A proof-theoretic bound extraction theorem for CAT(κ)-spaces

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Abstract

Starting in 2005, general logical metatheorems have been developed that guarantee the extractability of uniform effective bounds from large classes of proofs of theorems that involve abstract metric structures X. In this paper we adapt this to the class of CAT(κ)-spaces X for $\kappa > 0$ and establish a new metatheorem that explains specific bound extractions that recently have been achieved in this context as instances of a general logical phenomenon.

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1 Introduction

Beginning in 2005, general logical metatheorems have been developed that guarantee the extractability of explicit effective and highly uniform bounds from large classes of proofs in nonlinear analysis that work in the setting of abstract classes of metric spaces that are not assumed to be separable (see [10, 6, 11, 7]). Whereas in the separable context (studied already in [8]) uniformity from input data in general can only be expected to hold in the case of compactness, the abstract setting for sufficiently uniform classes of structures makes this possible as long as metrical bounds are imposed. Metric and normed structures to which this logic-based proof-theoretic approach has been adapted so far are: metric spaces, W-hyperbolic spaces, CAT(0)-spaces, uniformly convex W-hyperbolic spaces, δ-hyperbolic spaces, $\mathbb{R}$-trees, normed spaces, uniformly convex normed and Hilbert spaces, metric completions of these spaces, Banach lattices, abstract L^p- and $C(K)$-spaces and others. The logic-based approach towards bound extractions from given proofs, also called 'proof mining', has resulted in numerous new results obtained for theorems in nonlinear analysis that are formulated in the context of such spaces (see [13] for a survey and for references).

In recent years, the class of CAT(κ)-spaces for $\kappa > 0$ has received particular attention in fixed point and ergodic theory as well as in convex optimization (see e.g. [1, 5, 14, 15]). These spaces are defined via comparison properties for geodesic triangles and represent a generalization of smooth Riemannian manifolds of sectional curvature bounded above by κ (for a detailed introduction to CAT(κ)-spaces we refer to [3]). It has turned out that 'proof-mining'-results that previously had been obtained only for CAT(0)-spaces could be generalized to the CAT(κ)-setting (see [14] for a particularly striking instance of this and [12] for an application of proof mining for convex feasibility problems in CAT(κ)-spaces). This raises the natural question on whether these findings can be explained in general logical terms, i.e. whether one can formulate general logical metatheorems on bound extractions also for the class of CAT(κ)-spaces. In this note we give a positive answer to this question.

2 Main Results

In this paper, $\mathbb{N}$ always denotes the set $\{0, 1, 2, \ldots\}$. In the following, we make free use of the representation of real numbers (given as fast converging Cauchy sequences of rationals) by names in $\mathbb{N}^{\mathbb{N}}$ from [11] by which every function in $\mathbb{N}^{\mathbb{N}}$ represents a unique real number while every real number has many different names in $\mathbb{N}^{\mathbb{N}}$ and so actually corresponds to an equivalence class w.r.t. a Π^0_1-equivalence relation $f =_{\mathbb{R}} g$ on $\mathbb{N}^{\mathbb{N}}$. Noneffectively, though, one can select a unique 'canonical' representing function $(x)_\circ \in \mathbb{N}^{\mathbb{N}}$ of $x \in [0, \infty)$ by

$$(x)_\circ(n) := j(2k_0, 2^{n+1} - 1), \text{where } k_0 := \max k \left[\frac{k}{2^{n+1}} \leq x\right].$$

Here $j : \mathbb{N}^2 \to \mathbb{N}$ denotes the standard Cantor pairing function.

Remark 2.1. *Modulo the encoding of rational numbers by natural numbers as used in* [11], $(x)_\circ$ *is the Cauchy sequence whose n-th element is the largest dyadic rational number of the form $k/2^{n+1}$ that is $\leq x$.*

To represent quantification over $[0, 1]$ we use the operation $\tilde{(\cdot)} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, f \mapsto \tilde{f}$ from Definition 4.24 in [11] with the properties (provably in weak fragments of arithmetic in all finite types, see Lemma 4.25 in [11])

$$\begin{aligned}
&(i)\ 0_{\mathbb{R}} \leq_{\mathbb{R}} \tilde{x} \leq_{\mathbb{R}} 1_{\mathbb{R}},\\
&(ii)\ 0_{\mathbb{R}} \leq_{\mathbb{R}} x \leq_{\mathbb{R}} 1_{\mathbb{R}} \to \tilde{x} =_{\mathbb{R}} x,\\
&(iii)\ x >_{\mathbb{R}} 1_{\mathbb{R}} \to \tilde{x} =_{\mathbb{R}} 1_{\mathbb{R}},\ x <_{\mathbb{R}} 0_{\mathbb{R}} \to \tilde{x} =_{\mathbb{R}} 0_{\mathbb{R}}.
\end{aligned}$$

Metric spaces (X, d) are represented as quotients of pseudometric spaces where the latter are given by a constant d_X of type $X \times X \to \mathbb{N}^{\mathbb{N}}$ satisfying the axioms

$$\begin{aligned}
&(\text{d1})\ \forall x^X\, (d_X(x, x) =_{\mathbb{R}} 0_{\mathbb{R}}),\\
&(\text{d2})\ \forall x^X, y^X\, (d_X(x, y) =_{\mathbb{R}} d_X(y, x)),\\
&(\text{d3})\ \forall x^X, y^X, z^X\, (d_X(x, z) \leq_{\mathbb{R}} d_X(x, y) +_{\mathbb{R}} d_X(y, z)).
\end{aligned}$$

In order to axiomatize CAT(κ)-spaces (X,d) (for fixed $\kappa > 0$) with $diam(X) \le \pi/(2\sqrt{\kappa})$ we first add a constant W_X of type $X \times X \times \mathbb{N}^{\mathbb{N}} \to X$ representing a convexity operator $W : X \times X \times [0,1] \to X$ satisfying the axioms

$$\begin{aligned}
&(\text{W1})\ \forall x^X, y^X, z^X\, \forall \lambda^{\mathbb{N}^{\mathbb{N}}} \big(d_X(z, W_X(x,y,\lambda)) \le_{\mathbb{R}} (1_{\mathbb{R}} -_{\mathbb{R}} \tilde{\lambda}) \cdot_{\mathbb{R}} d_X(z,x) +_{\mathbb{R}} \tilde{\lambda} \cdot_{\mathbb{R}} d_X(z,y)\big),\\
&(\text{W2})\ \forall x^X, y^X\, \forall \lambda_1^{\mathbb{N}^{\mathbb{N}}}, \lambda_2^{\mathbb{N}^{\mathbb{N}}} \big(d_X(W_X(x,y,\lambda_1), W_X(x,y,\lambda_2)) =_{\mathbb{R}} |\tilde{\lambda}_1 -_{\mathbb{R}} \tilde{\lambda}_2|_{\mathbb{R}} \cdot_{\mathbb{R}} d_X(x,y)\big),\\
&(\text{W3})\ \forall x^X, y^X\, \forall \lambda^{\mathbb{N}^{\mathbb{N}}} \big(W_X(x,y,\lambda) =_X W_X(y,x,1_{\mathbb{R}} -_{\mathbb{R}} \lambda)\big)
\end{aligned}$$

(i.e. (X,d,W) is a space of hyperbolic type, see [10]) and - instead of (W4) used in [10] to define the class of (W)-hyperbolic spaces - we now have the axiom

$$(\text{W5}):\ \forall x^X, y^X, z^X\, \forall \lambda^{\mathbb{N}^{\mathbb{N}}}\ (d_X(W_X(x,z,\lambda), W_X(y,z,\lambda)) \le d_X(x,y)),$$

which expresses that $d(W(x,z,\lambda), W(y,z,\lambda)) \le d(x,y)$ for all $\lambda \in [0,1]$ and $x,y,z \in X$.

Remark 2.2. *The reason why in the axioms above we can write λ instead of $\tilde{\lambda}$ is that* (W2) *implies that $W_X(x,y,\lambda) =_X W_X(x,y,\tilde{\lambda})$ since, by the properties $(i),(ii)$ above, $\tilde{\tilde{\lambda}} =_{\mathbb{R}} \tilde{\lambda}$. So the intended meaning of $W_X(x,y,\lambda)$ is (given a convexity operator W): $W_X(x,y,\lambda)$ is $W(x,y,r)$ for the unique $r \in [0,1]$ that is represented by $\tilde{\lambda}$.*
In (W3) *we do not have to write $1 - \tilde{\lambda}$ instead of $1 - \lambda$ since, by the properties $(ii),(iii)$ above, $\widetilde{1-\lambda} =_{\mathbb{R}} 1 - \tilde{\lambda}$ and so*

$$W_X(x,y,1-\lambda) =_X W_X(x,y,\widetilde{1-\lambda}) =_X W_X(x,y,1-\tilde{\lambda}),$$

where the second equality follows from (W2) *since - by $(ii),(iii)$ - $\lambda_1 =_{\mathbb{R}} \lambda_2 \to \tilde{\lambda}_1 =_{\mathbb{R}} \tilde{\lambda}_2$.*

Next we add new constants c_κ of type $\mathbb{N} \to \mathbb{N}$ and $\overline{N}_\kappa$ of type $\mathbb{N}$ together with the following axioms (here we write for better readability the real number κ represented by c_κ rather than c_κ itself; note that all the operations used such as $\sqrt{\cdot}, \pi, \sin, \cos$ and the field operations on $\mathbb{R}$ can be explicitly written on the level of representatives of the respective reals and c_κ by primitive recursive terms, see [9]):

$$(\kappa 1)\ \kappa \ge_{\mathbb{R}} \frac{1}{\overline{N}_\kappa + 1},$$

i.e. $\overline{N}_\kappa$ is a witness for the strict positivity of $\kappa > 0$,

$$(\kappa 2)\ \forall x^X, y^X \big(d_X(x,y) \le_{\mathbb{R}} \frac{\pi}{2\sqrt{\kappa}}\big),$$

which expresses that $diam(X) \le \pi/(2\sqrt{\kappa})$,

$$(\kappa 3) \begin{cases} \forall a^X, b^X, p^X, q^X\, \forall n^{\mathbb{N}} \Big(d_X(a,p), d_X(b,q) >_{\mathbb{R}} \frac{1}{n+1} \to \\ \frac{\cos(\sqrt{\kappa} d_X(p,q)) + \cos(\sqrt{\kappa} d_X(a,p)) \cos(\sqrt{\kappa} d_X(b,q))}{\sin(\sqrt{\kappa} d_X(a,p)) \sin(\sqrt{\kappa} d_X(b,q))} \\ \qquad - \frac{\big(\cos(\sqrt{\kappa} d_X(a,p)) + \cos(\sqrt{\kappa} d_X(b,p))\big)\big(\cos(\sqrt{\kappa} d_X(b,q)) + \cos(\sqrt{\kappa} d_X(a,q))\big)}{\big(1 + \cos(\sqrt{\kappa} d_X(a,b)\big) \sin(\sqrt{\kappa} d_X(a,p)) \sin(\sqrt{\kappa} d_X(b,q))} \le_{\mathbb{R}} 1\Big), \end{cases}$$

which expresses that X satisfies the 'upper four point κ-quadrilateral cos-condition $\cos q_\kappa$' (see [2]).

Let us briefly notice that we can indeed define the quotients used in the axioms $(\kappa 2), (\kappa 3)$ by simple primitive recursive terms in the data: one can easily define a primitive recursive term $t : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $t(x^{\mathbb{N}^\mathbb{N}}, n)$ represents the reciprocal $1/r_x$ of the real number r_x represented by x provided that $r_x \geq \frac{1}{n+1}$. Such a lower bound on the denominator '$2\sqrt{\kappa}$' occurring in $(\kappa 2)$ can easily be obtained from $\overline{N}_\kappa$ in axiom $(\kappa 1)$, e.g. we may take $n := \lceil \overline{N}_\kappa / 2 \rceil$. For $(\kappa 3)$ we have to additionally observe that by $(\kappa 1), (\kappa 2)$ the function sin is only applied to arguments $x \in [\sqrt{\kappa}/(n+1), \pi/2] \subset [\sqrt{\kappa}/(n+1), 2]$ and that $\sin x \geq x/3$ for such x so that $\sin x \geq \sqrt{\kappa}/(3(n+1))$.

Definition 2.3. *We define the theory $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$ as the theory that results if we add to the theory $\mathcal{A}^\omega[X, d]$ from* [10] *(p.99) constants W_X, c_κ and $\overline{N}_\kappa$ of type $X \times X \times \mathbb{N}^\mathbb{N} \to X, \mathbb{N} \to \mathbb{N}$ and $\mathbb{N}$ respectively together with the axioms* (W1),(W2),(W3),(W5), $(\kappa 1)$, $(\kappa 2)$ *and* $(\kappa 3)$.

Remark 2.4. *The extra constant b_X of type $\mathbb{N}$ together with the axiom*

$$(iv)(4) \quad \forall x^X, y^X \, (d_X(x, y) \leq_\mathbb{R} (b_X)_\mathbb{R})$$

used in [10] *to express that (X, d) is bounded by b_X is now actually redundant (and so officially dropped from $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$) since, by $(\kappa 1), (\kappa 2)$, b_X can be defined in terms of $\overline{N}_\kappa$.*

Proposition 2.5. *Let (X, d) be a metric space, $W : X \times X \times [0, 1] \to X$ be a mapping, $\kappa \in (0, \infty)$ and $N_\kappa \in \mathbb{N}$. The full set-theoretic type structure $\mathcal{S}^{\omega, X}$ is a model of $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$ (in the sense of the interpretation as defined in* [11] *extended - for $\kappa > 0$ and $N_\kappa \in \mathbb{N}$ - by the interpretation of $[c_\kappa]_{\mathcal{S}^{\omega,X}} := (\kappa)_\circ$ and $[\overline{N}_\kappa]_{\mathcal{S}^{\omega,X}} := N_\kappa$) iff (X, d) is a CAT(κ)-space with $\kappa \geq 1/(N_\kappa + 1)$ and $diam(X) \leq \pi/(2\sqrt{\kappa})$ and W is defined in terms of the unique geodesic segment in X joining given points x, y and $r_\lambda \in [0, 1]$.*

Proof: '$\Rightarrow$': Let $(X, d), W, \kappa, N_\kappa$ be such that $\mathcal{S}^{\omega, X}$ satisfies the axioms listed above. By (W1),(W2), clearly (X, d) is geodesically connected with

$$\gamma : [0, d(x, y)] \to X, \; \gamma(\alpha) := W(x, y, \alpha/d(x, y))$$

for $x \neq y$. Moreover, the axioms $(\kappa 1), (\kappa 2)$ imply that $\kappa \geq 1/(N_\kappa + 1)$ and $diam(X) \leq \pi/(2\sqrt{\kappa})$. $(\kappa 3)$ implies that (X, d) satisfies the 'upper four point $\cos q_\kappa$ condition'. Hence by Theorem 1.1 in [2], (X, d) is a CAT(κ)-space.
'$\Leftarrow$': Let (X, d) be a CAT(κ)-space with $\kappa \geq 1/(N_\kappa + 1)$ and $diam(X) \leq \pi/(2\sqrt{\kappa})$ and let $W(x, y, \lambda) := \gamma(\lambda \cdot d(x, y))$ for the unique geodesic $\gamma : [0, d(x, y)] \to X$ joining x, y. Then the axioms $(\kappa 1), (\kappa 2)$ are satisfied (with the interpretation of the constant c_κ and $\overline{N}_\kappa$ as specified in the proposition) and (X, d) is uniquely geodesic which implies that the W_X defined in terms of this unique geodesic satisfies (W2),(W3). Moreover, since for any $x_0 \in X$

the function $x \mapsto d(x, x_0)$ is convex (see Ex.2.3 in [3], p.176), (W1) also holds. Again by Theorem 1.1 in [2] one has that (X, d) satisfies the 'upper four point cosq_κ condition' so that axiom (κ3) holds. (W5) follows from Lemma 4.1 in [14] (see also [15]). $\square$

One crucial restriction for the logical metatheorems referred to in the introduction to hold is that instead of a full extensionality axiom, which for the type X would be

$$\forall f^{X \to X}\, \forall x^X, y^X\, (x =_X y \to f(x) =_X f(y)),$$

one only has a rule which allows one to infer that $f(t) =_X f(s)$ from a proof that $t =_X s$. Since $x =_X y$ is defined as $d_X(x, y) =_{\mathbb{R}} 0$, the very conclusion of such a metatheorem when applied to the extensionality of f would imply a uniform quantitative form of that extensionality which is nothing else but the uniform continuity of f. Note that in the model-theoretic approach to metric structures as in continuous or positive bounded logic, the uniform continuity of the functions in question is a basic assumption (see, however, the recent paper [4] which relaxes this) while this is not necessary in the proof-theoretic context (see [11, 13] for extensive discussions of this issue).

In our current situation, we do have sufficient uniform continuity of W_X as a consequence of its axioms to be able to derive full extensionality:

Proposition 2.6. *$\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$ proves the extensionality of W_X, i.e.*

$$\forall \lambda_1^1, \lambda_2^1, x_1^X, x_2^X, y_1^X, y_2^X\, (\lambda_1 =_{\mathbb{R}} \lambda_2 \wedge x_1 =_X x_2 \wedge y_1 =_X y_2 \to W_X(x_1, y_1, \lambda_1) =_X W(x_2, y_2, \lambda_2)).$$

Proof: By Lemma 4.25.6) in [11], $\lambda_1 =_{\mathbb{R}} \lambda_2$ implies that $\tilde{\lambda}_1 =_{\mathbb{R}} \tilde{\lambda}_2$ and so by (W2)

$$W_X(x_1, y_1, \lambda_1) =_X W_X(x_1, y_1, \lambda_2).$$

By (W5), $x_1 =_X x_2$ implies

$$W_X(x_1, y_1, \lambda_2) =_X W_X(x_2, y_1, \lambda_2).$$

Using (W3) and again (W5), $y_1 =_X y_2$ yields

$$W_X(x_2, y_1, \lambda_2) =_X W_X(x_2, y_2, \lambda_2).$$

The transitivity of $=_X$ now implies that

$$W_X(x_1, y_1, \lambda_1) =_X W_X(x_2, y_2, \lambda_2).$$

$\square$

Proposition 2.7. *$[c_\kappa]_{\mathcal{S}^{\omega}, X} = (\kappa)_\circ$ is majorized by $c_\kappa^*(n) := j(b \cdot 2^{n+2}, 2^{n+1} - 1)$, where $b \in \mathbb{N}$ is such that $b \geq \kappa$.*

Proof: See Lemma 17.8 in [11]. $\square$

Remark 2.8. *By rescaling one usually can reduce things to the case of $\kappa = 1$ in which we may simply interpret c_κ by $(1_{\mathbb{R}})_\circ$ and and $\overline{N}_\kappa$ by 0.*

Definition 2.9 ([10])**.** We say that a finite type ρ over the base types $\mathbb{N}$ and X has degree 1 if $\rho = \mathbb{N} \to \ldots \to \mathbb{N}$ (including $\rho = \mathbb{N}$). ρ has degree $(\mathbb{N}, X)$ if $\rho = \mathbb{N} \to \ldots \to \mathbb{N} \to X$ (including $\rho = X$). A type ρ has degree $(1, X)$ if it has the form $\tau_1 \to \ldots \to \tau_k \to X$ (including $\rho = X$), where τ_i has degree 1 or $(\mathbb{N}, X)$.

Definition 2.10 ([10])**.** A formula F is called $\forall$-formula (resp. $\exists$-formula) if it has the form $F \equiv \forall \underline{a}^{\underline{\sigma}} F_{qf}(\underline{a})$ (resp. $F \equiv \exists \underline{a}^{\underline{\sigma}} F_{qf}(\underline{a})$) where F_{qf} does not contain any quantifier and the types in $\underline{\sigma}$ are of degree 1 or $(1, X)$.

One can now easily adapt the proof of the main logical metatheorem for bounded metric, W-hyperbolic and CAT(0)-spaces from Theorem 3.7 in [10] to the case of CAT(κ)-spaces for $\kappa > 0$:

Theorem 2.11. Let σ, ρ be types of degree 1 and τ be a type of degree $(1, X)$.
Let $s^{\sigma \to \rho}$ be a closed term of $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$ and $B_\forall(x^\sigma, y^\rho, z^\tau, u^{\mathbb{N}})$ $(C_\exists(x^\sigma, y^\rho, z^\tau, v^{\mathbb{N}}))$ be a $\forall$-formula containing only x, y, z, u free (resp. a $\exists$-formula containing only x, y, z, v free).
If

$$\forall x^\sigma \, \forall y \leq_\rho s(x) \, \forall z^\tau \left(\forall u^{\mathbb{N}} B_\forall(x, y, z, u) \to \exists v^{\mathbb{N}} C_\exists(x, y, z, v) \right)$$

is provable in $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$, then one can extract a computable functional
$\Phi : \mathcal{S}_\sigma \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathcal{S}_\sigma$[1] all $b, N \in \mathbb{N}$

$$\forall y \leq_\rho s(x) \forall z^\tau \left[\forall u \leq \Phi(x, b, N) \, B_\forall(x, y, z, u) \to \exists v \leq \Phi(x, b, N) \, C_\exists(x, y, z, v) \right]$$

holds in any (non-empty) CAT(κ)-space (X, d) with $1/(N+1) \leq \kappa \leq b$ and $diam(X) \leq \pi/(2\sqrt{\kappa})$ (with $\overline{N}_\kappa$ being interpreted by N).
The computational complexity of Φ can be estimated in terms of the strength of the $\mathcal{A}^\omega$-principle instances actually used in the proof (see Remark 2.12 below).
Instead of single variables x, y, z, u, v we may also have finite tuples of variables $\underline{x}, \underline{y}, \underline{z}, \underline{u}, \underline{v}$ as long as the elements of the respective tuples satisfy the same type restrictions as x, y, z, u, v.
Moreover, instead of a single premise of the form '$\forall u^{\mathbb{N}} B_\forall(x, y, z, u)$' we may have a finite conjunction of such premises.

Proof: We only have to augment the proof from Theorem 3.7 in [10] by the following observations (together with Remark 2.4):

1. The new axioms $(W5), (\kappa 1), (\kappa 2)$ and $(\kappa 3)$ are all (logically equivalent to) purely universal sentences, where the quantified variables are of the types $\mathbb{N}, \mathbb{N} \to \mathbb{N}$ or X for which the full set-theoretic model $\mathcal{S}^{\omega, X}$ and the model of strongly majorizable functionals $\mathcal{M}^{\omega, X}$ coincide. For $(\kappa 3)$ note that the premise '$\ldots >_{\mathbb{R}} 1/(n+1)$' is purely existential so that the whole expression '$\big(\ldots \big)$' prenexes into a purely universal formula.

2. $[c_\kappa]_{\mathcal{M}^{\omega,X}} := [c_\kappa]_{\mathcal{S}^{\omega,X}} := (\kappa)_\circ$ is majorized by the simple function c_κ^* whose definition only uses b. $[\overline{N}_\kappa]_{\mathcal{M}^{\omega,X}} := [\overline{N}_\kappa]_{\mathcal{S}^{\omega,X}} := N$ is trivially majorized by itself.

[1]Note that $\mathcal{S}_0 = \mathbb{N}$ and $\mathcal{S}_\sigma$ is the set of all functions $\mathbb{N}^k \to \mathbb{N}$ for σ being the type of k-ary number theoretic functions.

$\square$

Remark 2.12. *1. The proof of Theorem 2.11 actually provides an extraction algorithm for Φ. The functional Φ can always be defined in the calculus T+BR of so-called bar recursive functionals, where T refers to Gödel's primitive recursive functionals T and BR refers to Spector's schema of bar recursion. However, for concrete proofs usually only small fragments of $\mathcal{A}^\omega[X, d, W, CAT(\kappa)]$ (corresponding to fragments of $\mathcal{A}^\omega$) will be needed to formalize the proof guaranteeing bounds of much lower complexity (see Remark 3.8 in* [10] *and the references given there as well as* [11]*).*

2. *It is well-known that any CAT(κ)-space is also a CAT(κ')-space for all $\kappa' \geq \kappa$ (see e.g.* [3][Theorem 1.12(1)]*). So to be CAT(κ) for a $\kappa > 0$ that is very close to* 0 *(resulting in a large bound $\overline{N}_\kappa$) is a better condition than being CAT(κ) for a κ that is larger while the fact that our bound will depend on $\overline{N}_\kappa$ does not seem to be in line with this. However, one has to note that the condition $diam(X) \leq \pi/(2\sqrt{\kappa})$ on the diameter of X becomes the more liberal the smaller κ is and a uniform bound extraction theorem in the generality of Theorem 2.11 does require (already in the CAT(0)-case) that X is bounded (see* [10][Theorem 3.7]*; for the unbounded case, treated in* [6][Theorem 4.10]*, one needs extra conditions on z to guarentee the majorizability of z) and the bound has to depend on an upper bound on $diam(X)$. So assume now that we have a B-bounded CAT(κ)-space X. Then X is also a CAT(κ')-space for $\kappa' := (\pi/2B)^2$ satisfying $diam(X) \leq \pi/(2\sqrt{\kappa'})$ provided that $\kappa \leq \kappa'$ and we can apply the extracted bound with $\overline{N}_{\kappa'}$ being any natural number such that $1/(\overline{N}_{\kappa'}+1) \leq (\pi/2B)^2$ no matter how small $\kappa > 0$ was (and with $b \geq \kappa'$). In the case where $\kappa \geq \kappa'$ we can take $\overline{N}_\kappa := \overline{N}_{\kappa'}$ in the bound and may use any $b \geq \kappa$.*

The most common definition of CAT(κ)-spaces is via an inequality for comparison triangles and so it would be beneficial for the purpose of mining proofs based on this property to have direct access to it (rather than having to go through the proof in [2] that it is implied by the upper four point cosq_κ condition). Let us consider one version of such a characterization (given in Proposition 1.7.(2) in [3], p.161): let $x_1, x_2, x_3 \in X$ and consider a comparison triangle $\Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in M^2_κ, i.e. $\overline{x_1}, \overline{x_2}, \overline{x_3} \in \mathbb{S}^2$ (here $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$) with

$$(+)\ d(x_i, x_j) = d_{M^2_\kappa}(\overline{x_i}, \overline{x_j}), \text{ where } d_{M^2_\kappa}(\overline{x_i}, \overline{x_j}) = \frac{1}{\sqrt{\kappa}} \arccos(\langle \overline{x_i}, \overline{x_j} \rangle),$$

for $i, j \in \{1, 2, 3\}$. Then

$$(++)\ \forall t \in [0,1]\ \big(d(x_1, (1-t)x_2 + tx_3) \leq d_{M^2_\kappa}(\overline{x_1}, (1-t)\overline{x_2} + t\overline{x_3})\big).$$

Unfortunately, due to the universal quantifier hidden in the premise $d(x_i, x_j) = d_{M^2_\kappa}(\overline{x_i}, \overline{x_j})$ this characterization '(+) $\to$ (++)' is not universal but prenexes into the form $\forall\exists$. So in order to bring it into a purely universal form we have to see that it in fact implies already a seemingly stronger quantitative form where $\forall\exists$-is realized by an explicit function (definable in our system). We will now show that this can be done in a highly uniform way: one can define a function $\delta : \mathbb{N} \to \mathbb{N}$ such that any $1/(\delta(k)+1)$-comparison triangle, i.e. (for $i, j \in \{1,2,3\}$)

$$(+)_k \quad \left|d(x_i, x_j) - d_{M^2_\kappa}(\overline{x_i}, \overline{x_j})\right| < \frac{1}{\delta(k)+1}$$

satisfies $(++)$ up to the error $1/(k+1)$, i.e.

$$(++)_k \quad \forall t \in [0,1] \left(d(x_1, (1-t)x_2 + tx_3) \le d_{M^2_\kappa}(\overline{x_1}, (1-t)\overline{x_2} + t\overline{x_3}) + \frac{1}{k+1}\right).$$

Now

$$\text{(W6)} \quad \forall x_1, x_2, x_3 \in X \, \forall \overline{x_1}, \overline{x_2}, \overline{x_3} \in \mathbb{S}^2 \, \forall k \in \mathbb{N} \, \left((+)_k \to (++)_k\right),$$

where $(1-t)x_2 + tx_3$ is to be understood as $W_X(x_2, x_3, t)$, is (equivalent to) a purely universal statement and so can be added simply as an axiom to our formal system. Here we wrote things for simplicity in normal mathematical laguage but (W6) can easily be formalized using W_X, d_X as before while quantification over $\mathbb{S}^2$ can be reduced to $\mathbb{R}^3$ (and hence in turn to quantification over triples of objects of type 1) without introducing the purely universal premise '$\langle x, x\rangle = 1$' by writing instead of '$\forall x \in \mathbb{S}^2 \ \Phi(x)$'

$$\forall x \in \mathbb{R}^3 \ (\|x\|_E > \frac{1}{2} \to \Phi(\widehat{x})),$$

where $\widehat{x} := x/\max\{1/2, ||x||_E\}$. With Φ also then $\Phi'(x) := \|x\|_E > \frac{1}{2} \to \Phi(\widehat{x})$ is (equivalent to) a purely universal formula since $>_\mathbb{R}$ is existential.

Clearly, the quantitative form (W6) immediately implies back the original characterization from [3]. The existence of such a uniform bound δ in fact in itself is an instance of the logical metatheorem on uniform bound extractions when applied to a proof of the characterization given in [3] from the different one due to [2] used further above.

Remark 2.13. *To have* $(W6)$ *- and hence the qualitative inequality - stated for the specific geodesic selected by* W *implies already (given the condition on the diameter being* $\le \pi/(2\sqrt{\kappa}) < \pi/\sqrt{\kappa}$*) that* X *is uniquely geodesic so that to state* $(W6)$ *w.r.t.* W *implies the seemingly stronger version for arbitrary geodesics: suppose* x, y *are joined by two geodesic segments and let* m_1 *and* m_2 *be the respective midpoints. Apply the comparison inequality for the triangles* $\Delta(x, m_1, m_2)$ *and* $\Delta(y, m_1, m_2)$ *(having a geodesic segment selected by* W *for each edge in these triangles). Then, if* m *is a midpoint of* m_1 *and* m_2 *one gets (if* $m_1 \neq m_2$*)*

$$d(x, m) \le d(\overline{x}, \overline{m}) < d(\overline{x}, \overline{m_1}) = d(x, m_1).$$

Applying the same argument in $\Delta(y, m_1, m_2)$ *gives*

$$d(x, y) \le d(x, m) + d(y, m) < d(x, y)$$

and hence a contradiction.

Definition 2.14. *Let (X, d) be a CAT(κ)-space with $\kappa > 0$ and diam$(X) \leq \pi/(2\sqrt{\kappa})$. Take $x_1, x_2, x_3 \in X$. Having $\delta > 0$, a δ-comparison triangle for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in M_κ^2 such that*

$$\left|d(x_i, x_j) - d_{M_\kappa^2}(\overline{x_i}, \overline{x_j})\right| \leq \frac{\delta}{\sqrt{\kappa}} \quad \text{for } i, j \in \{1, 2, 3\}.$$

Proposition 2.15. *In the setting of Definition 2.14, for every $\varepsilon \in (0, 1)$ there exists $\delta := \frac{\varepsilon^2}{108} \sin \frac{\varepsilon^2}{36}$ such that for every δ-comparison triangle $\Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ we have that*

$$\forall t \in [0, 1] \ \left(d(x_1, (1-t)x_2 + tx_3) \leq d_{M_\kappa^2}(\overline{x_1}, (1-t)\overline{x_2} + t\overline{x_3}) + \frac{\varepsilon}{\sqrt{\kappa}}\right).$$

Proof: We give the proof for $\kappa = 1$ (the general case follows by a simple rescaling). For $\Delta(x_1, x_2, x_3)$, let $\Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ and $\Delta(\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3})$ be a δ-comparison triangle and a comparison triangle, respectively. Fix $t \in [0, 1]$ and denote

$$\overline{a} = d_{\mathbb{S}^2}(\overline{x_1}, \overline{x_2}), \quad \overline{b} = d_{\mathbb{S}^2}(\overline{x_1}, \overline{x_3}), \quad \overline{c} = d_{\mathbb{S}^2}(\overline{x_2}, \overline{x_3}), \quad \overline{m} = d_{\mathbb{S}^2}(\overline{x_1}, (1-t)\overline{x_2} + t\overline{x_3}),$$

and

$$\widetilde{a} = d_{\mathbb{S}^2}(\widetilde{x_1}, \widetilde{x_2}), \quad \widetilde{b} = d_{\mathbb{S}^2}(\widetilde{x_1}, \widetilde{x_3}), \quad \widetilde{c} = d_{\mathbb{S}^2}(\widetilde{x_2}, \widetilde{x_3}), \quad \widetilde{m} = d_{\mathbb{S}^2}(\widetilde{x_1}, (1-t)\widetilde{x_2} + t\widetilde{x_3}).$$

Then $|\cos \overline{a} - \cos \widetilde{a}| \leq |\overline{a} - \widetilde{a}| \leq \delta$, $|\cos \overline{b} - \cos \widetilde{b}| \leq |\overline{b} - \widetilde{b}| \leq \delta$ and $|\cos \overline{c} - \cos \widetilde{c}| \leq |\overline{c} - \widetilde{c}| \leq \delta$. Note that if $\widetilde{m} \leq \overline{m}$, then $d(x_1, (1-t)x_2 + tx_3) \leq \widetilde{m} \leq \overline{m}$. Thus, we assume in the following that $\widetilde{m} > \overline{m}$, so $\cos \widetilde{m} < \cos \overline{m}$. Denote $\varepsilon' = \varepsilon^2/18$. Then $\delta = (\varepsilon'/6) \sin(\varepsilon'/2)$.
Suppose first that $\overline{c} \geq \varepsilon'/2$. By Lemma 3.1 in [2],

$$\cos \overline{m} = \frac{\sin((1-t)\overline{c})}{\sin \overline{c}} \cos \overline{a} + \frac{\sin(t\overline{c})}{\sin \overline{c}} \cos \overline{b} \tag{1}$$

and

$$\cos \widetilde{m} = \frac{\sin((1-t)\widetilde{c})}{\sin \widetilde{c}} \cos \widetilde{a} + \frac{\sin(t\widetilde{c})}{\sin \widetilde{c}} \cos \widetilde{b}. \tag{2}$$

The function $f : (0, \pi) \to \mathbb{R}$, $f(x) = \sin(tx)/\sin x$ is increasing. Since $\overline{c} \leq \widetilde{c} + \delta$, we obtain that

$$\frac{\sin(t\overline{c})}{\sin \overline{c}} \leq \frac{\sin(t(\widetilde{c} + \delta))}{\sin(\widetilde{c} + \delta)} = \frac{\sin(t\widetilde{c})\cos(t\delta) + \cos(t\widetilde{c})\sin(t\delta)}{\sin(\widetilde{c} + \delta)} < \frac{\sin(t\widetilde{c})}{\sin(\widetilde{c} + \delta)} + \frac{\delta}{\sin(\widetilde{c} + \delta)}.$$

Note that $\varepsilon'/2 \leq \overline{c} \leq \widetilde{c} + \delta \leq \pi/2 + \delta < \pi - \varepsilon'/2$, so $\dfrac{\delta}{\sin(\widetilde{c} + \delta)} < \dfrac{\delta}{\sin(\varepsilon'/2)} = \dfrac{\varepsilon'}{6}$. At the same time, $\sin(\widetilde{c} + \delta) \geq \cos \delta \sin \widetilde{c} \geq (1 - \delta) \sin \widetilde{c}$ and $1 - \delta \geq \sin(\varepsilon'/2)$, so

$$\begin{aligned} \frac{\sin(t\widetilde{c})}{\sin(\widetilde{c} + \delta')} &\leq \frac{\sin(t\widetilde{c})}{\sin \widetilde{c}} \frac{1}{1-\delta} = \frac{\sin(t\widetilde{c})}{\sin \widetilde{c}} \left(1 + \frac{\delta}{1-\delta}\right) \leq \frac{\sin(t\widetilde{c})}{\sin \widetilde{c}} + \frac{\delta}{1-\delta} \\ &\leq \frac{\sin(t\widetilde{c})}{\sin \widetilde{c}} + \frac{\delta}{\sin(\varepsilon'/2)} = \frac{\sin(t\widetilde{c})}{\sin \widetilde{c}} + \frac{\varepsilon'}{6}. \end{aligned}$$

This means that

$$\frac{\sin(t\overline{c})}{\sin\overline{c}} < \frac{\sin(t\widetilde{c})}{\sin\widetilde{c}} + \frac{\varepsilon'}{3}.$$

Thus, by (1) and (2),

$$\begin{aligned}\cos\overline{m} &< \left(\frac{\sin((1-t)\widetilde{c})}{\sin\widetilde{c}} + \frac{\varepsilon'}{3}\right)(\cos\widetilde{a} + \delta) + \left(\frac{\sin(t\widetilde{c})}{\sin\widetilde{c}} + \frac{\varepsilon'}{3}\right)(\cos\widetilde{b} + \delta) \\ &< \cos\widetilde{m} + 2\delta + \frac{2\varepsilon'}{3}(1+\delta).\end{aligned}$$

One can easily see that $2\delta + 2\varepsilon'(1+\delta)/3 < \varepsilon'$ and so

$$\cos d(x_1, (1-t)x_2 + tx_3) \geq \cos\widetilde{m} > \cos\overline{m} - \varepsilon'.$$

Suppose now $\overline{c} < \varepsilon'/2$. We may assume that $\widetilde{a} \leq \widetilde{b}$. Then

$$\cos\widetilde{m} \geq \cos\widetilde{b} \geq \cos\overline{b} - \delta \geq \cos\overline{b} - \varepsilon'/6.$$

If $\overline{c} = 0$, then $\overline{b} = \overline{m}$ and we are done. Otherwise, note first that since $|\overline{a} - \overline{b}| < \varepsilon'/2$, we have that $|\cos\overline{a} - \cos\overline{b}| < \varepsilon'/2$. By (1),

$$\cos\overline{m} \leq \frac{\sin((1-t)\overline{c})}{\sin\overline{c}}\left(\cos\overline{b} + \frac{\varepsilon'}{2}\right) + \frac{\sin(t\overline{c})}{\sin\overline{c}}\cos\overline{b} \leq \frac{\sin((1-t)\overline{c}) + \sin(t\overline{c})}{\sin\overline{c}}\cos\overline{b} + \frac{\varepsilon'}{2}.$$

If $\cos\overline{b} < 0$, then, because $\sin((1-t)\overline{c}) + \sin(t\overline{c}) \geq \sin\overline{c}$, we have that $\cos\overline{m} \leq \cos\overline{b} + \varepsilon'/2$. Otherwise,

$$\begin{aligned}\cos\overline{m} &\leq \frac{2\sin(\overline{c}/2)}{\sin\overline{c}}\cos\overline{b} + \frac{\varepsilon'}{2} = \frac{1}{\cos(\overline{c}/2)}\cos\overline{b} + \frac{\varepsilon'}{2} \leq \frac{1}{\cos(\varepsilon'/4)}\cos\overline{b} + \frac{\varepsilon'}{2} \\ &\leq \frac{1}{1-\varepsilon'/4}\cos\overline{b} + \frac{\varepsilon'}{2} = \left(1 + \frac{\varepsilon'}{4-\varepsilon'}\right)\cos\overline{b} + \frac{\varepsilon'}{2} \leq \cos\overline{b} + \frac{\varepsilon'}{4-\varepsilon'} + \frac{\varepsilon'}{2}.\end{aligned}$$

Finally,

$$\cos d(x_1, (1-t)x_2 + tx_3) \geq \cos\widetilde{m} \geq \cos\overline{m} - \frac{\varepsilon'}{4-\varepsilon'} - \frac{\varepsilon'}{2} - \frac{\varepsilon'}{6} > \cos\overline{m} - \varepsilon'.$$

Thus, in both cases we have that $\cos\overline{m} - \cos d(x_1, (1-t)x_2 + tx_3) < \varepsilon^2/18$. Since cos is strictly decreasing on $[0,\pi]$ with modulus $\varepsilon^2/18$, i.e.

$$\forall \alpha, \beta \in [0,\pi]\, \forall \varepsilon > 0\, (\beta + \varepsilon \leq \alpha \rightarrow \cos\beta - \cos\alpha \geq \varepsilon^2/18)$$

(see e.g. [9], p.31), it follows that $d(x_1, (1-t)x_2 + tx_3) \leq \overline{m} + \varepsilon$.

□

Remark 2.16. *Instead of the axiom* (W6) *one could have also axiomatized the characterization of CAT(κ)-spaces by comparison triangles in the following form (here $B_1(0)$ denotes the closed unit ball in $\mathbb{R}^3$):*

$$(*)\begin{cases} \forall x_1, x_2, x_3 \in X\, \exists \overline{x_1}, \overline{x_2}, \overline{x_3} \in B_1(0)\, \forall t \in [0,1] \\ \quad \Big(\bigwedge\limits_{i,j\in\{1,2,3\}} (\|\overline{x_i}\|_E = 1 \wedge d(x_i, x_j) = d_{M^2_\kappa}(\overline{x_i}, \overline{x_j})) \wedge \\ \quad\; d(x_1, (1-t)x_2 + tx_3) \leq d_{M^2_\kappa}(\overline{x_1}, (1-t)\overline{x_2} + t\overline{x_3})\Big). \end{cases}$$

By the unique existence (up to isometry) of comparison triangles, this formulation which states the existence of a comparison triangle satisfying the CAT(κ)-inequality is equivalent to the characterizing property that this inequality holds for all comparison triangles. While not being purely universal, $(*)$ *has the form of a so-called axiom Δ (see* [7]*) which has a trivial monotone functional interpretation and so may be taken as an axiom in our formal framework preserving the logic metatheorem. In fact, our quantitative version* (W6) *of the characterization* $(+) \to (++)$ *may be viewed as a quantitative analysis (in the sense of such a metatheorem) of the proof that* $(*)$ *implies this characterization i.e. of the proof of the uniqueness (up to isometry) of comparison triangles.*

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