A quantitative version of a theorem due to Borwein-Reich-Shafrir

Ulrich Kohlenbach

BRICS[*]
Department of Computer Science
University of Aarhus
Ny Munkegade
DK-8000 Aarhus C, Denmark
kohlenb@brics.dk

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Abstract

We give a quantitative analysis of a result due to Borwein, Reich and Shafrir on the asymptotic behaviour of the general Krasnoselski-Mann iteration for nonexpansive self-mappings of convex sets in arbitrary normed spaces. Besides providing explicit bounds we also get new qualitative results concerning the independence of the rate of asymptotic regularity of that iteration from various input data. In the special case of bounded convex sets, where by well-known results of Ishikawa, Edelstein/O'Brien and Goebel/Kirk the norm of the iteration converges to zero, we obtain uniform bounds which do not depend on the starting point of the iteration and the nonexpansive function, but only depend on the error ε, an upper bound on the diameter of C and some very general information on the sequence of scalars λ_k used in the iteration. Only in the special situation, where $\lambda_k := \lambda$ is constant, uniform bounds were known in that bounded case. For the unbounded case, no quantitative information was known before. Our results were obtained in a case study of analysing non-effective proofs in analysis by certain logical methods. General logical meta-theorems of the author guarantee (at least under some additional restrictions) the extractability of such bounds from proofs of a certain kind and provide an algorithm to extract them. Our results in the present paper (which we present here without any reference to that logical background) were

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found by applying that method to the original proof of the Borwein/Reich/Shafrir theorem. The general logical method which led to these results will be discussed (with further examples) in [22].

1 Introduction

This paper is the offspring of a case study in the project of analyzing non-effective proofs in analysis by logical tools with the aim of extracting new numerically relevant information (e.g. effective uniform bounds or algorithms etc.) hidden in the proofs.[1]

Let us discuss in more detail what kind of numerical information we are aiming at.
Many problems in numerical (functional) analysis are concerned with the construction of solutions x of certain equations $A(x)$, where x is element of some Polish space (typically with additional structure) and $A(x)$ can be written as $F(x) = 0$ for some continuous function $F : X \to \mathbb{R}$ (usually A, and hence F, will depend on certain parameters a which again belong to Polish spaces). The construction of a solution for A quite often involves two distinct steps:

1) Approximate solutions (also called 'ε-solutions') $x_n \in X$ satisfying $A_{\frac{1}{n}}(x_n)$,

$$A_\varepsilon(x) :\equiv (|F(x)| < \varepsilon),$$

are constructed (uniformly in the parameters of A).

2) One shows, using e.g. compactness arguments, that either $(x_n)_{n\in\mathbb{N}}$ itself or some subsequence converges to a solution of $A(x)$.

It is the non-effectivity of the second step which in many cases prevents one from being able to compute a solution $\widehat{x}$ of A effectively within a prescribed error $\frac{1}{k}$, i.e. to compute a function $n(k)$ such that $d_X(x_{n(k)}, \widehat{x}) < \frac{1}{k}$. Even when $X := K$ is compact and $\widehat{x}$ is uniquely determined, so that (x_n) itself converges to $\widehat{x}$, explicit a-priori bounds (in particular not depending on $\widehat{x}$ itself) on the rate of convergence of that sequence are often not provided in numerical analysis (due to the ineffectivity of the proof of the uniqueness of $\widehat{x}$).[2]
In several papers we have shown how proof theoretic techniques can be applied to extract certain quantitative information (so-called uniform moduli of uniqueness which generalize the concept of strong unicity as used e.g. in Chebycheff approximation theory) even from highly non-constructive uniqueness proofs and how effective rates of convergence can be obtained using this information (see [20] for an introduction to this and [17],[18],[19],[23]

[1]For other case studies in analysis in the context of best approximation theory see [17],[18],[23]. For general information on 'proof mining' in analysis see [19],[20].

[2]See also [25] for an interesting discussion of this and related points.

for concrete applications to approximation theory).

In this paper we are concerned with applications to the first of the two steps mentioned above in situations where an effective solution of 2) is not possible (mainly due to the lack of uniqueness). Again these applications are instances of general logical meta-theorems (proved in [21]). These meta-theorems provide an algorithm which transforms non-constructive proofs p (of a certain kind) into numerically enriched proofs p^*. The resulting proof p^* is again an ordinary mathematical proof which does not rely on any logical tools (it is only the method used to find it which does). In the present paper we just present the main result of our case study in ordinary mathematical terms without any reference to the general logical mechanism which produced it. For the latter see [22] where it is shown that the general form of the result is an instance of a quite universal logical scheme.

The proof we are going to treat in this paper is taken from the fixed point theory of nonexpansive mappings $f : C \to C$, for certain sets C in normed spaces X. The well-known Banach fixed point theorem tells us that contractive mappings f always have a unique fixed point if X is complete and C is closed and that the sequence $x_{k+1} := f(x_k)$ starting from any $x_0 \in C$ effectively converges to that fixed point. For nonexpansive functions f (i.e. functions which are Lipschitz continuous with Lipschitz constant $\lambda = 1$), in general fixed points only exist if X is uniformly convex, C is closed convex and bounded (by the famous Browder-Göhde-Kirk fixed point theorem, see [5],[11],[14]).

If X is a uniformly convex Banach space, $C \subset X$ is closed convex and bounded and $f(C)$ is a compact subset of C, then a well-known fixed point theorem due to Krasnoselski ([24]) states that a fixed point of f can be approximated by the following Krasnoselski iteration[3]

$$x_{k+1} := \frac{1}{2}(x_k + f(x_k)),\ x_0 \in C \text{ arbitrary.}$$

However, the situation still is quite different from the Banach fixed point theorem since

1) f may have several fixed points,

2) a fact closely related to the non-uniqueness of the fixed point is, that the rate of convergence of the Krasnoselski iteration to its limit is not computable uniformly in f and x_0 (see [22]).

So the Krasnoselski iteration does not provide an algorithm for the computation of a fixed point of f (with prescribed precision) but it can be used to find effectively approximate fixed points. Since (x_k) converges to some fixed point of f and f is continuous it is clear that for a sufficiently large n on, x_m $(m \geq n)$ will be an approximate fixed point:

$$(*) \forall \varepsilon > 0 \exists n \in \mathbb{N} \forall m \geq n (\|x_m - f(x_m)\| < \varepsilon).$$

[3]Due to a much more general result from [13], which we will discuss below, the assumption of X being uniformly convex actually is superfluous.

Because of the simple monotonicity property (see lemma 2.4.1) below)

$$\|x_{m+1} - f(x_{m+1})\| \leq \|x_m - f(x_m)\|$$

the formula

$$\forall m \geq n(\|x_m - f(x_m)\| < \varepsilon)$$

is equivalent to

$$\|x_n - f(x_n)\| < \varepsilon,$$

which (given a representation of real numbers as Cauchy sequences of rational numbers with fixed rate of convergence) is purely existential which is crucial for the possibility to extract an algorithm for n in $(*)$ uniformly in x_0 and f (here it is assumed that X, C have a computable so-called standard representation, see e.g. [17]).

In [22] we obtained various quantitative forms of Krasnoselski's theorem. In the present paper we consider strong generalizations of Krasnoselski's result due to [13],[6],[8] and [4]. In [13] it is shown that Krasnoselski's fixed point theorem even holds without the assumption of X being uniformly convex. Moreover, very general so-called Krasnoselski-Mann iterations

$$x_{k+1} := (1 - \lambda_k)x_k + \lambda_k f(x_k)$$

are allowed, where λ_k is a sequence in $[0, 1]$ which is divergent in sum and satisfies $\limsup_{k\to\infty} \lambda_k < 1$.

In particular, it is proved in [13] that for such iterations

$$\text{(I)} \lim_{k\to\infty} \|x_k - f(x_k)\| = 0,$$

where X is an arbitrary normed linear space, C a bounded convex subset of X and $f : C \to C$ is nonexpansive.

This result is further generalized in [4] to the case where C no longer is required to be bounded.[4] Then one has

$$\text{(II)} \lim_{k\to\infty} \|x_k - f(x_k)\| = r_C(f),$$

where

$$r_C(f) := \inf_{x\in C} \|x - f(x)\|$$

will in general be strictly positive.

We give a complete quantitative analysis of (II) (see theorem 2.7) as an instance of our

[4]Unbounded sets C in connection with Ishikawa's theorem were apparently first considered in [2] (tmh.2.1).

general results on the extractability of bounds from proofs using non-effective tools like the convergence of bounded monotone sequences of reals (see [21]) which in the case at hand is just applied to $(\|x_k - f(x_k)\|)_{k\in\mathbb{N}}$. We then specialize the resulting bound to the case where C is bounded and derive a uniform bound for (I) which only depends on ε, an upper bound d_C for the diameter $d(C)$ of C and some quite weak information on (λ_k) (see corollary 2.8).

None of the papers [13],[6],[8],[4] contains any bounds and in fact [6] and [8] use non-trivial functional theoretic embeddings to show (ineffectively) the existence of a common number $k \in \mathbb{N}$ which satisfies $\|x_k - f(x_k)\| < \varepsilon$ uniformly for all starting points x_0 ([6])[5] and all nonexpansive functions f ([8]). This uniformity comes for free out of our proof analysis. Moreover, as already mentioned we also have a new strong uniformity concerning C as the bounds only depend on d_C and to some extent also a uniformity w.r.t. (λ_k) (corollary 2.10). Even the non-effective existence of such uniform bounds was considered to be 'unlikely' in [9] (p.101). All this shows that the authors of the papers listed were not aware of the uniform bounds hidden in their proofs (note that the proof in [13] essentially is contained as a special case in the proof from [4] we are analyzing, so that the logical analysis of the former is even simpler than our proof analysis for the stronger result (II)).

This clearly indicates the usefulness of analysing non-effective proofs logically even if one is not particularly interested in the numerical details of the bounds themselves. In many cases such explicit bounds immediately show the independence of the quantity in question from certain input data (uniformity of the bound).[6]

2 Effective uniform bounds on the Krasnoselski-Mann iteration in arbitrary normed spaces

Definition 2.1 *Let $(X, \|\cdot\|)$ be a normed linear space and $S \subseteq X$ be a subset of X. A function $f : S \to S$ is called nonexpansive if*

$$(*)\ \forall x, y \in S(\|f(x) - f(y)\| \le \|x - y\|).$$

Definition 2.2 *Let $(X, \|\cdot\|)$ be a normed linear space, S a subset of X, $f : S \to S$ and $\varepsilon > 0$. A point $x \in S$ is called ε-fixed point of f if $\|x - f(x)\| \le \varepsilon$.*

[5] This paper only considers the special case where $\lambda_k := \lambda$ is constant.

[6] Another example for this: the explicit uniform constants of strong unicity for Chebycheff approximation which we extracted in [17],[18] by analysing classical uniqueness proofs for the best Chebycheff approximation (known already since about 1905-1917) immediately implied the existence of a common constant of unicity for compact sets K of functions $f \in C[a, b]$, if $\inf_{f\in K} dist(f, H) > 0$ (H a Haar space). This fact was proved in approximation theory only in 1976 (see [12]) without providing any bounds. Yet another example for this in the context of L_1-approximation can be found in [23].

Lemma 2.3 *Let $(X, \|\cdot\|)$ be a normed linear space, $\emptyset \neq C \subseteq X$ convex with bounded diameter $d(C) < \infty$ and $f : C \to C$ nonexpansive. Then f has ε-fixed points in C for every $\varepsilon > 0$.*

Proof: The situation reduces to the Banach fixed point theorem by the following well-known construction (see e.g. [4] but also [11] and [10](prop.1.4)): $f_t(x) := (1-t)f(x) + tc$ for some $c \in C$ and $t \in (0,1]$ since $f_t : C \to C$ is a contraction and therefore Banach's fixed point theorem applies. Since we are only interested in approximate fixed points it is not necessary to assume that X is complete or that C is closed (see [22] for details). □

In the following, $(X, \|\cdot\|)$ will be an arbitrary normed linear space, $C \subseteq X$ a non-empty convex subset of X and $f : C \to C$ a nonexpansive mapping.
We consider the so-called Krasnoselski-Mann iteration (which is more general than the Krasnoselski iteration and due to Mann [26]) generated starting from an arbitrary $x \in C$ by

$$x_0 := x, \quad x_{k+1} := (1-\lambda_k)x_k + \lambda_k f(x_k),$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of real numbers in $[0,1]$.

Lemma 2.4 ([4]) *For all $k \in \mathbb{N}$ and $x, x^* \in C$:*

1) $\|x_{k+1} - f(x_{k+1})\| \le \|x_k - f(x_k)\|$,

2) $\|x_{k+1} - x^*_{k+1}\| \le \|x_k - x^*_k\|$.

We assume (following [4]) that $(\lambda_k)_{k\in\mathbb{N}}$ is divergent in sum, which can be expressed (since $\lambda_k \ge 0$) as

$$(A)\ \forall n, i \in \mathbb{N} \exists k \in \mathbb{N} \left(\sum_{j=i}^{i+k} \lambda_j \ge n \right),$$

and that

$$(B)\ \limsup_{k\to\infty} \lambda_k < 1.$$

Define

$$r_C(f) := \inf_{x\in C} \|x - f(x)\|.$$

Theorem 2.5 ([4]) [7] *Suppose that $(\lambda_k)_{k\in\mathbb{N}}$ satisfies the conditions (A) and (B). Then for any starting point $x \in C$ and the Krasnoselski-Mann iteration (x_n) starting from x we have*

$$\|x_n - f(x_n)\| \overset{n\to\infty}{\to} r_C(f).$$

[7]With the additional assumption that λ_k is bounded away from zero, this result is also proved in [27].

Corollary 2.6 ([13],[8],[4]) [8] *Under the assumptions of theorem 2.5 plus the additional assumption that C has bounded diameter $d(C) < \infty$ the following holds:*

$$\forall x \in C \forall \varepsilon > 0 \exists n \mathbb{N} \forall m \geq n(\|x_m - f(x_m)\| \leq \varepsilon).$$

Proof: Follows from theorem 2.5 and lemma 2.3. □

Theorem 2.7 *Let $(X, \|\cdot\|)$ be a normed linear space, $C \subseteq X$ a non-empty convex subset and $f : C \to C$ a nonexpansive mapping. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $[0,1]$ which is divergent in sum and satisfies*

$$\forall k \in \mathbb{N}(\lambda_k \leq 1 - \frac{1}{K})$$

for some $K \in \mathbb{N}$.[9]
Let $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be such that[10]

$$\forall i, n \in \mathbb{N}(\alpha(i,n) \leq \alpha(i+1,n)) \text{ and}$$

$$\forall i, n \in \mathbb{N}(n \leq \sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s).$$

Let $(x_n)_{n\in\mathbb{N}}$ be the Krasnoselski-Mann iteration

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n f(x_n),\ x_0 := x$$

starting from $x \in C$. Then the following holds

$$\forall x, x^* \in C \forall \varepsilon > 0 \forall n \geq h(\varepsilon, x, x^*, f, K, \alpha)(\|x_n - f(x_n)\| < \|x^* - f(x^*)\| + \varepsilon),$$

where[11]

$$h(\varepsilon, x, x^*, f, K, \alpha) := \widehat{\alpha}(\lceil 2\|x - f(x)\| \cdot \exp(K(M+1))\rceil \dot{-} 1, M),$$
$$\text{with } M := \left\lceil \frac{1+2\|x-x^*\|}{\varepsilon} \right\rceil \text{ and}$$
$$\widehat{\alpha}(0,M) := \tilde{\alpha}(0,M),\ \widehat{\alpha}(m+1,M) := \tilde{\alpha}(\widehat{\alpha}(m,M),M) \text{ with}$$
$$\tilde{\alpha}(m,M) := m + \alpha(m,M) \quad (m \in \mathbb{N})$$

[8] See the discussion at the end of our paper for historical information on this result.

[9] The condition $\limsup\limits_{n\to\infty} < 1$ in the Borwein-Reich-Shafrir theorem is sightly less restrictive as it only implies the existence of such a K from some index k_0 on. However, our result can easily be extended to this situation just by letting the iteration start from x_{k_0} instead of x_0.

[10] Using the reasoning from the proof of corollary 2.10 below, we can easily use any function $\beta : \mathbb{N} \to \mathbb{N}$ satisfying $n \leq \sum\limits_{s=0}^{\beta(n)} \lambda_s$ and then define $\alpha(i,n) := \max\limits_{j\leq i}(\beta(n+j) - j + 1)$ to get an α satisfying these conditions.

[11] $n \dot{-} 1 = \max(0, n-1)$.

(*Instead of M we may use any upper bound $\mathbb{N} \ni \tilde{M} \geq \frac{1+2\|x-x^*\|}{\varepsilon}$). Likewise, $\|x - f(x)\|$ may be replaced by any upper bound.*)

Proof:

$$(1)\ \gamma := \|x^* - f(x^*)\|.$$

Let furthermore $\varepsilon > 0$ and $x \in C$ be arbitrary and let $M \in \mathbb{N}$ be such that

$$(2)\ M \geq \frac{1 + 2\|x - x^*\|}{\varepsilon}.$$

Let $\delta > 0$ be so small that

$$(3)\ \delta \exp(K(M + 1)) < 1,$$

where $K \in \mathbb{N}$ satisfies

$$(4)\ \forall k \in \mathbb{N}(\lambda_k \leq 1 - \frac{1}{K}).$$

Let $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be such that[12]

$$(5)\ \forall i, n \in \mathbb{N}(n \leq S_{i,\alpha(i,n)} \leq n + 1),$$

where

$$(6)\ S_{i,n} := \sum_{s=i}^{i+n-1} \lambda_s.$$

Consider the Krasnoselski-Mann iteration $(x_n)_{n\in\mathbb{N}}$ starting from x. By lemma 2.4.1), the sequence $(\|x_n - f(x_n)\|)_{n\in\mathbb{N}} \subset [0, \|x - f(x)\|]$ is monotone decreasing and hence convergent. Thus there exists an i such that

$$(7)\ \|x_i - f(x_i)\| - \|x_{i+\alpha(i,M)} - f(x_{i+\alpha(i,M)})\| \leq \delta.$$

Suppose that

$$(8)\ \|x_i - f(x_i)\| \geq \gamma + \varepsilon.$$

The proof in [4] uses the following inequality which is derived from a fundamental inequality due to [8] and which holds for all $i, n \in \mathbb{N}$

$$(9)\ S_{i,n} \cdot \|x_i - f(x_i)\| \leq \|x_i - x_{i+n}\| + P_{i,n} \cdot (\|x_i - f(x_i)\| - \|x_{i+n} - f(x_{i+n})\|),$$

where

$$(10)\ P_{i,n} := \prod_{s=i}^{i+n-1} \frac{1}{1 - \lambda_s}.$$

[12]Since $\lambda_k \in [0, 1)$ this can always be achieved. Below we show how this requirement on α can be replaced by the more handy one stated in the theorem.

As in [13] one shows that

$$(11)\ \forall i,n\in\mathbb{N}(P_{i,n}\le \exp(K\cdot S_{i,n})).$$

In [4] (p.23), the following inequality is established ((x^*_k) is the Krasnoselski-Mann iteration starting from x^*)

$$(12)\ \|x^*_i - x^*_{i+n}\| \le S_{i,n}\cdot\|x^*_i - f(x^*_i)\| \overset{l.2.4.1)}{\le} S_{i,n}\cdot\|x^*-f(x^*)\|.$$

Together with lemma 2.4.2) we obtain

$$(13)\ \begin{cases} S_{i,\alpha(i,M)}\cdot(\gamma+\varepsilon) \overset{(8)}{\le} S_{i,\alpha(i,M)}\cdot\|x_i - f(x_i)\| \overset{(9,7)}{\le} \|x_i - x_{i+\alpha(i,M)}\| + \delta P_{i,\alpha(i,M)} \\ \le \|x_i - x^*_i\| + \|x^*_i - x^*_{i+\alpha(i,M)}\| + \|x^*_{i+\alpha(i,M)} - x_{i+\alpha(i,M)}\| + \delta P_{i,\alpha(i,M)} \\ \overset{2.4.2)}{\le} 2\|x-x^*\| + \|x^*_i - x^*_{i+\alpha(i,M)}\| + \delta P_{i,\alpha(i,M)} \\ \overset{(12)}{\le} 2\|x-x^*\| + S_{i,\alpha(i,M)}\cdot\|x^*-f(x^*)\| + \delta P_{i,\alpha(i,M)}. \end{cases}$$

Hence

$$(14)\ \begin{cases} 1+2\|x-x^*\| \overset{(2)}{\le} M\cdot\varepsilon \overset{(5)}{\le} \varepsilon S_{i,\alpha(i,M)} \\ \overset{(1)}{\le} S_{i,\alpha(i,M)}(\gamma+\varepsilon-\|x^*-f(x^*)\|) \overset{(13)}{\le} 2\|x-x^*\| + \delta P_{i,\alpha(i,M)} \\ \overset{(11)}{\le} 2\|x-x^*\| + \delta\exp(K\cdot S_{i,\alpha(i,M)}) \\ \overset{(5)}{\le} 2\|x-x^*\| + \delta\exp(K(M+1)) \overset{(3)}{<} 2\|x-x^*\|+1, \end{cases}$$

which is a contradiction. Therefore $\|x_i - f(x_i)\| < \gamma+\varepsilon$.
It remains to construct a function $h(x,f,K,\alpha,M)$ which is a bound $i\le h(x,f,K,\alpha,M)$ for i in (7):
Define

$$\tilde{\alpha}(i,M) := i+\alpha(i,M)$$

and the m-times iteration $\widehat{\alpha}$ of $\lambda i.\tilde{\alpha}(i,M)$

$$\widehat{\alpha}(0,M) := \tilde{\alpha}(0,M) \text{ and } \widehat{\alpha}(m+1,M) := \tilde{\alpha}(\widehat{\alpha}(m,M),M).$$

It is clear that

$$(15)\ \forall i(\widehat{\alpha}(i,M)\le\widehat{\alpha}(i+1,M)).$$

Claim:

$$\exists i\le \left\lceil \frac{\|x-f(x)\|}{\delta}\right\rceil \dot{-} 1(\|x_{\widehat{\alpha}(i,M)} - f(x_{\widehat{\alpha}(i,M)})\| - \|x_{\widehat{\alpha}(i+1,M)} - f(x_{\widehat{\alpha}(i+1,M)}\| \le \delta).$$

Proof of Claim: Let $j := \left\lceil \frac{\|x-f(x)\|}{\delta} \right\rceil \dot{-} 1$ and suppose the claim is false. Then

$$\forall i \leq j(\|x_{\widehat{\alpha}(i,M)} - f(x_{\widehat{\alpha}(i,M)})\| - \|x_{\widehat{\alpha}(i+1,M)} - f(x_{\widehat{\alpha}(i+1,M)}\| > \delta).$$

Since by lemma 2.4.1) the sequence $(\|x_i - f(x_i)\|)_{i\in\mathbb{N}}$ is decreasing and – by (15) – $\lambda i.\widehat{\alpha}(i, M)$ is monotone, we obtain

$$\begin{aligned} &\|x_{\widehat{\alpha}(0,M)} - f(x_{\widehat{\alpha}(0,M)})\| - \|x_{\widehat{\alpha}(j+1,M)} - f(x_{\widehat{\alpha}(j+1,M)})\| \\ &> \delta \cdot (j+1) \geq \|x - f(x)\|, \end{aligned}$$

which is a contradiction to the fact that $\forall n \in \mathbb{N}(\|x_n - f(x_n)\| \in [0, \|x - f(x)\|])$ and finishes the proof of the claim.
Using that

$$(16)\ \forall i(\widehat{\alpha}(i+1, M) = \widehat{\alpha}(i, M) + \alpha(\widehat{\alpha}(i, M), M),$$

the claim yields

$$(17) \begin{cases} \exists i \leq \left\lceil \frac{\|x-f(x)\|}{\delta} \right\rceil \dot{-} 1 \\ (\|x_{\widehat{\alpha}(i,M)} - f(x_{\widehat{\alpha}(i,M)})\| - \|x_{\widehat{\alpha}(i,M)+\alpha(\widehat{\alpha}(i,M),M)} - f(x_{\widehat{\alpha}(i,M)+\alpha(\widehat{\alpha}(i,M),M)}\| \leq \delta). \end{cases}$$

Hence – using again the monotonicity of $\lambda i.\widehat{\alpha}(i, M)$ – a bound for i in $(*)$ is given by $\widehat{\alpha}(\lceil(\|x - f(x)\|/\delta)\rceil \dot{-} 1, M)$. Since we can put $\delta := \frac{1}{2\exp(K(M+1))}$ we obtain

$$h(x, f, K, \alpha, M) := \widehat{\alpha}(\lceil 2\|x - f(x)\| \exp(K(M + 1))\rceil \dot{-} 1, M)$$

with $M \in \mathbb{N}$ such that

$$M \geq \frac{1 + 2\|x - x^*\|}{\varepsilon}$$

as bound for an i such that

$$\|x_i - f(x_i)\| < \gamma + \varepsilon$$

and therefore (using again lemma 2.4.1)

$$\forall i \geq h(x, f, K, \alpha, M)(\|x_i - f(x_i)\| < \gamma + \varepsilon).$$

We now show that we can replace (5) by the more flexible requirement[13]

$$(5)'\ \forall i, n \in \mathbb{N}(\alpha(i, n) \leq \alpha(i + 1, n) \wedge n \leq S_{i,\alpha(i,n)}).$$

[13]Note that the first conjunct can always be achieved without violating the second one by using $\alpha_+(i, n) := \max\limits_{j\leq i}(\alpha(j, n))$.

Assume that α satisfies $(5)'$. Define

$$\alpha^*(i,n) := \min m \in \mathbb{N}[n \leq \sum_{s=i}^{i+m-1} \lambda_s].$$

Then

$$\forall i, n \in \mathbb{N}(n \leq S_{i,\alpha^*(i,n)} \leq n+1),$$

since $\lambda_s \leq 1$. Hence by the logical analysis carried out so far we obtain the bound $h(\varepsilon, x, x^*, f, K, \alpha^*)$. In this bound, α^* can be replaced by α since

$$h(\varepsilon, x, x^*, f, K, \alpha^*) \leq h(\varepsilon, x, x^*, f, K, \alpha),$$

which is a consequence of

$$\forall i, n \in \mathbb{N}(\widehat{\alpha^*}(i,n) \leq \widehat{\alpha}(i,n)),$$

which can be proved by an easy induction on i. □

Corollary 2.8
Let $(X, \|\cdot\|)$ be a normed linear space, $C \subseteq X$ a non-empty convex subset with bounded (positive) diameter $d(C) < \infty$ and $f : C \to C$ a nonexpansive mapping. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $[0,1]$ which is divergent in sum and satisfies

$$\forall k \in \mathbb{N}(\lambda_k \leq 1 - \frac{1}{K})$$

for some $K \in \mathbb{N}$.
Let $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be such that

$$\forall i, n \in \mathbb{N}(\alpha(i,n) \leq \alpha(i+1,n)) \text{ and}$$

$$\forall i, n \in \mathbb{N}(n \leq \sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s).$$

Let $(x_n)_{n\in\mathbb{N}}$ be the Krasnoselski-Mann iteration

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n f(x_n), \; x_0 := x$$

starting from $x \in C$. Then the following holds

$$\forall x \in C \forall \varepsilon > 0 \forall n \geq h(\varepsilon, d(C), K, \alpha)(\|x_n - f(x_n)\| \leq \varepsilon),$$

where

$$h(\varepsilon, d(C), K, \alpha) := \widehat{\alpha}(\lceil 2d(C) \cdot \exp(K(M+1)) \rceil - 1, M), \text{ with } M := \left\lceil \frac{1+2d(C)}{\varepsilon} \right\rceil \text{ and}$$

$$\widehat{\alpha}(0, M) := \tilde{\alpha}(0, M), \ \widehat{\alpha}(m+1, M) := \tilde{\alpha}(\widehat{\alpha}(m, M), M) \text{ with}$$

$$\tilde{\alpha}(m, M) := m + \alpha(m, M) \quad (m \in \mathbb{N})$$

(Instead of $M, d(C)$ we may use any upper bounds $\mathbb{Q}_+^* \ni d_C \geq d(C)$ and $\mathbb{N} \ni \tilde{M} \geq \frac{1+2d_C}{\varepsilon}$).

Proof: The corollary follows from theorem 2.7 and lemma 2.3 by noticing that $\|x - f(x)\|, \|x - x^*\| \leq d(C)$. □

Remark 2.9 *By renorming the space with the factor $1/d(C)$ one can improve the $d(C)$-dependency of the bound above to $h(\varepsilon/d(C), 1, K, \alpha)$.*

Corollary 2.10 *Let $d, \varepsilon > 0$, $K \in \mathbb{N}$ and $\beta : \mathbb{N} \to \mathbb{N}$ an arbitrary function. Then there exists an $n \in \mathbb{N}$ such that for any normed space X, any convex set $C \subseteq X$ such that $d(C) \leq d$, any nonexpansive function $f : C \to C$, any sequence $\lambda_k \in [0, 1 - \frac{1}{K}]$ satisfying $n \leq \sum_{s=0}^{\beta(n)} \lambda_s$ (for all $n \in \mathbb{N}$) and any starting point $x_0 \in C$ of the corresponding Krasnoselski-Mann iteration the following holds*

$$\forall m \geq n(\|x_m - f(x_m)\| < \varepsilon).$$

Proof: Follows immediately from corollary 2.8 noticing that if $n \leq \sum_{s=0}^{\beta(n)} \lambda_s$, then also $n \leq \sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s$, where $\alpha(i, n) := \beta(n+i) - i + 1$. But this implies $n \leq \sum_{s=i}^{i+\alpha_+(i,n)-1} \lambda_s$, where $\alpha_+(i, n) := \max_{j \leq i}(\alpha(j, n))$ satisfies $\alpha_+(i, n) \leq \alpha_+(i+1, n)$. □

Corollary 2.11 *Let $(X, \|\cdot\|)$, C, f be as in corollary 2.8, $k \in \mathbb{N}, k \geq 2$ and $\lambda_n \in [\frac{1}{k}, 1 - \frac{1}{k}]$ for all $n \in \mathbb{N}$. Consider the Krasnoselski-Mann iteration $x_{n+1} := (1 - \lambda_n)x_n + \lambda_n f(x_n)$ starting from $x_0 := x \in C$. Then the following holds:*

$$\forall x \in C \forall \varepsilon > 0 \forall n \geq g(\varepsilon, d(C))(\|x_n - f(x_n)\| \leq \varepsilon),$$

where

$$g(\varepsilon, d(C)) := kM \cdot \lceil 2d(C) \exp(k(M+1)) \rceil \text{ with } M := \left\lceil \frac{1 + 2d(C)}{\varepsilon} \right\rceil.$$

Proof: We can put in corollary 2.8 $\alpha(i, M) := kM$. One easily proves that $\widehat{\alpha}(i, M) := k(i+1)M$. The corollary now follows from corollary 2.8. □

3 Summary of the results

For the first time we obtain explicit bounds for Ishikawa's result on the asymptotic behaviour of the general Krasnoselski-Mann iteration in arbitrary normed spaces X and for bounded sets C (corollary 2.8). Moreover, our bounds are uniform in the sense that they only depend on the error ε and an upper bound d_C of the diameter of C (and some data from the sequence of scalars λ_k used in defining the iteration) but not on the nonexpansive function f, the starting point $x_0 \in C$ of the iteration or other C-data. Only the non-effective existence of a bound independent of f and x_0 was known before (see [8] where a non-trivial functional theoretic embedding is used to obtain this uniformity after $\|x_k - f(x_k)\| \to 0$ has been established by the proof we are analysing)[14]. In fact, [16] explicitly mentions the non-effectivity of all these results and states that 'it seems unlikely that such estimates would be easy to obtain in general setting' (p.191) and therefore only studies the special 'tractable' (p.191) case of uniformly convex spaces due to Krasnoselski. Not even the ineffective existence of bounds which (like our result in 2.10) depend on C only via d_C was known so far (for general Krasnoselski-Mann iterations) and in fact still in [9] (p.101) conjectured as 'unlikely' to be true (not that the proof of $\|x_k - f(x_k)\| \to 0$ given in [8] by the same authors does yield such a bound by logical analysis!). Only in the special case of $\lambda_k := \lambda \in (0,1)$ being constant, a uniform (and in fact optimal quadratic) bound was recently discovered using computer aided proofs (see [1], where again the non-effectivity of all known proofs of the full Ishikawa result is stressed) and only for $\lambda_k := \frac{1}{2}$ a classically proved result of that type has been obtained subsequently (see [3]). This result, of course, is numerically better than our exponential bound in corollary 2.11 when specialised to $\lambda = \frac{1}{2}$. However, as the authors concede, their extremely complicated method does not extend to the case of non-constant sequences (λ_k). Our result in theorem 2.7 on the general case of unbounded C (as treated in [4]) is apparently all new.

4 Final comments and open problems

1) Recently, Kirk ([15]) obtained a new proof of the uniform (w.r.t. x_0 and f) Ishikawa result for the special case $\lambda_k = \lambda$ (again using a functional theoretic embedding) even in the more general setting of so-called directionally nonexansive mappings. It would be interesting to see what quantitative results a logical analysis of that proof would provide.

2) The results of Ishikawa and Borwein-Reich-Shafrir even hold in the more general setting of hyperbolic spaces in the sense of [10] (see e.g. [4],[27]). We expect that the quantitative analysis carried out in the present paper extends to this setting in a suitable form.

3) In [22] we have shown that the rate of convergence of the Krasnoselski iteration towards

[14] See [7] for a recent interesting application of this uniformity.

a fixed point (in the compact case) cannot be computed uniformly in the nonexpansive function f. This phenomenon, which already appears in most trivial cases and for the unit interval, is due to the non-uniqueness of the fixed point. In [17] we showed that by contrast one can compute unique solutions of functional equations in rather general settings by extracting certain effective data ('strong uniqueness') from the uniqueness proof. In this connection the recent results in [28] on cases where the existence of a unique fixed point of certain nonexpansive operators is guaranteed are of interest to apply this logical methodology to in order to possibly get computable bounds on the rate of convergence of the iteration and not just on the asymptotic regularity.

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