On the computational content of the Krasnoselski and Ishikawa fixed point theorems

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Abstract. This paper is part of a case study in proof mining applied to non-effective proofs in nonlinear functional analysis. More specifically, we are concerned with the fixed point theory of nonexpansive selfmappings f of convex sets C in normed spaces. We study Krasnoselski and more general so-called Krasnoselski-Mann iterations which converge to fixed points of f under certain compactness conditions. But, as we show, already for uniformly convex spaces in general no bound on the rate of convergence can be computed uniformly in f. However, the iterations yield even without any compactness assumption and for arbitrary normed spaces approximate fixed points of arbitrary quality for bounded C (asymptotic regularity, Ishikawa 1976). We apply proof theoretic techniques (developed in previous papers) to non-effective proofs of this regularity and extract effective uniform bounds (with elementary proofs) on the rate of the asymptotic regularity. We first consider the classical case of uniformly convex spaces which goes back to Krasnoselski (1955) and show how a logically motivated modification allows to obtain an improved bound. Moreover, we get a completely elementary proof for a result which was obtained in 1990 by Kirk and Martinez-Yanez only with the use of the deep Browder-Göhde-Kirk fixed point theorem.

In section 4 we report on new results ([29]) we established for the general case of arbitrary normed spaces including new quantitative versions of Ishikawa's theorem (for bounded C) and its extension due to Borwein, Reich and Shafrir (1992) to unbounded sets C. Our explicit bounds also imply new qualitative results concerning the independence of the rate of aymptotic regularity from various data.

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1 General introduction

This paper (and its companion [29]) is another case study in the project of 'proof mining' [1] in analysis by which we mean the logical analysis of mathematical proofs (typically using non-effective analytical tools) with the aim of extracting new numerically relevant information (e.g. effective uniform bounds or algorithms etc.) hidden in the proofs.[2]

Many problems in numerical (functional) analysis can be seen as instances of the following general task: construct a solution x of an equation

$$A(x) :\equiv (F(x) = 0),$$

where x is an element of some Polish (i.e. complete separable metric) space (typically with additional structure) and $F : X \to \mathbb{R}$ (usually F will depend on certain parameters a which again belong to Polish spaces). Quite often the construction of such a solution is obtained in two steps:

1) One shows how to construct (uniformly in the parameters of A) approximate solutions (sometimes called 'ε-solutions') $x_\varepsilon \in X$ for an ε-version of the original equation
$$A_\varepsilon(x) :\equiv \big(|F(x)| < \varepsilon\big).$$
2) Exploiting compactness conditions on X one concludes that either $(x_{\frac{1}{n}})_{n\in\mathbb{N}}$ itself or some subsequence of it converges to a solution of $A(x)$.

The first step usually is constructive. However, the non-effectivity of the second step in many cases prevents one from being able to compute a solution $\widehat{x}$ of A effectively within a prescribed error $\frac{1}{k}$, i.e. to compute a function $n(k)$ such that $d_X(x_{n(k)}, \widehat{x}) < \frac{1}{k}$. In many cases $X := K$ is compact and $\widehat{x}$ is uniquely determined. Then (x_n) itself converges to $\widehat{x}$ so that no subsequence needs to be selected. However, the problem of how to get a-priori bounds (in particular not depending on $\widehat{x}$ itself) on the rate of convergence of that sequence remains. In numerical analysis, often such rates are not provided (due to the ineffectivity of the proof of the uniqueness of $\widehat{x}$).[3]
In a series of papers we have demonstrated the applicability of proof theoretic techniques to extract so-called uniform moduli of uniqueness (which generalize

[1] The term 'proof mining' (instead of G. Kreisel's 'unwinding of proofs') for the activity of extracting additional information hidden in given proofs using proof theoretic tools was suggested to the author by Professor Dana Scott.
[2] For a different case study in analysis in the context of best approximation theory see [21],[22]. For other kinds of logical analyses of specific proofs see [33] and [36].
[3] In interesting critical discussion of this and related issues can be found in [32].

the concept of strong unicity as used e.g. in Chebycheff approximation theory) from non-constructive uniqueness proofs and to use them to get effective rates of convergence (e.g. [25], [21],[22],[23],[30] for concrete applications to approximation theory).

In this paper we carry out a logical analysis of examples for the first of the two steps mentioned above in situations where an effective solution of 2) is not possible (mainly due to the lack of uniqueness) and already the fact that the sequence $y_n := x_{\frac{1}{n}}$ yields better and better approximate solutions is proved non-constructively (using sequential compactness).

These applications to 1) fall under (an extension of) the same general logical scheme as our previous applications to 2). In a series of papers ([23],[24],[25],[27] among others) we have developed general meta-theorems which guarantee the extractability of uniform bounds from proofs which are allowed to make use of substantial parts of analysis. In particular, we specified situations where (due to the fact that only weak forms of induction are used) exponential and even polynomial bounds are guaranteed. Furthermore, these results show that many lemmas used in such proofs do not need to be analysed (since they do not contribute to the bound) because of their logical form. The proofs of these meta-theorems actually provide an extraction algorithm (based on certain proof-theoretic transformations of the specific proof to be analyzed). So applied to a given proof p in analysis we get another proof p^* which provides more numerical information. When this transformation is carried out explicitly we obtain a new ordinary mathematical proof of a stronger statement which no longer relies on any logical tools at all. Of course, the general proof-theoretic algorithm will usually be used only as a guideline but not followed step by step in the actual construction of p^* (unless this is necessary).

The special case of our general meta-theorems which is relevant for the present paper has the following form:
Let X be a Polish space, K a compact metric space and A_1 a purely existential property. If a theorem of the form

$$(*)\ \forall n \in \mathbb{N} \forall x \in X \forall y \in K \exists m \in \mathbb{N} A_1(n, x, y, m)$$

has been proved in certain formal systems $\mathcal{T}$ for (fragments of) analysis, then one can extract a computable uniform bound $\Phi(n, x)$[4] for $\exists m$, i.e.

$$(**)\ \forall n \in \mathbb{N} \forall x \in X \forall y \in K \exists m \leq \Phi(n, x) A_1(n, x, y, m).$$

An important feature of the bound $\Phi(n, x)$ is that it does not depend on $y \in K$. Typically, $\exists m A_1$ is monotone in m so that the bound $\Phi(n, x)$ actually realizes

[4] This bound (as well as the logical form of A_1) will in general depend on the specific representation of $x \in X$ used in $\mathcal{T}$.

the quantifier. In [25] we have specified a system **PBA** of polynomially bounded analysis which guarantees that $\Phi(n,x)$ will be a polynomial. If we add the exponential function to **PBA** we obtain a system **EBA** which guarantees that Φ uses at most a finite iteration of exp (so if exp is not iterated at all the bound will be exponential in n relative to x). Whereas for our first application in the present paper (theorem 4 below) this result for **PBA** is already sufficient for providing the general logical framework, our analysis of a proof from [4] carried out in theorem 7 needs an extended version due to the use of a principle used in that proof which is not available in **PBA** or **EBA**. Whereas these systems contain quite some parts of non-constructive analysis, principles based on sequential compactness are not included. The significant and highly non-trivial impact of such principles for the extraction of bounds has been determined completely in [27] and [28]. We only discuss the results for the particular simple case of the principle

$$\mathrm{PCM}(a_k) :\equiv [\forall n(0 \le a_{n+1} \le a_n) \to \exists a \in \mathbb{R}_+(\lim_{n\to\infty} a_n = a)]$$

of convergence for bounded monotone sequences $(a_n)_{n\in\mathbb{N}}$ of reals, as it is this principle which is used in the proof from [4] we are going to analyse. In systems like **PBA**, real numbers are represented as Cauchy sequences of rational numbers with fixed rate of convergence. Because of this representation $\mathrm{PCM}(a_n)$ is a fairly strong principle equivalent to

$$(+)\ \forall n\big(0 \le a_{n+1} \le a_n\big) \to \exists f : \mathbb{N} \to \mathbb{N} \forall k, m\big(m \ge f(k) \to a_{f(k)} - a_m \le \frac{1}{k+1}\big).$$

Because of the existence of the 'Cauchy modulus f', '$\forall (a)_n\ \mathrm{PCM}(a_n)$' is equivalent to the principle of so-called arithmetical comprehension which potentially creates bounds of huge complexity when added to systems like **PBA**, **EBA** (see [28]). What we showed in [27] is that things are quite different when $\mathrm{PCM}(a_n)$ is only applied to sequences (a_n) in a given proof of a theorem $(*)$ which can be explicitly defined in terms of the parameters n, x, y of $(*)$. Then, relative to **PBA** and **EBA**, the use of $\mathrm{PCM}(a_n)$ can be reduced to its arithmetical version[5]

$$\mathrm{PCM}_{ar}(a_n) :\equiv \left[\forall n\big(0 \le a_{n+1} \le a_n\big) \to \forall k \exists n \forall m\big(m \ge n \to a_n - a_m \le \frac{1}{k+1}\big)\right].$$

By further proof theoretic considerations, the use of PCM_{ar} can even be reduced to that of its so-called 'no-counterexample interpretation' (or 'Herbrand normal

[5] Because of (+), $\mathrm{PCM}(a_n)$ essentially is the so-called Skolem normal form of $\mathrm{PCM}_{ar}(a_n)$. In general it is NOT possible (in a context like **PBA** or **EBA**) to reduce the use of the Skolem normal form of an arithmetical principle A to A itself. The fact that this IS possible for PCM_{ar} makes profound use of the fact that this principle satisfies a strong monotonicity property, see [26].

form') $\mathrm{PCM}^H_{ar}(a_n) :\equiv$

$$\left[\forall n\big(0 \le a_{n+1} \le a_n\big) \to \forall k, g \exists n\big(g(n) \ge n \to a_n - a_{g(n)} \le \frac{1}{k+1}\big)\right].$$

The computational significance of this reduction is, that in contrast to the quantifier dependency '$\forall k \exists n$' in PCM_{ar}, which in general has no computable bound, the quantifier '$\exists n$' in PCM^H_{ar} can be bounded (uniformly in k, g and an upper bound $N \in \mathbb{N}$ of a_0) by $\widehat{\Psi}(k, g, N) := \max_{i \le (k+1)N} \Psi(i, g)$, where $\Psi(k, g)$ is the k-times iteration of g applied to 0, i.e.

$$\Psi(0, g) := 0, \ \Psi(k+1, g) := g(\Psi(k, g))$$

(see [27] for details on all this). We like to stress, that this quantitative bound for $\mathrm{PCM}^H_{ar}(a_n)$ only depends on (a_n) via an upper bound $N \ge a_0$ whereas a bound for $\exists n$ in $\mathrm{PCM}_{ar}(a_n)$ of course has to depend on (a_n). So the reduction from $\mathrm{PCM}_{ar}(a_n)$ to $\mathrm{PCM}^H_{ar}(a_n)$ also provides an important independence from (a_n) which will play a crucial role in our proof of corollaries 3 and 4 below.
We have seen that the use of $\mathrm{PCM}(a_n)$ for definable (a_n) contributes to the bound Φ in $(**)$ by $\widehat{\Psi}$. By finite iteration, Ψ (and hence $\widehat{\Psi}$) is able to produce arbitrary primitive recursive growth rates. However, in practice it will usually only be applied to some fixed functions g which can explicitly be defined in terms of the parameters of the problem. It is the construction of these functions which plays a crucial role in the process of proof mining. This fact is clearly reflected in the bounds we obtain in theorem 7 and corollary 3 below (see the definition of $\widehat{\alpha}$ in these results).[6]

We have discussed the logical background of our results in order to convince the reader, that these results are instances of a general scheme for proof mining. Once one is familiar with this scheme, one can produce improvements of existence theorems of the kind we illustrate here using examples from fixed point theory also in other areas of analysis.

[6] In the concrete application in this paper it is mainly the reduction from PCM_{ar} to PCM^H_{ar} which plays a significant role in the proof mining. The (in general much more complicated) reduction from PCM to PCM_{ar} is almost straightforward. However, we expect that this will be different for other examples. In any case, we believe that the fact that our applications to concrete proofs reflect crucial steps of the general proof-theoretic reduction and are instances of a general meta-theorem (which at least under an additional compactness assumption predicts the type of results we obtain), makes it justified to call them genuine applications of logic in the sense discussed in [10].

2 Applications to the fixed point theory of nonexpansive mappings

In this paper we will analyse proofs from the fixed point theory of nonexpansive mappings $f : C \to C$ for certain sets C in normed spaces X.

Definition 1. *Let $(X, \|\cdot\|)$ be a normed linear space and $S \subseteq X$ be a subset of X. A function $f : S \to S$ is called nonexpansive if*

$$(*)\ \forall x, y \in S\big(\|f(x) - f(y)\| \leq \|x - y\|\big).$$

Whereas the fixed point theory for mappings with Lipschitz constant < 1 (i.e. contractions) is essentially trivial (even from a computational point of view) because of the well-known Banach fixed point theorem, the fixed point theory for nonexpansive mappings has been one of the most active research areas in nonlinear functional analysis from the 50's until today. Let us indicate how the picture known for contractions breaks down for nonexpansive mappings:

1) Whereas in Banach's fixed point theorem no boundedness conditions are necessary, fixed points of a nonexpansive mapping will not exist unless the set C is at least bounded: take $X := C := \mathbb{R}$ and $f(x) := x + 1$.
2) Even when C is compact (and therefore fixed points exist by the fixed point theorems of Brouwer and Schauder), they are not uniquely determined: take $X := \mathbb{R}, C := [0, 1]$ and $f(x) = x$.
3) Even when the fixed point is uniquely determined, it will in general not be approximated by the Banach iteration $x_{n+1} := f(x_n)$: take $X := \mathbb{R}, C := [0, 1], f(x) := 1 - x$ and $x_0 := 0$. Then x_n alternates between 0 and 1.

The early history of the fixed point theory for nonexpansive mappings rests on two main theorems which both use a geometric assumption on the normed space X, namely that it is uniformly convex:

Definition 2 ([6]). *A normed linear space $(X, \|\cdot\|)$ is uniformly convex if*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X\big(\|x\|, \|y\| \leq 1 \wedge \|x - y\| \geq \varepsilon \to \|\frac{1}{2}(x + y)\| \leq 1 - \delta\big).$$

A function $\eta : (0, 2] \to (0, 1]$ providing such a $\delta := \eta(\varepsilon) > 0$ for given $\varepsilon > 0$ is a modulus of uniform convexity.

The following fundamental existence theorem for fixed points of nonexpansive mappings and uniformly convex Banach spaces was proved independently by Browder, Göhde and Kirk (note that no compactness assumption is made in this result):

Theorem 1 ([5],[13],[17]). *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $C \subseteq X$ a non-empty convex closed and bounded subset of X and $f : C \to C$ a nonexpansive mapping. Then f has a fixed point.*

Remark 1. In 1975, a counterexample showing that the assumption of X being uniformly convex in theorem 1 cannot be omitted was found (see [7], p. 37).

Another fundamental theorem in the fixed point theory for nonexpansive mappings is the following result due to Krasnoselski, which shows that (under an additional compactness condition, which by the Schauder fixed point theorem guarantees the existence of a fixed point) a fixed point of f can be approximated by a special iteration sequence:

Theorem 2 (Krasnoselski [31]). *Let K be a non-empty convex closed and bounded set in a uniformly convex Banach space $(X, \|\cdot\|)$ and f a nonexpansive mapping of K into a compact subset of K. Then for every $x_0 \in K$, the sequence*

$$x_{k+1} := \frac{x_k + f(x_k)}{2}$$

converges to a fixed point $z \in K$ of f.

Remark 2. Due to a much more general result from [16], which we will discuss below, the assumption of X being uniformly convex is actually superfluous in theorem 2.

We will show below that there cannot be an effective procedure to compute a rate of convergence of the iteration in the Krasnoselski fixed point theorem uniformly in f and the starting point $x \in K$ of the iteration. This already holds for the special case of $X := \mathbb{R}$ and $K := [0,1]$ and the fixed starting point $x_0 := 0$ as there exists no computable function F from the set of all nonexpansive functions $f : [0,1] \to [0,1]$ into $[0,1]$ which computes uniformly in f a fixed point of f (this is closely related to the fact that such a function F cannot be continuous with respect to the maximum norm $\|f\|_\infty$). Logically, this ineffectivity in Krasnoselski's theorem corresponds to the fact the statement that $(x_k)_{k\in\mathbb{N}}$ converges is Π^0_3.
On the other hand if we consider the weaker question of how far we have to go in the iteration to obtain an ε-fixed point, then we notice that the logical form of the statement

$$(*)\ \forall k \in \mathbb{N} \exists n \in \mathbb{N} \big(\|x_n - f(x_n)\| < \frac{1}{k+1} \big)$$

is Π^0_2 (assuming that real numbers are represented as Cauchy sequences with fixed rate of convergence so that $<_{\mathbb{R}} \in \Sigma^0_1$). That is why we are able to extract

an algorithm for n in $(*)$ uniformly in x_0 and f (if X, C have a computable representation).
The following crucial monotonicity property holds (see lemma 2 below):

$$\|x_{m+1} - f(x_{m+1})\| \leq \|x_m - f(x_m)\|$$

Hence the formula

$$\|x_n - f(x_n)\| < \frac{1}{k+1}$$

is equivalent to

$$\forall m \geq n\big(\|x_m - f(x_m)\| < \frac{1}{k+1}\big).$$

Thus any bound on $(*)$ provides a rate of convergence for

$$(**) \ \|x_n - f(x_n)\| \stackrel{n\to\infty}{\to} 0,$$

where $(**)$ is called the asymptotic regularity of (x_n) (we will see below that this asymptotic regularity holds without any compactness assumption).
Let us assume for the moment that $C := K$ itself is compact. Then the standard proof (as given in e.g. [3]) of the Krasnoselski fixed point theorem (more precisely of its consequence $(**)$ above) directly fits into the general extraction scheme discussed above. Besides basic arithmetical reasoning only the existence of a fixed point $y \in K$ of f (which follows from the Schauder fixed point theorem) is used to show $(*)$. Since the statement

$$(a) \ \exists y \in K\big(\|f(y) - y\| = 0\big)$$

has the logical form of those assumptions which do not contribute to the growth of extractable bounds and which furthermore can be reduced to their ε-weakening

$$(b) \ \forall \varepsilon > 0 \exists y \in K\big(\|f(y) - y\| < \varepsilon\big)$$

and since furthermore the starting point $x_0 \in K$ belongs to a compact set and the set of all nonexpansive mappings $f : K \to K$ is also compact, we know a-priori that the extractability of a uniform bound (of low complexity) for n in $(*)$ which does not depend on x_0 or f (but only on ε and a modulus of uniform convexity) is guaranteed and its verification only uses (b). The actual extraction shows that instead of the compactness only the boundedness of K is needed. This is even true for the reduction of (a) to (b) which allows furthermore to remove the assumption on K being closed (and X being complete), since the existence of approximate fixed points (but not of fixed points) can be shown without these assumptions. This, of course, is a-posteriori information which was

not guaranteed by a general logical result. Nevertheless, the a-priori information provided for the special case with K being compact prompted the search for such uniform bounds. As a result we get for an arbitrary convex bounded subset $C \subset X$ a uniform bound for $(*)$ depending only on ε, a modulus of uniform convexity η of X and an upper bound for the diameter of C. The bound itself is not new: for the special compact case it is essentially already contained in Krasnoselski's original paper ([31]) and was proved for the case of closed bounded convex sets in [18] using the deep Browder-Göhde-Kirk fixed point theorem. We nevertheless carry out the analysis because it shows two phenomena:
1) the possibility of replacing the existence of fixed points by the existence of ε-fixed points which permits a completely elementary verification of the bound (without assuming X complete or C closed), which does not even rely on the Schauder fixed point theorem used by Krasnosleski in the compact case. Even the qualitative asymptotic regularity had been obtained before only either with the use of the Browder-Göhde-Kirk fixed point theorem or as a corollary of a much more general result due to Ishikawa which we discuss below.
2) There is a logical modification of the proof from [3] which makes use of the above mentioned no-counterexample interpretation PCM^H_{ar} of PCM_{ar} and allows the use of a certain multiplicative property typically satisfied by moduli η, by which we obtain (for such moduli) a numerically better result. As a special instance of this we get a bound which is polynomial in ε of degree p for the spaces L_p with $p \geq 2$ (a result which (for this special case only) was first obtained in 1990 in [18] by an ad hoc calculation).[7] For $X := \mathbb{R}, C := [0,1]$ we even get a linear bound (see also [18], p.192).

We now move on to vast extensions of Krasnoselski's fixed point theorem. In [16] it is shown that Krasnoselski's fixed point theorem even holds without the assumption of X being uniformly convex and for much more general so-called Krasnoselski-Mann iterations

$$x_{k+1} := (1-\lambda_k)x_k + \lambda_k f(x_k),$$

where λ_k is a sequence in $[0,1]$ which is divergent in sum and satisfies $\limsup\limits_{k\to\infty} \lambda_k < 1$.
Furthermore, [16] establishes that for such iterations

$$\text{(I)} \lim_{k\to\infty} \|x_k - f(x_k)\| = 0,$$

where X is an arbitrary normed linear space, C any bounded convex subset of X and $f : C \to C$ is nonexpansive.

[7] For $p = 2$, no better bound is known. For $p \geq 3$ a better bound was only obtained by the extremely complicated work of [1].

In [4], a generalization of this result to the case of unbounded convex sets C is proved:

$$\text{(II)} \quad \lim_{k\to\infty} \|x_k - f(x_k)\| = r_C(f),$$

where

$$r_C(f) := \inf_{x\in C} \|x - f(x)\|$$

will in general be strictly positive. Note that (by lemma 1 below) $r_C(f) = 0$ for bounded convex C. Hence (II) entails (I) as a special case.

In section 4, we will report on results from [29] which for the first time provide a quantitative analysis of (II) (see theorem 7). These results were obtained as an instance of our general result on the extractability of bounds from proofs using PCM(a_k) for a sequence $(a_k)_{k\in\mathbb{N}}$ which is definable in the parameters of the problem. In the case at hand $(a_k)_{k\in\mathbb{N}}$ is just $(\|x_k - f(x_k)\|)_{k\in\mathbb{N}}$. By specializing the resulting bound to the case where C is bounded we get a uniform bound for (I) which only depends on ε, an upper bound d_C for the diameter $d(C)$ of C and some rather general information on (λ_k) (see corollary 3). In particular the bound is independent of f, the starting point x_0 and the space X. Such uniformity results are of great interest in the area of nonlinear functional analysis. In the final section of this paper we discuss the long history of partial results towards our new full uniformity result from [29].

These applications clearly show the usefulness of logical proof mining even if one is not primarily interested in quantitative results like the numerical quality of the bounds (or the bounds extractable happen to be too large to be useful in practice) but is interested in new qualitative results on the independence of the quantity in question from certain input data.[8]

3 Effective uniform bounds on the Krasnoselski iteration in uniformly convex spaces

We start by showing that the rate of convergence of the Krasnosleski iteration in Krasnolselski's fixed point theorem is in general not computable (uniformly in the input data). We then show in the main part of this section that, in contrast to this negative result, one can obtain computable rates of convergence (of low complexity) for the asymptotic regularity of x_n, i.e. for $\|x_n - f(x_n)\| \to 0$. This

[8] For another instance of this see our explicit uniform constants of strong unicity for Chebycheff approximation which were extracted in [21],[22] from classical uniqueness proofs for the best Chebycheff approximation (known already since about 1905-1917). These constants in particular imply the existence of a common constant of unicity for compact sets K of functions $f \in C[a,b]$, if $\inf_{f\in K} dist(f,H) > 0$ (H a Haar space), a fact that was proved in approximation theory only in 1976 and non-effectively (see [15]).

is even true without any compactness, closedness or completeness assumptions on the convex set C or the space X.

Let $(X, \|\cdot\|)$ be a uniformly convex normed space, $K \subset X$ a compact and convex set and $f : K \to K$. By the Krasnoselski fixed point theorem we know that the Krasnoselski iteration (x_n) converges to a fixed point of f. We now show that already for $X := \mathbb{R}, K := [0,1]$ and a very simple class of nonexpansive mappings f, no rate of convergence for (x_n) (starting from $x_0 := 0$) can be computed uniformly in f.

Theorem 3. *There is no Turing machine M^α which uniformly in $\alpha : \mathbb{N} \to \{0,1\}$ as an oracle computes a number m such that*

$$\forall j \geq m\big(|x_j - x_m| < \frac{1}{2}\big),$$

where

$$x_0 := 0,\ x_{n+1} := \tfrac{x_n + f_{\lambda_\alpha}(x_n)}{2} \text{ and}$$
$$f_{\lambda_\alpha}(x) := \lambda_\alpha x + 1 - \lambda_\alpha,\ \text{where } \lambda_\alpha := \sum_{i=0}^{\infty} \alpha(i) 2^{-i-1}.$$

Proof: Assume that there is a Turing machine M^α which computes an m satisfying

$$\forall j \geq m\big(|x_j - x_m| < \frac{1}{2}\big).$$

One easily verifies that

$$\begin{array}{l}(1)\ \lambda_\alpha < 1 \Rightarrow x_n \to 1 \Rightarrow x_m \in [\frac{1}{2}, 1] \text{ and} \\ (2)\ \lambda_\alpha = 1 \Rightarrow \forall n(x_n = 0) \Rightarrow x_m = 0.\end{array}$$

Since with m also x_m is computable uniformly in α and one can decide whether $x_m \in [\frac{1}{2}, 1]$ or $x_m = 0$, one can decide uniformly in α whether $\lambda_\alpha < 1$ or $\lambda_\alpha = 1$, i.e. whether $\exists n(\alpha(n) \neq 1)$ or $\forall n(\alpha(n) = 1)$, which is impossible. $\dashv$

Remark 3. The representation of $\lambda \in [0,1]$ via α in the proof above is very strong in that it provides a rather special Cauchy sequence of rationals with fixed rate of convergence which in general cannot be uniformly computed in an arbitrary Cauchy sequence of rationals with fixed rate of convergence. However, this makes the non-computability result even stronger in that not even this strong representation of the input allows one to compute a fixed point.

Definition 3. *Let $(X, \|\cdot\|)$ be a normed linear space, S a subset of X, $f : S \to S$ and $\varepsilon > 0$. A point $x \in S$ is called an ε-fixed point of f if $\|x - f(x)\| \leq \varepsilon$.*

Lemma 1. *Let $(X, \|\cdot\|)$ be a normed linear space, let $\emptyset \neq C \subseteq X$ be convex with bounded diameter $d(C) < \infty$ and let $f : C \to C$ be nonexpansive. Then f has ε-fixed points in C for every $\varepsilon > 0$.*

Proof: Since the lemma is trivial for $\varepsilon > d(C)$, we may assume that $\varepsilon \le d(C)$. To reduce the situation to the Banach fixed point theorem we use the following well-known construction (see e.g. [4] but also [13]): $f_t(x) := (1-t)f(x) + tc$ for some $c \in C$ and $t \in (0,1]$. $f_t : C \to C$ is a contraction and therefore Banach's fixed point theorem applies. Note furthermore that the completeness assumption in Banach's theorem is needed only to guarantee the existence of a limit of the Cauchy sequence $(f_t^n(c))_{n\in\mathbb{N}}$, where f_t^n denotes the n-times iteration of f_t, which is not necessary to ensure that $f_t^n(c)$ is an ε-fixed point of f_t for sufficiently large n and hence (for $t := \varepsilon/d(C)$) a 2ε-fixed point of f. That is why we do not have to assume that X is complete or that C is closed. $\dashv$

The following lemma belongs to the 'folklore' of the subject. We include its simple proof for the sake of completeness.

Lemma 2. *Let $(X, \|\cdot\|)$ be a normed linear space, let $C \subseteq X$ be a convex subset of X and let $f : C \to C$ be a nonexpansive function. Let $x_0 \in C$ be arbitrary and define $x_{k+1} := \frac{x_k + f(x_k)}{2}$. Then*

$$\forall k\big(\|x_{k+1} - f(x_{k+1})\| \le \|x_k - f(x_k)\|\big).$$

Proof:

$$\begin{array}{l}
\|x_{k+1} - f(x_{k+1})\| = \|\frac{1}{2}x_k + \frac{1}{2}f(x_k) - f(\frac{1}{2}x_k + \frac{1}{2}f(x_k))\| = \\
\|(\frac{1}{2}x_k - \frac{1}{2}f(x_k)) + (f(x_k) - f(\frac{1}{2}x_k + \frac{1}{2}f(x_k)))\| \le \\
\|\frac{1}{2}x_k - \frac{1}{2}f(x_k)\| + \|f(x_k) - f(\frac{1}{2}x_k + \frac{1}{2}f(x_k))\| \le \\
\|\frac{1}{2}x_k - \frac{1}{2}f(x_k)\| + \|x_k - (\frac{1}{2}x_k + \frac{1}{2}f(x_k))\| = \\
\frac{1}{2}\|x_k - f(x_k)\| + \frac{1}{2}\|x_k - f(x_k)\| = \|x_k - f(x_k)\|.
\end{array}$$

$\dashv$

Quantitative analysis of the proof of theorem 2 in [3]:

We now give two quantitative versions of the consequence

$$(*)\ \forall k \in \mathbb{N} \exists n \in \mathbb{N}\big(\|x_n - f(x_n)\| < \frac{1}{k+1}\big)$$

of theorem 2 discussed in the previous section. The first one follows directly the proof of the theorem as given in [3]. The second one uses a logical modification of that proof which is motivated by our general elimination procedure for PCM_{ar}. This second analysis allows to take into account in a very easy way a property which is satisfied by many moduli of uniform convexity, e.g. for all spaces L_p with $p \ge 2$, which makes it possible to improve the results obtained from the first, direct analysis for such spaces.

General logical preliminaries:

Let us for the moment assume that K itself is compact. In Bonsall's [3] proof of theorem 2, the following is established (where $x_0 := x,\ x_{k+1} := (x_k + f(x_k))/2$ is the Krasnoselski iteration starting from x):

$$\forall x, y \in K \forall \varepsilon > 0 \big(f(y) = y \wedge \forall k (\|f(x_k) - x_k\| \geq \varepsilon) \to \lim_{n\to\infty} \|x_n - y\| = 0 \big)$$

and hence

$$\forall x, y \in K \forall \varepsilon > 0 \big(f(y) = y \wedge \forall k (\|f(x_k) - x_k\| \geq \varepsilon) \to \exists n (\|x_n - y\| < \varepsilon) \big),$$

where the existence of a fixed point $y \in K$ is derived from the Schauder fixed point theorem. This can be rephrased in the following form

$$\forall x, y \in K \forall \varepsilon > 0 \exists k, n, l \in \mathbb{N} \\ \underbrace{\left(\|f(y) - y\| \leq \frac{1}{l+1} \wedge \|f(x_k) - x_k\| \geq \varepsilon \to \exists \tilde{n} \leq n (\|x_{\tilde{n}} - y\| < \varepsilon) \right)}_{\in \Sigma^0_1}.$$

By our general results on the extractability of uniform bounds we know a priori (using the compactness of K as well as of the space of all nonexpansive mappings $f : K \to K$) that we can extract bounds $K(\varepsilon), N(\varepsilon), L(\varepsilon)$ (and hence because of the monotonicity in k, n, l of the formula above, which follows from lemma 2, also realizations) for k, n, l which are independent of $x, y \in K$ and f and only depend on $\varepsilon > 0$ (and a modulus of uniform convexity η of X). Since we may assume that $L(\varepsilon) > \frac{1}{\varepsilon}$ and since by the nonexpansivity of f

$$\|f(y) - y\| \leq \varepsilon \wedge \|x_{\tilde{n}} - y\| \leq \varepsilon \to \|f(x_{\tilde{n}}) - x_{\tilde{n}}\| \leq 3\varepsilon,$$

this yields

$$\exists n \leq \max\big(K(\varepsilon), N(\varepsilon)\big)\big(\|f(x_n) - x_n\| \leq 3\varepsilon\big),$$

and so again by lemma 2

$$\forall n \geq \max\big(K(\varepsilon), N(\varepsilon)\big)\big(\|f(x_n) - x_n\| \leq 3\varepsilon\big).$$

Thus we have obtained a uniform bound and at the same time reduced the assumption '$\exists y \in K(f(y) = y)$' to '$\forall \varepsilon > 0 \exists y \in K(\|f(y) - y\| < \varepsilon)$'. In particular, as the bound does not depend on y, the computation of such an approximate fixed point and hence an analysis of the proof of its existence is not needed.
The actual extraction of the bound carried out below reveals that such uniform bounds K, N, L even exist when the compactness assumption on K is replaced by the boundedness of K. Since by lemma 1 the existence of approximate fixed points (but not of fixed points) in this much more general setting is even guaranteed for spaces X which are not complete, we can remove this assumption as well and the result is proved without appeal to any fixed point theorem other than Banach's (actually only its ε-version):

Theorem 4 (Direct analysis of Bonsall's [3] proof of theorem 2). *Let $(X, \|\cdot\|)$ be a uniformly convex normed space with modulus of convexity $\eta : (0,2] \to (0,1]$ and $C \subseteq X$ be a non-empty convex set with*

$$d(C) := \sup_{x_1, x_2 \in C} \|x_1 - x_2\| \le d_C \in \mathbb{Q}^*_+.$$

Let $f : C \to C$ be a nonexpansive function.
Define for arbitrary $x \in C$

$$x_0 := x, \quad x_{k+1} := \frac{x_k + f(x_k)}{2}.$$

Then

$$\forall x \in C \forall \varepsilon > 0 \forall k \ge h(\varepsilon, d_C)\big(\|x_k - f(x_k)\| \le \varepsilon\big),$$

where $h(\varepsilon, d_C) := \left\lceil \frac{\ln(4d_C) - \ln(\varepsilon)}{\eta(\varepsilon/(d_C+1))} \right\rceil$ for $\varepsilon < d_C$ and $h(\varepsilon, d_C) := 0$ otherwise.

Proof: The theorem is trivial for $\varepsilon \ge d_C$. So we can assume that $\varepsilon < d_C$. By lemma 1, f has ε-fixed points $x_\varepsilon \in C$, $\|f(x_\varepsilon) - x_\varepsilon\| < \varepsilon$ for every $\varepsilon > 0$. Let $\delta > 0$ be such that $\delta < \min(1, \frac{\varepsilon}{12h(\varepsilon, d_C)})$ and let $y \in C$ be a δ-fixed point of f, i.e.

$$(1)\ \|y - f(y)\| < \delta.$$

Assume that

$$(2)\ \|x_k - f(x_k)\| = \|(x_k - y) - (f(x_k) - y)\| > \varepsilon.$$

Then

$$(3)\ \left\| \frac{x_k - y}{\|x_k - y\| + \delta} - \frac{f(x_k) - y}{\|x_k - y\| + \delta} \right\| > \frac{\varepsilon}{\|x_k - y\| + \delta} \ge \frac{\varepsilon}{d_C + 1}.$$

Because of

$$(4)\ \|f(x_k) - y\| \overset{(1)}{\le} \|f(x_k) - f(y)\| + \delta \le \|x_k - y\| + \delta,$$

we have

$$(5)\ \left\| \frac{x_k - y}{\|x_k - y\| + \delta} \right\|, \left\| \frac{f(x_k) - y}{\|x_k - y\| + \delta} \right\| \le 1$$

and therefore

$$(6)\ \left\| \frac{1}{2} \left(\frac{x_k - y}{\|x_k - y\| + \delta} + \frac{f(x_k) - y}{\|x_k - y\| + \delta} \right) \right\| \le 1 - \eta(\varepsilon/(d_C + 1)).$$

Hence

$$(7)\ \begin{cases} \|x_{k+1}-y\| = \|\frac{1}{2}(x_k+f(x_k))-y\| = \|\frac{1}{2}(x_k-y+f(x_k)-y)\| \leq \\ \big(1-\eta(\varepsilon/(d_C+1))\big)(\|x_k-y\|+\delta). \end{cases}$$

Therefore, if (2) holds for all $k \leq k_0 := h(\varepsilon, d_C) - 1$ then

$$(8)\ \begin{cases} \|x_{k_0+1}-y\| \\ \leq \big(1-\eta(\varepsilon/(d_C+1))\big)^{k_0+1}\|x_0-y\| + \sum\limits_{i=1}^{k_0+1}(1-\eta(\varepsilon/(d_C+1)))^i \cdot \delta \\ \leq \big(1-\eta(\varepsilon/(d_C+1))\big)^{k_0+1} \cdot d_C + (k_0+1)\delta \\ \leq \big(1-\eta(\varepsilon/(d_C+1))\big)^{k_0+1} \cdot d_C + \frac{\varepsilon}{12}. \end{cases}$$

We now show that

$$(9)\ \big(1-\eta(\varepsilon/(d_C+1))\big)^{k_0+1} \cdot d_C \leq \frac{\varepsilon}{4}.$$

Proof of (9): If $\eta(\varepsilon/(d_C+1)) = 1$, then the claim holds trivially. Otherwise, (9) is equivalent to

$$k_0 + 1 \geq \frac{\ln(\varepsilon/4d_C)}{\ln(1-\eta(\varepsilon/(d_C+1)))}.$$

Since $\ln(1) = 0$ and $\frac{d}{dx}ln(x) = \frac{1}{x} \geq 1$ for all $x \in (0,1]$, we get

$$-\ln(1-\eta(\varepsilon/(d_C+1))) \geq \eta(\varepsilon/(d_C+1)).$$

Together with $-\ln(\varepsilon/4d_C) = \log(4d_C) - \ln(\varepsilon)$ this yields (9).
(8) and (9) together imply

$$(10)\ \forall k \leq h(\varepsilon, d_C) - 1\big(\|x_k - f(x_k)\| > \varepsilon\big) \to \|x_{h(\varepsilon,d_C)} - y\| \leq \frac{\varepsilon}{3}.$$

Since f is nonexpansive and y is an $\frac{\varepsilon}{3}$-fixed point of f the right-hand side of the implication yields $\|x_{h(\varepsilon,d_C)} - f(x_{h(\varepsilon,d_C)})\| \leq \varepsilon$. So

$$(11)\ \exists k \leq h(\varepsilon, d_C)\big(\|x_k - f(x_k)\| \leq \varepsilon\big)$$

and hence by lemma 2 above

$$(12)\ \forall k \geq h(\varepsilon, d_C)\big(\|x_k - f(x_k)\| \leq \varepsilon\big),$$

which concludes the proof of the theorem. $\dashv$

Theorem 5 (Analysis of a modification of Bonsall's [3] proof of thm.2).

Under the same hypotheses as in theorem 4 we obtain

$$\forall x \in C \forall \varepsilon > 0 \forall k \geq h(\varepsilon, d_C)\big(\|x_k - f(x_k)\| \leq \varepsilon\big),$$

where $h(\varepsilon, d_C) := \left\lceil \frac{4 \cdot d_C}{\varepsilon \cdot \eta(\frac{\varepsilon}{d_C+1})} \right\rceil$ *for* $\varepsilon < d_C$ *and* $h(\varepsilon, d_C) := 0$ *otherwise.*
Moreover, if $\eta(\varepsilon)$ *can be written as* $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ *with*

$$(*)\ \forall \varepsilon_1, \varepsilon_2 \in (0,2]\big(\varepsilon_1 \geq \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \geq \tilde{\eta}(\varepsilon_2)\big),$$

then the bound $h(\varepsilon, d_C)$ *can be replaced (for* $\varepsilon < d_C$*) by*

$$\tilde{h}(\varepsilon, d_C) := \left\lceil \frac{2 \cdot d_C}{\varepsilon \cdot \tilde{\eta}(\frac{\varepsilon}{d_C+1})} \right\rceil.$$

Proof: By lemma 1, f has ε-fixed points $x_\varepsilon \in C$, $\|f(x_\varepsilon) - x_\varepsilon\| < \varepsilon$ for every $\varepsilon > 0$.
Let $\delta > 0$ be such that $\delta < \min(1, \frac{\varepsilon}{3}, \frac{\varepsilon}{12} \cdot \eta(\varepsilon/(d_C+1)))$ and let $y \in C$ be a δ-fixed point of f, i.e.

$$(1)\ \|y - f(y)\| < \delta.$$

Assume that

$$(2)\ \|x_k - y\| \geq \frac{\varepsilon}{3} \text{ and}$$

$$(3)\ \|x_k - f(x_k)\| = \|(x_k - y) - (f(x_k) - y)\| > \varepsilon.$$

As in the proof of theorem 4 one shows that

$$(4)\ \left\| \frac{1}{2}\left(\frac{x_k - y}{\|x_k - y\| + \delta} + \frac{f(x_k) - y}{\|x_k - y\| + \delta}\right)\right\| \leq 1 - \eta(\varepsilon/(d_C+1)).$$

Hence

$$(5) \begin{cases} \|x_{k+1} - y\| = \|\frac{1}{2}(x_k + f(x_k)) - y\| = \|\frac{1}{2}(x_k - y + f(x_k) - y)\| \leq \\ \|x_k - y\| + \delta - (\|x_k - y\| + \delta) \cdot \eta(\varepsilon/(d_C+1)) \overset{(2)}{\leq} \\ \|x_k - y\| + \delta - \frac{\varepsilon}{3} \cdot \eta(\varepsilon/(d_C+1)) \leq \|x_k - y\| - \frac{\varepsilon}{4} \cdot \eta(\varepsilon/(d_C+1)). \end{cases}$$

Define

$$n_\varepsilon := \left\lceil \frac{d_C}{\frac{\varepsilon}{4} \cdot \eta(\varepsilon/(d_C+1))} \right\rceil = \left\lceil \frac{4 \cdot d_C}{\varepsilon \cdot \eta(\varepsilon/(d_C+1))} \right\rceil.$$

If $(2), (3)$ both hold for all $k \leq n_\varepsilon$, then (5) yields

$$(6)\ \|x_{n_\varepsilon+1} - y\| < \|x_0 - y\| - d_C,$$

which contradicts the choice of d_C by which $\|x_k - y\| \in [0, d_C]$ for all $k \in \mathbb{N}$. Hence

$$(7)\ \exists k \le n_\varepsilon \big(\|x_k - y\| \le \frac{\varepsilon}{3} \vee \|x_k - f(x_k)\| \le \varepsilon\big).$$

By the choice of δ,(1) and the nonexpansivity of f, the first disjunct also implies that $\|f(x_k) - x_k\| \le \varepsilon$ and so by the preceding lemma

$$(8)\ \forall k \ge n_\varepsilon \big(\|x_k - f(x_k)\| \le \varepsilon\big).$$

The last claim in the theorem follows by choosing $y \in C$ as a δ-fixed point of f with $\delta < \min(1, \frac{\varepsilon}{3}, \frac{\varepsilon}{2} \cdot \tilde{\eta}(\varepsilon/(d_C + 1)))$ and the following modifications of (4), (5) to

$$(4)^* \quad \left\| \frac{1}{2} \left(\frac{x_k - y}{\|x_k - y\| + \delta} + \frac{f(x_k) - y}{\|x_k - y\| + \delta} \right) \right\| \le 1 - \eta(\varepsilon/(\|x_k - y\| + \delta)).$$

$$(5)^* \begin{cases} \|x_{k+1} - y\| = \|\frac{1}{2}(x_k + f(x_k)) - y\| = \|\frac{1}{2}(x_k - y + f(x_k) - y)\| \le \\ \|x_k - y\| + \delta - (\|x_k - y\| + \delta) \cdot \eta(\varepsilon/(\|x_k - y\| + \delta)) = \\ \|x_k - y\| + \delta - \varepsilon \cdot \tilde{\eta}(\varepsilon/(\|x_k - y\| + \delta)) \overset{(*)}{\le} \|x_k - y\| + \delta - \varepsilon \cdot \tilde{\eta}(\varepsilon/(d_C + 1)) \\ \le \|x_k - y\| - \frac{\varepsilon}{2} \cdot \tilde{\eta}(\varepsilon/(d_C + 1)) \end{cases}$$

(note that we can apply η to $\varepsilon/(\|x_k - y\| + \delta)$ since (3) and

$$\|f(x_k) - y\| \overset{(1)}{\le} \|f(x_k) - f(y)\| + \delta \le \|x_k - y\| + \delta$$

imply

$$\varepsilon \le \|x_k - y\| + \|f(x_k) - y\| \le 2(\|x_k - y\| + \delta)$$

and therefore

$$\varepsilon/(\|x_k - y\| + \delta) \in (0, 2]).$$

$\dashv$

If we disregard for a moment the diameter estimate d_C in the bounds in theorems 4 and 5 and put $\varepsilon := 2^{-n}$, then we see that the bound from theorem 4 essentially is $n/\eta(2^{-n})$, whereas the first bound in theorem 5 is only about $2^n/\eta(2^{-n})$. If, however, $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ satisfying $(*)$, then theorem 5 roughly gives $1/\eta(2^{-n})$ which is better than the bound from theorem 4. It is this fact that we will use in the example below to obtain a polynomial bound for L_p $(p \ge 2)$ which is of degree p.

Examples: It is well-known that the Banach spaces L_p with $1 < p < \infty$ are uniformly convex (this was first proved in [6], see also [20]). For $p \ge 2$, the following explicit modulus η_p of uniform convexity was obtained in [14]

$$\eta_p(\varepsilon) := 1 - (1 - (\varepsilon/2)^p)^{1/p}.$$

One easily shows (using the derivative of $x^{1/p}$) that (for $\varepsilon \in (0,2]$)

$$\eta_p(\varepsilon) \geq \frac{\varepsilon^p}{p2^p}.$$

Hence $\frac{\varepsilon^p}{p2^p}$ is a modulus of convexity as well. Since

$$\frac{\varepsilon^p}{p2^p} = \varepsilon \cdot \tilde{\eta_p}(\varepsilon)$$

with

$$\tilde{\eta_p}(\varepsilon) = \frac{\varepsilon^{p-1}}{p2^p}$$

satisfying $(*)$ in the theorem above, we obtain the following

Corollary 1. *Let $p \geq 2$, $C \subseteq L_p$ a non-empty convex subset with $d(C) \leq d_C \in Q^*_+$, $f : C \to C$ nonexpansive and $(x_k)_{k\in\mathbb{N}}$ defined as in the theorem. Then*

$$\forall x \in C \forall \varepsilon > 0 \forall k \geq \left\lceil \frac{d_C p (d_C+1)^{p-1} 2^{p+1}}{\varepsilon^p} \right\rceil \big(\|x_k - f(x_k)\| \leq \varepsilon \big).$$

Note that the bound in corollary 1 only depends on p, ε and an upper bound d_C of $d(C)$ but not on $x \in C$ or f.

For the case $X := \mathbb{R}, C := [0,1]$, theorem 5 even gives a linear bound, since $\varepsilon/2$ is a modulus of uniform convexity in this case and $\tilde{\eta}(\varepsilon) := \frac{1}{2}$ satisfies $(*)$.

Remark 4. Our result in corollary 1 can easily be improved by replacing (d_C+1) by $(d_C+\delta)$ for any $\delta > 0$ and so in the limit by d_C. In [18], using a direct calculation based on the modulus of uniform convexity for L_p, essentially the same result is obtained (only with a better constant as the factor '$p2^{p+1}$' is missing). For a linear bound in the case $[0,1]$, [18] refers to an unpublished result of J. Alexander. Note, however, that our bounds in these examples were derived just as special instances of the general bound in theorem 5.

4 Effective uniform bounds on the Krasnoselski-Mann iteration in arbitrary normed spaces

In this section we discuss some of the results from [29]. Throughout this section, $(X, \|\cdot\|)$ will be an arbitrary normed linear space, $C \subseteq X$ a non-empty convex subset of X and $f : C \to C$ a nonexpansive mapping.
We consider the so-called Krasnoselski-Mann iteration (which is more general

than the Krasnoselski iteration and due to Mann [34]) generated starting from an arbitrary $x \in C$ by

$$x_0 := x, \quad x_{k+1} := (1-\lambda_k)x_k + \lambda_k f(x_k),$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of real numbers in $[0,1]$. For the background information on this iteration and related references see [4].

Lemma 3 ([4]). *For all $k \in \mathbb{N}$ and $x \in C$:*
$\|x_{k+1} - f(x_{k+1})\| \leq \|x_k - f(x_k)\|$.

For the results in this section we assume (following [4]) that $(\lambda_k)_{k\in\mathbb{N}}$ is divergent in sum, which can be expressed (since $\lambda_k \geq 0$) as

$$(A)\ \forall n, i \in \mathbb{N} \exists k \in \mathbb{N} \left(\sum_{j=i}^{i+k} \lambda_j \geq n \right).$$

We also assume (again as in [4]) that

$$(B)\ \limsup_{k\to\infty} \lambda_k < 1.$$

Define

$$r_C(f) := \inf_{x\in C} \|x - f(x)\|.$$

Theorem 6 ([4]). [9] *Suppose that $(\lambda_k)_{k\in\mathbb{N}}$ satisfies the conditions (A) and (B). Then for any starting point $x \in C$ and the Krasnoselski-Mann iteration (x_n) starting from x we have*

$$\|x_n - f(x_n)\| \overset{n\to\infty}{\to} r_C(f).$$

By lemma 1, the theorem implies

Corollary 2 ([16],[11],[4]). *Under the assumptions of theorem 6 plus the additional assumption that C has bounded diameter $d(C) < \infty$ the following holds:*

$$\|x_n - f(x_n)\| \overset{n\to\infty}{\to} 0.$$

Remark 5. In [11] it is actually shown that one can choose n in the corollary independently of $x \in C$ and f. Whereas in [11] a complicated functional theoretic embedding into the space of all nonexpansive mappings is used to derive this uniformity statement, it trivially follows from our quantitative analysis in

[9] With the additional assumption that λ_k is bounded away from zero, this result is also proved in [35].

corollary 3 below which even provides an explicit effective description of such a uniform n. For a more restricted iteration the existence of a bound n independent of x was also obtained by [8] using, however, also a universal embedding theorem (due to Banach and Mazur). The use of non-trivial functional theoretic arguments in [11] and [8] to obtain the (ineffective) existence of a uniform n clearly indicates that the authors were not aware of explicit effective uniform bounds hidden in the proof of $\lim\limits_{k\to\infty} \|f(x_k)-x_k\| = 0$ as given e.g. in [11] and its generalization in [4].[10]

We will now show how the proof of theorem 6 as given in [4] fits under the general schema of logical proof mining discussed in the introduction. The actual extraction of the bound itself will be carried out in [29].

General logical form of the quantitative analysis of the proof of theorem 6 in [4]:

As we have discussed above, we only can expect to be able to extract a bound $\forall x \exists y \le \Phi(x) A(x,y)$ from a non-constructive proof if A is a purely existential formula. Since the statement in theorem 6 involves two implicative assumptions on $(\lambda_k)_{k\in\mathbb{N}}$ as well as the existence of $r_C(f)$, it prima facie does not have the required form. However, it can be reformulated so as to have the right logical form by enriching the input $(\lambda_k)_{k\in\mathbb{N}}, f, x, \varepsilon$ by additional data $K \in \mathbb{N}, \alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ and $x^* \in C$.
Let us first examine conditions (A) and (B) on $(\lambda_k)_{k\in\mathbb{N}}$:
An explicit version of (A) asks for a function $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ realizing the existential quantifier, i.e.

$$(A_\alpha)\ \forall n, i \in \mathbb{N} \left(\sum_{j=i}^{i+\alpha(i,n)} \lambda_j \ge n \right).$$

(B) states the existence of a $K \in \mathbb{N}$ such that

$$\lambda_k \le 1 - \frac{1}{K}$$

from some index k_0 on. Since k_0 only contributes an additive constant to our bound we may assume for simplicity that $k_0 = 0$. So let

$$(B_K)\ \forall k \in \mathbb{N}\big(\lambda_k \le 1 - \frac{1}{K}\big).$$

We now formulate the theorem more explicitly as follows:

$$(*) \begin{cases} \forall (\lambda_k) \in [0,1]^{\mathbb{N}} \forall f : C \to C \forall x, x^* \in C \forall K, \alpha \forall \varepsilon > 0 \exists n \in \mathbb{N} \\ \big(f \text{ nonexpans.} \wedge (A_\alpha) \wedge (B_K) \to \|x_n - f(x_n)\| < \|x^* - f(x^*)\| + \varepsilon\big). \end{cases}$$

[10] For a more detailed discussion, see the final section of this paper.

Note that by lemma 3, $(*)$ immediately implies theorem 6.
By our representation of real numbers by which $\leq_{\mathbb{R}} \in \Pi^0_1$ and $<_{\mathbb{R}} \in \Sigma^0_1$, the implication

$$\big(f \text{ nonexpansive} \wedge (A_\alpha) \wedge (B_K) \to \|x_n - f(x_n)\| < \|x^* - f(x^*)\| + \varepsilon\big)$$

is equivalent to a purely existential formula. The proof of $(*)$ only uses tools formalizable in **EBA** plus the principle PCM($\|x_k - f(x_k)\|$) (discussed in the introduction) applied to $(\|x_k - f(x_k)\|)_{k\in\mathbb{N}}$ and a complicated inequality due to [11]. This inequality can be treated just as another purely universal implicative premise and does therefore not increase the logical complexity of the theorem (nor does its proof need to be analysed). Since, furthermore, the Hilbert cube $[0,1]^{\mathbb{N}}$ is a compact space, our general results discussed in the introduction guarantee (at least for complete separable X and definable C)[11] the existence of an effective bound for n which does not depend on (λ_k) directly but which may possibly depend on $K, \alpha, x, x^*, f, \varepsilon$ and γ. This information on what type of result we should look for is a significant application of our logical approach to the specific proof of theorem 6 which would not have been visible without the reformulation of the theorem focusing on its logical form.
We also know a-priori from our general logical meta-theorem, that a uniform bound on n which does not depend on $x, x^* \in C, \gamma > 0$ or f is extractable **if** C is compact (and hence has bounded diameter). For the bound actually extracted, the dependence on x, x^*, f, γ can already be eliminated as soon as we have an upper bound on the diameter $d(C)$ of C. This stronger uniformity result is a-postiori information we get for free just by examining the extracted bound.

The extraction itself will be published in another paper [29]. We present here only the results:

Theorem 7 ([29]). *Let $(X, \|\cdot\|)$ be a normed linear space, $C \subseteq X$ a non-empty convex subset and $f : C \to C$ a nonexpansive mapping. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $[0,1]$ which is divergent in sum and satisfies*

$$\forall k \in \mathbb{N}\big(\lambda_k \leq 1 - \frac{1}{K}\big)$$

for some $K \in \mathbb{N}$.
Let $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be such that

$$\forall i, n \in \mathbb{N}\big(\alpha(i,n) \leq \alpha(i+1,n)\big) \textit{ and}$$

[11] The actually extracted bound will in fact turn out to be valid for arbitrary normed linear spaces X and convex subsets $C \subset X$. Note that the convexity assumption on C is purely universal.

$$\forall i, n \in \mathbb{N}\big(n \leq \sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s\big).$$

Let $(x_n)_{n\in\mathbb{N}}$ *be the Krasnoselski-Mann iteration*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n f(x_n),\ x_0 := x$$

starting from $x \in C$. *Then the following holds*

$$\forall x, x^* \in C \forall \varepsilon > 0 \forall n \geq h(\varepsilon, x, x^*, f, K, \alpha)(\|x_n - f(x_n)\| < \|x^* - f(x^*)\| + \varepsilon),$$

where

$$\begin{array}{l} h(\varepsilon, x, x^*, f, K, \alpha) := \widehat{\alpha}(\lceil 2\|x - f(x)\| \cdot \exp(K(M+1))\rceil \dot{-} 1, M), \\ \text{with } M := \left\lceil \frac{1+2\|x-x^*\|}{\varepsilon} \right\rceil \text{ and} \\ \widehat{\alpha}(0, M) := \tilde{\alpha}(0, M),\ \widehat{\alpha}(m+1, M) := \tilde{\alpha}(\widehat{\alpha}(m, M), M) \text{ with} \\ \tilde{\alpha}(m, M) := m + \alpha(m, M) \quad (m \in \mathbb{N}) \end{array}$$

(*Instead of* M *we may use any upper bound* $\mathbb{N} \ni \tilde{M} \geq \frac{1+2\|x-x^*\|}{\varepsilon}$). *Likewise, we may replace* $\|x - f(x)\|$ *by any upper bound).*

Corollary 3 ([29]).
Under the same assumptions as in theorem 7 plus the assumption that C *has a positive*[12] *bounded diameter* $d(C) < \infty$ *the following holds:*

$$\forall x \in C \forall \varepsilon > 0 \forall n \geq h(\varepsilon, d(C), K, \alpha)\big(\|x_n - f(x_n)\| \leq \varepsilon\big),$$

where

$$h(\varepsilon, d(C), K, \alpha) := \widehat{\alpha}(\lceil 2d(C) \cdot \exp(K(M+1))\rceil - 1, M),\ \text{with } M := \left\lceil \frac{1+2d(C)}{\varepsilon} \right\rceil$$

and $\widehat{\alpha}$ *as in the previous theorem.*

Remark 6. The behaviour of the bound in corollary 3 w.r.t. $d(C)$ can be improved as follows: if $d(C)$ is different from 1 we renorm the space by the multiplicative factor $\frac{1}{d(C)}$. Then $h(\varepsilon, 1, K, \alpha)$ gives the rate of the asymptotic regularity w.r.t. this new norm and hence $h(\frac{\varepsilon}{d(C)}, 1, K, \alpha)$ for the original norm.

Corollary 4 ([29]). *Let* $d, \varepsilon > 0$, $K \in \mathbb{N}$ *and* $\beta : \mathbb{N} \to \mathbb{N}$ *an arbitrary function. Then there exists an* $n \in \mathbb{N}$ *such that for any normed space* X, *any non-empty convex set* $C \subseteq X$ *such that* $d(C) \leq d$, *any nonexpansive function* $f : C \to$

[12] For $d(C) = 0$ things are trivial.

C, any sequence $\lambda_k \in [0, 1-\frac{1}{K}]$ satisfying $n \leq \sum_{s=0}^{\beta(n)} \lambda_s$ (for all $n \in \mathbb{N}$) and any starting point $x_0 \in C$ of the corresponding Krasnoselski-Mann iteration the following holds

$$\forall m \geq n\big(\|x_m - f(x_m)\| < \varepsilon\big).$$

Corollary 5 ([29]). *Let $(X, \|\cdot\|)$, $C, d(C), f$ be as in corollary 3, $k \in \mathbb{N}, k \geq 2$ and $\lambda_n \in [\frac{1}{k}, 1-\frac{1}{k}]$ for all $n \in \mathbb{N}$. Consider the Krasnoselski-Mann iteration $x_{n+1} := (1-\lambda_n)x_n + \lambda_n f(x_n)$ starting from $x_0 := x \in C$. Then the following holds:*

$$\forall x \in C \forall \varepsilon > 0 \forall n \geq g(\varepsilon, d(C))\big(\|x_n - f(x_n)\| \leq \varepsilon\big),$$

where

$$g(\varepsilon, d(C)) := kM \cdot \lceil 2d(C)\exp(k(M+1))\rceil \text{ with } M := \left\lceil \frac{1+2d(C)}{\varepsilon} \right\rceil.$$

5 Evaluation of the results of the case study

We have seen that there are interesting proofs in non-linear functional analysis (and specifically in fixed point theory) which fall under general proof theoretic results on the extractability of uniform bounds we had obtained in previous papers.
We applied these results to essentially two proofs

1) A standard proof from [3] (from the year 1962)[13] of the well-known Krasnoselski fixed point theorem.
2) A proof from [4] (which contains as a special case a proof from [11] from 1982) for a general result on the asymptotic behaviour of the Krasnoselski-Mann iteration in arbitrary normed spaces (generalizing a result from Ishikawa [16]).

Results on 1): Logical analysis of a proof from 1955/62 yielded uniform bounds together with an elementary verification for arbitrary bounded convex sets C. Under slightly less general conditions and with the use of the deep Browder-Göhde-Kirk fixed point theorem our result in theorem 4 was obtained only in 1990 ([18]) (The compact case is due already to Krasnoselski). Moreover, a logical modification of the proof using PCM^H_{ar} (with $g(n) = n+1$ as the Herbrand index function in '$\forall g$' of $\mathrm{PCM}^H_{ar}(a_n)$) allowed to improve this bound under a further condition usually satisfied by moduli of uniform convexity (theorem 5). Applying

[13] Krasnoselski's original proof from 1955 is very similar to that as far as we can judge from the Russian text.

this general bound to L_p ($p \geq 2$) resulted in a polynomial bound of degree p (a result which for this special case was obtained in [18] by an ad hoc calculation). For $X := \mathbb{R}$ and $C := [0,1]$ we even get a linear bound out of our general result (see also [18], p.192).

Results on 2): The logical analysis of the proof in [4] (resp. [11]) carried out in [29] provides the first effective bound for Ishikawa's theorem on the asymptotic behaviour of **general** Krasnoselski-Mann iterations in arbitrary normed spaces X and for bounded sets C (corollary 3). Moreover, our bound is uniform in the sense that it only depends on the error ε and an upper bound d_C of the diameter of C (and some quite general data from the sequence of scalars λ_k used in defining the iteration). I.e. it is independent of the normed space $(X, \|\cdot\|)$. the starting point $x_0 \in C$ of the iteration, the nonexpansive function f and C-data other than d_C. Moreover, it is to a certain extent independent of λ_k. Our result has a long history of partial results: In [8] the ineffective existence of a bound which is independent of x_0 was shown in the special case of constant $\lambda_k = \lambda$. In [11] the non-effective existence of a bound independent of x_0 and f was shown for the case of general λ_k (both [8] and [11] use non-trivial functional theoretic embeddings to obtain these uniformities. Recently, W.A. Kirk ([9]) found an interesting application of this uniformity). In [18] the non-effectivity of all these results is explicitly mentioned and it is stated that 'it seems unlikely that such estimates would be easy to obtain in general setting' (p.191) and therefore in [18] only the special 'tractable' (p.191) classical case of uniformly convex spaces is studied (see the discussion in remark 4 above). Not even the ineffective existence of bounds which, moreover, only depend on C via d_C (corollary 4), was known before and actually in [12] (p.101) conjectured as 'unlikely' to be true (by the same authors whose proof of $\|x_k - f(x_k)\| \to 0$ in [11] does yield such a bound by logical analysis!). Only in the special case of $\lambda_k := \lambda \in (0,1)$ being constant, a uniform (and in fact optimal quadratic) bound was recently obtained using extremely complicated computer aided proofs involving hypergeometric functions (see [1], where once more the non-effectivity of all known proofs of the full Ishikawa result is emphasized). Subsequently, only for the even more special case of $\lambda_k := \frac{1}{2}$ a classically proved result of that type has been obtained (see [2]). This result, of course, is – for that highly special case – numerically better than the exponential bound we obtain in [29] for the much more general case of $\lambda_k \in [\frac{1}{n}, 1 - \frac{1}{n}]$ ($n \geq 2$). The authors stress, however, that their method as it stands does not apply to non-constant sequences (λ_k). [14] Our bound for the general case of unbounded C treated in [4] (theorem 7) is apparently all new.

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[14] Professor Kirk ([19]) communicated to us a new proof of the uniform (w.r.t. x_0 and f) Ishikawa result for the special case $\lambda_k = \lambda$ (again using a functional theoretic embedding). He works in the even more general setting of so-called directionally nonexpansive mappings. It seems interesting to apply a logical analysis to that proof as well. We are grateful to Professor Kirk for bringing the work of Baillon and Bruck to our attention and for communicating to us his recent papers [9],[19].

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