptions $\forall x \exists y \leq sx \forall z A_0$, the truth of (2) is not sufficient for the extraction.

4.17 Theorem

1) Assume ($\rho = 0 \wedge \tau$ arbitrary) or ($\rho = 1 \wedge \tau = 0$) and

(i) $E-PA^{\omega}+AC^{\rho,\tau}$–qf $\vdash \forall \alpha^1 \exists \beta \leq_1 r\alpha \forall n^0 A_0(\alpha,\beta,n) \to$

$$\forall x^1 \big(\forall w^0 B_0(x,w) \to \forall z^0 \exists y \leq_1 sx \bigwedge_{j=0}^{z} C_0(x,y,j) \big) \quad \text{and}$$

(ii) $E-PA^{\omega}+AC^{\rho,\tau}$–qf $\vdash \forall x \big(\forall w^0 B_0(x,w) \to \exists y \leq_1 sx \forall z^0 C_0(x,y,z) \big)$

$$\to \forall u^1; v \leq_1 tu \exists k^0 D_0(u,v,k) \big).$$

Then from (i) one can extract a closed term $\chi \in T$ such that

$(i)^*$ $WE-HA^\omega \vdash \forall \alpha, n^0 \exists \beta \le r\alpha \bigwedge\limits_{i=0}^{n} A_0(\alpha,\beta,i) \to$

$$\forall x, z\big(\bigwedge\limits_{i=0}^{\chi xz} B_0(x,i) \to \exists y \le sx \bigwedge\limits_{j=0}^{z} C_0(x,y,j)\big).$$

¿From (ii)–using χ–one can extract a closed $\Psi \in T$ such that

$$(ii)^*\ WE-HA^\omega \vdash \forall \alpha, n \exists \beta \le r\alpha \bigwedge\limits_{i=0}^{n} A_0(\alpha,\beta,i) \to \forall u; v \le tu \bigvee\limits_{k=0}^{\Psi u} D_0(u,v,k).$$

2) An analogous result holds for $\widehat{E-PA}^\omega\!\restriction$, $\widehat{PR}$ and $\widehat{WE-HA}^\omega\!\restriction$.

Proof:

1) ¿From (i) one concludes (using elimination of extensionality)

$WE-PA^\omega + AC^{\rho,\tau}$–qf $\vdash \forall \alpha^1 \exists \beta \le_1 r\alpha \forall n A_0(\alpha,\beta,n) \to \forall x^1, z^0 \exists w^0 \Big(B_0(x,w) \to \exists y \le_1 sx \bigwedge\limits_{j=0}^{z} C_0\Big)$.

By 3.5 there exists (eff.) a quantifier–free formula $F_0 \in \mathcal{L}(WE-PA^\omega)$ such that

$$WE-PA^\omega \vdash F_0(x,w,z) \leftrightarrow \Big(B_0(x,w) \to \exists y \le sx \bigwedge_{j=0}^{z} C_0\Big).$$

2.13.2 yields closed terms $\xi, \chi \in T$ such that

$$WE-HA^\omega \vdash \forall x, z\Big(\forall \alpha \exists \beta \le_1 r\alpha \bigwedge_{i=0}^{\xi xz} A_0 \to \Big(\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \exists y \le sx \bigwedge_{j=0}^{z} C_0\Big)\Big)$$

(by 3.6, $b-AC-\forall^b$ is not needed to prove this conclusion),
which implies $(i)^*$.
(ii) implies (using again elimination of extensionality)

$$WE-PA^\omega + AC^{\rho,\tau}\text{–qf} \vdash \forall x \exists y \le sx \forall z \exists w \Big(\bigwedge_{i=0}^{w} B_0(x,i) \to \bigwedge_{j=0}^{z} C_0(x,y,j)\Big) \to \forall u; v \le tu \exists k D_0.$$

Hence

$$WE\text{–}PA^\omega + AC^{\rho,\tau}\text{–qf} \vdash \exists Y \le s, W^{001} \forall x, z\Big(\bigwedge_{i=0}^{Wxz} B_0(x,i) \to \bigwedge_{j=0}^{z} C_0(x,Yx,j)\Big) \to \forall u; v \le tu \exists k D_0.$$

Therefore

$$WE-PA^{\omega}+AC^{\rho,\tau}\text{–qf}\vdash \forall Y\leq s;W,u;v\leq tu\exists x,z,k\Big(\Big(\bigwedge_{i=0}^{Wxz}B_0(x,i)\to\bigwedge_{j=0}^{z}C_0(x,Yx,j)\Big)$$

$$\to\exists kD_0(u,v,k)\Big).$$

Using functional interpretation (see Kohlenbach (1992),3.3) one can construct $\tilde{\Phi},\tilde{\Psi}\in T$ from the given proof such that

$$WE-HA^\omega \vdash \forall Y \le s; W,u; v \le tu\Big(\forall x\Big(\bigwedge_{i=0}^{Wx(\tilde{\Phi}YWuv)} B_0(x,i) \to \bigwedge_{j=0}^{\tilde{\Phi}YWuv} C_0(x,Yx,j)\Big)$$

$$\to \bigvee_{k=0}^{\tilde{\Psi}YWuv} D_0(u,v,k)\Big)$$

$\Longrightarrow$

$$(*)\ WE-HA^\omega \vdash \forall Y \le s; u; v \le tu\Big(\forall x\Big(\bigwedge_{i=0}^{\chi x(\tilde{\Phi}Y\chi uv)} B_0(x,i) \to \bigwedge_{j=0}^{\tilde{\Phi}Y\chi uv} C_0(x,Yx,j)\Big)$$

$$\to \bigvee_{k=0}^{\tilde{\Psi}Y\chi uv} D_0(u,v,k)\Big).$$

By Kohlenbach (1992),2.15 one can construct $\tilde{\Phi}^*, \tilde{\Psi}^*, \chi^*, s^*, t^* \in T$ such that

$$WE-HA^\omega \vdash \tilde{\Phi}^*\ maj\ \tilde{\Phi} \wedge \tilde{\Psi}^*\ maj\ \tilde{\Psi} \wedge \chi^*\ maj\ \chi \wedge s^*\ maj\ s \wedge t^*\ maj\ t.$$

Define $\Phi := \lambda u.\tilde{\Phi}^* s^* \chi^* u(t^* u)$, $\Psi := \lambda u.\tilde{\Psi}^* s^* \chi^* u(t^* u)$, then

$$WE-HA^\omega \vdash \forall Y \le s; u; v \le tu\big(\Phi u \ge_0 \tilde{\Phi}Y\chi uv \wedge \Psi u \ge_0 \tilde{\Psi}Y\chi uv\big).$$

Together with $(*)$ this implies

$$WE-HA^\omega \vdash \forall Y \le s; u; v \le tu\Big(\forall z \le \Phi u; x\Big(\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \bigwedge_{j=0}^{z} C_0(x,Yx,j)\Big)$$

$$\to \bigvee_{k=0}^{\Psi u} D_0(u,v,k)\Big)$$

$\Longrightarrow$

$$WE-HA^\omega \vdash \forall u\Big(\exists Y \le s \forall x; z \le \Phi u\Big(\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \bigwedge_{j=0}^{z} C_0(x,Yx,j)\Big)$$

$$\to \forall v \le tu \bigvee_{k=0}^{\Psi u} D_0(u,v,k)\Big)$$

$\overset{3.6}{\Longrightarrow}$

$$(**)\ WE-HA^\omega \vdash \forall u\Big(\forall x \exists y \le sx \forall z \le \Phi u\Big(\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \bigwedge_{j=0}^{z} C_0(x,y,j)\Big)$$

$$\to \forall v \le tu \bigvee_{k=0}^{\Psi u} D_0(u,v,k)\Big).$$

It remains to show

$$WE-HA^\omega \vdash \forall x,z\Big(\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \exists y \le sx \bigwedge_{j=0}^{z} C_0(x,y,j)\Big) \to$$

$$\forall u,x\exists y \le_1 sx\forall z \le_0 \Phi u\Big(\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \bigwedge_{j=0}^{z} C_0(x,y,j)\Big)$$

(Together with $(**)$ and $(i)^*$ this implies $(ii)^*$).

Assume $(+)\ \forall x,z\Big(\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \exists y \le sx \bigwedge_{j=0}^{z} C_0(x,y,j)\Big)$. Define primitive recursively in x,u (in the sense of T) $z_{u,x}$ such that $z_{u,x} = max\left\{z \le_0 \Phi u | \bigwedge_{i=0}^{\chi xz} B_0(x,i)\right\}$. Then $\bigwedge_{i=0}^{\chi x z_{u,x}} B_0(x,i)$. By (+) there exists an $y \le sx$ with $(++)$ $\bigwedge_{j=0}^{z_{u,x}} C_0(x,y,j)$. We show

$$(+++)\ \forall z \le_0 \Phi u\Big(\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \bigwedge_{j=0}^{z} C_0(x,y,j)\Big):$$

Case 1: $z \le z_{u,x}$. Then by $(++)$ $\bigwedge_{j=0}^{z} C_0(x,y,j)$.

Case 2: $\Phi u \ge z > z_{u,x}$. By the maximality of $z_{u,x}$ it follows that $\neg \bigwedge_{i=0}^{\chi xz} B_0(x,i)$ and hence $\bigwedge_{i=0}^{\chi xz} B_0(x,i) \to \bigwedge_{j=0}^{z} C_0(x,y,j)$.

2) The proof is analogous.

4.18 Remark

1) The above theorem is usefull for the analysis of proofs which can be split into the parts

$(i)\ \forall\alpha\exists\beta \le r\alpha\forall n A_0 \to \forall x\big(\forall w B_0 \to \exists y \le sx\forall z C_0\big)$ and

$(ii)\ \forall x\big(\forall w B_0 \to \exists y \le sx\forall z C_0\big) \to \forall u; v \le tu\exists k D_0$:

One analyses separately the proof of (i), which is in particular a proof of

$$(i)^*\ \forall\alpha\exists\beta \le r\alpha\forall n A_0 \to \forall x\big(\forall w B_0 \to \forall z\exists y \le sx \bigwedge_{j=0}^{z} C_0\big),$$

and the proof of (ii) and combines the results to a bound for "$\exists k$".

In classical analysis there are interesting examples of proofs having this structure, e.g. the proof of the uniqueness of best Chebycheff approximation from de La Vallee Poussin (1919)/Natanson (1949), which we analyse in a subsequent paper using exactly the above strategy.

2) Theorem 4.17 implies immediately (without 4.7) that $E-PA^\omega + AC^{\alpha,\beta}$–qf+$WKL$ is conservative over $WE-HA^\omega$ w.r.t. $\forall u^1; v \leq_1 tu \exists k^0 A_0$–sentences (analogous for the systems with restricted induction), since

$$\widehat{WE-HA}^{\omega}\upharpoonright \vdash WKL \leftrightarrow \forall f,g\Big(Tf\wedge\forall x(lth(gx)=x\wedge f(gx)=0)\rightarrow \exists b\leq_1 \lambda k.1\forall x\big(f(\bar{b}x)=0\big)\Big)$$

and

$$\widehat{WE-HA}^{\omega}\upharpoonright \vdash \forall f,g\Big(Tf\wedge\forall x\big(lth(gx)=x\wedge f(gx)=0\big)\rightarrow \forall x\exists b\leq \lambda k.1\bigwedge_{j=0}^{x}\big(f(\bar{b}j)=0\big)\Big).$$

[Correction (1993): Replace the condition '($\alpha=0\wedge\beta$ arbitrary) or ($\alpha=1\wedge\beta=0$)' on $AC^{\alpha,\beta}-qf$ in 2.7, 3.8, 4.9.2 and 4.17 by '($\alpha=0\wedge\beta\leq 1$) or ($\alpha=1\wedge\beta=0$)'.]

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