On the computational content of moduli of regularity and their logical strength

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Abstract

We continue the investigation into the computational status of the existence of moduli of regularity (and their use for rates of convergence) in the sense of [21], carried out w.r.t. classical reverse mathematics and Weihrauch degrees in [17], and determine the amount of LEM involved. We also show that the existence of a modulus of regularity always yields an algorithm for the computation of a zero in the case of continuous functions $F : K \to \mathbb{R}$ on a compact metric space (in F equipped with a modulus of uniform continuity and K given in standard representation) whenever such a zero exists. If $K \subset X$ is a compact subset of a uniformly convex Banach space X and the zero set of F is convex one can compute even the zero of minimal norm. A modulus of regularity can also be used to compute the left-most infinite path of an infinite 0/1-tree. We also show that there is no proof-theoretically tame nonstandard uniformity principle which would make it possible to replace in the regularity assumption compactness by metric boundedness and still guarantee classically correct bounds.

1 Introduction

In this paper, we continue our investigation from [17] into the computational and logical strength of (the existence of) of the general form of (metric) regularity introduced in [21] as a unifying concept of many related notions studied in continuous optimization. Roughly speaking, the assumption of regularity of a solution set allows one to conclude that a sufficiently good approximate solution must be close to an actual solution. If the solution set is a singleton set this has been studied under the name of strong or uniform uniqueness (also with moduli) e.g. in [9] (see also [16]). The concept of regularity generalizes this to the nonunique case. In the context of compact metric spaces X and continuous functions $F : X \to \mathbb{R}$ the regularity of the set zer F of zeros of F always holds by a result in [21]. However, as shown in [17], one usually - in contrast to the case of uniqueness where proof-theoretic techniques can be used to extract an effective modulus of uniqueness - cannot hope for a computable modulus of regularity. In fact, the existence of the latter is - even for Lipschitz continuous functions $F : [0,1] \to \mathbb{R}$ - equivalent to arithmetical comprehension ACA_0 while the $\forall\exists$-form of regularity (without a modulus) only requires WKL_0. The reason for this difference is that the $\forall\exists$-form of regularity (while proof-theoretic being weak) implies intuitionistically (and actually is equivalent to) the Σ^0_1-law-of-excluded-middle principle Σ^0_1-LEM: see Theorem 2.11 below.[1] When strengthened into a modulus, this use of Σ^0_1-LEM then becomes even classically visible in the form of Σ^0_1-comprehension (and so - by iteration - as ACA_0).

In [21] it is shown that a modulus of regularity yields a rate of convergence whenever we have a Fejér monotone (w.r.t. the solution set) algorithm for the computation of approximate solutions together with an approximate solution bound for this algorithm.[2] In this paper we show that in the situation above with a compact metric space X one unconditionally can construct a primitive recursive functional which computes uniformly in a modulus of regularity, a standard representation of X and a name for F (given by the restriction of F to a countable sense subset and a modulus of uniform continuity) a zero of F provided that zer $F \neq \emptyset$ (Theorem 2.3).

As a special case one can subsume the problem of finding an infinite path of an infinite binary (i.e. 0/1-)tree. Here there is a Kalmar elementary functional which uniformly in (the characteristic function of) such a tree and a modulus of regularity (w.r.t. the set of infinite paths as solution set) computes the leftmost branch of the tree (Theorem 2.8).

If $K \subset X$ is a compact subset of a uniformly convex Banach space X, $F : K \to \mathbb{R}$ is continuous and zer F is convex (a situation which frequently occurs in convex optimization) then one can compute in the above data augmented with a modulus of convexity of X even a zero of minimal norm (Theorem 2.4).

In proof mining one often can allow the use of nonstandard arguments which replace a compactness assumption by metric boundedness. The uniform boundedness principle $\exists$-UB^X introduced in [15] (see also [16] and the connected discussion in [5]) systematically makes this possible and can - though classically being false - be eliminated without any complexity contribution from the verification of the bounds extracted from proofs which make use of this principle. This raises the question whether some combination of, say, arithmetical comprehension (which is an admissible principle in the logical metatheorems on proof mining if one allows for bar recursive bounds) with $\exists$-UB^X (or some other 'tame' nonstandard principle) implies regularity even in the absence of compactness. In Proposition 2.5 we show that this is not the case.

2 Main Results

Definition 2.1 ([21]). *Let (X,d) be a metric space and let be $F : X \to \mathbb{R}$ a mapping. Let zer $F := \{x \in X : F(x) = 0\} \neq \emptyset$ and $r > 0$. We say that F is regular w.r.t. zer F and*

[1] A particularly striking example of such a situation is given by Ramsey's theorem for pairs and two colors which - though proof-theoretically weak - implies (and is equivalent to) even the principle Σ^0_3-LLPO (see [3]) which by [1] is strictly stronger than Σ^0_2-LEM. See also [4] for a generalization to k-many colors.

[2] For a recent extension of this result to a generalized form of Fejér monotonicity see [22].

$\overline{B}(z,r)$ *for* $z \in zer\, F$ *if*

$$\forall n \in \mathbb{N}\, \exists k \in \mathbb{N}\, \forall x \in \overline{B}(z,r)\, \left(|F(x)| < 2^{-k} \to \exists z' \in zer\, F\, (d(x,z') < 2^{-n}) \right).$$

If this holds with '$\forall x \in \overline{B}(z,r)$*' replaced by '*$\forall x \in X$*' we say that* F *is regular w.r.t. zer* F*. A function* $\rho : \mathbb{N} \to \mathbb{N}$ *providing given* n *a number* $k = \rho(n)$ *satisfying the above is called a modulus of regularity of* F *w.r.t. zer* F *and* $\overline{B}(z,r)$ *resp. w.r.t. zer* F *(short:* ρ *mreg* $zer\, F, r$*)*

Proposition 2.2 ([21])**.** *If* X *is proper and* F *is continuous, then for any* $z \in zer\, F$ *and* $r > 0$*,* F *has a modulus of regularity w.r.t. zer* F *and* $\overline{B}(z,r)$*.*

Let (X,d) be compact metric space and (a_n) be a sequence in X and $\alpha : \mathbb{N} \to \mathbb{N}$ both together witnessing the total boundedness of X, i.e.

$$\forall x \in X\, \forall k \in \mathbb{N}\, \exists\, 0 \le i \le \alpha(k)\, \left(d(x,a_i) < 2^{-k} \right)$$

(compare $(TOTI)$ in [20]).
Let $g : \mathbb{N}^2 \to \mathbb{N}^{\mathbb{N}}$ be such that for all $i,j \in \mathbb{N}$, $g(i,j)$ is a name (in the sense of [16]) of $d(a_i,a_j)$.
Let $F : X \to \mathbb{R}$ be a continuous function with a modulus $\omega : \mathbb{N} \to \mathbb{N}$ of uniform continuity, i.e.

$$\forall k \in \mathbb{N}\, \forall x,y \in X\, \left(d(x,y) < 2^{-\omega(k)} \to d(F(x),F(y)) < 2^{-k} \right).$$

Let $h : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ be such that for each $i \in \mathbb{N}$, $h(i)$ is a name of $F(a_i)$.

Theorem 2.3. *Let* $(X,d), F$ *be as above. One can define a primitive recursive functional (in the sense of Kleene's S1-S8 from* [8]*)* Φ *such that for all functions* $\omega, \alpha, g, h, \rho$*, it holds for* $\beta := \Phi(\omega,\alpha,g,h,\rho) : \mathbb{N} \to \mathbb{N}$ *that* $(a_{\beta(k)})_{k\in\mathbb{N}}$ *converges with rate* 2^{-k} *to a zero of* F *provided that zer* $F \neq \emptyset$*, the functions* ω, α, g, h *satisfy the above requirements and* ρ *is a modulus of regularity* ρ *for zer* F*.*

Proof: We show how to compute primitive recursively in $\omega, \alpha, g, h, \rho$ satisfying the above requirements a function $\beta : \mathbb{N} \to \mathbb{N}$ such that for $x_k := a_{\beta(k)}$

$$(*)\ \forall k \in \mathbb{N}\, \left(|F(x_k)| < 2^{-\max\{k,\rho(k+2)\}} \wedge (k > 0 \to d(x_k,x_{k-1}) \le 2^{-k}) \right).$$

$(*)$ clearly implies that (x_k) is a Cauchy sequence with rate 2^{-k} since for $m \ge n \ge k$

$$d(x_m,x_n) \le \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \le \sum_{k=n}^{m-1} 2^{-k-1} \le \sum_{k=n}^{\infty} 2^{-k-1} \le 2^{-n}.$$

Since F is continuous, $(*)$ - moreover - implies that $x := \lim x_k$ is a zero of F.
We prove $(*)$ by induction on k and simultaneously define β : Let $k = 0$. Since zer $F \neq \emptyset$, one can search primitive recursively in $\omega, \alpha, g, h, \rho$ for an n with

$$|F(a_n)| < 2^{-\rho(2)}$$

so that $(*)$ holds with $\beta(0) := n$ (for definiteness we may stipulate that $x_{-1} := a_0$). Indeed, let $z \in \text{zer } F$. There there exists an $n \leq \alpha(\omega(\rho(2)+2))$ such that

$$d(a_n, z) < 2^{-\omega(\rho(2)+2)}.$$

Then in turn

$$|F(a_n)| < 2^{-\rho(2)-2}$$

and so for the $2^{-\rho(2)-2}$-rational approximation $(F(a_n))(\rho(2)+2)$ of $F(a_n)$ provided by $\widehat{h(n)}(\rho(2)+2)$

$$(1)\ |(F(a_n))(\rho(2)+2)| < 2^{-\rho(2)-1}.$$

Since (1) is a primitive recursively decidable property, one can search primitive recursively for the least $n \leq \alpha(\omega(\rho(2)+2))$ such that (1) holds which in turn implies

$$(2)\ |F(a_n)| < 2^{-\rho(2)}.$$

$k \mapsto k+1, k \geq 0$: By the induction hypothesis we have defined already $\beta(0), \ldots, \beta(k)$ such that

$$(3)\ k > 0 \rightarrow d(x_k, x_{k-1}) \leq 2^{-k}$$

and

$$(4)\ |F(x_k)| < 2^{-\rho(k+2)}.$$

By (4) we get

$$(5)\ \exists z \in X\ \Big(F(z) = 0 \wedge d(z, x_k) < 2^{-k-2}\Big).$$

Similarly to the above argument we can search primitive recursively in $\omega, \alpha, g, h, \rho, k$ and $\beta(k)$ for an n_k such that

$$d(a_{n_k}, x_k) \leq 2^{-k-1} \wedge |F(a_{n_k})| < 2^{-\max\{k+1, \rho(k+3)\}}.$$

Indeed: let $N := \max\{k+1, \rho(k+3)\}$. Let z be as in (5). There exists an $n_k \leq \alpha\left(\max\{k+4, \omega(N+2)\}\right)$ such that

$$d(a_{n_k}, z) < 2^{-\max\{k+4, \omega(N+2)\}}$$

and so

$$d(a_{n_k}, x_k) < 2^{-k-4} + 2^{-k-2} \wedge |F(a_{n_k})| < 2^{-N-2}.$$

Hence

$$(6)\ (d(a_{n_k}, x_k))(k+4) < 2^{-k-3} + 2^{-k-2} \wedge (|F(a_{n_k})|)(N+2) < 2^{-N-1},$$

where $(d(a_{n_k}, x_k))(k+4)$ and $(F(a_{n_k}))(N+2)$ are the 2^{-k-4}- and 2^{-N-2}- rational approximations of $d(a_{n_k}, x_k)$ and $F(a_{n_k})$ provided by $g\widehat{(n_k, \beta(k))}(k+4)$ and $\widehat{h(n_k)}(N+2)$

respectively. As (6) is a primitive recursively decidable property we can search for the least $n_k \leq \alpha\left(\max\{k+4, \omega(N+2)\}\right)$ such that (6) holds which implies that

$$(7)\ d(a_{n_k}, x_k) < 2^{-k-1} \wedge |F(a_{n_k})| < 2^{-N}.$$

Now take $\beta(k+1) := n_k$. □

Let $(X, \|\cdot\|)$ be a uniformly convex Banach space with modulus of convexity $\eta : \mathbb{N} \to \mathbb{N}$ as in (10*) in [16] (p. 4.12). Let $K \subset X$ be a compact subset and $F : K \to \mathbb{R}$ be continuous with modulus ω of uniform continuity. Let (a_n) be a dense sequence in K and $\alpha : \mathbb{N} \to \mathbb{N}$ be a modulus of total boundedness for K as above, $h : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ be a sequence of names for $(F(a_n))_{n\in\mathbb{N}}$ and $g : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ be such that for each $n \in \mathbb{N}$, $g(n)$ is a name for $\|a_n\|$. We now assume that $C :=$ zer F is nonempty closed and convex and that ρ is a modulus of regularity for F w.r.t. zer F.
It is well-known that the metric projection of X onto C is well-defined and single-valued.
Let $D \in \mathbb{N}$ be an upper bound on the norm of some zero of F.

Theorem 2.4. *There exists a primitive recursive functional in the sense of Kleene Ψ which uniformly in functions $\eta, \alpha, \omega, g, h, \rho$ and D satisfying the above requirements computes the unique zero $\Psi(\eta, \alpha, \omega, g, h, \rho, D) \in$ zer F of F which has minimal norm among all zeros, i.e. the metric projection of 0 onto zer F.*

Proof: In η and D one can easily compute primitive recursively a modulus of uniqueness $\Phi(k) := \Phi(\eta, D, k) \in \mathbb{N}$ for the metric projection of 0 onto C, see e.g. [16, Proposition 17.4], where we here write this modulus with ε/δ of the form 2^{-k}, i.e.

$$\forall k \in \mathbb{N}\, \forall y_1, y_2 \in C \left(\bigwedge_{i=1}^{2} (\|y_i\| \leq \inf_{y\in C} \|y\| + 2^{-\Phi(k)}) \to \|y_1 - y_2\| \leq 2^{-k} \right).$$

Let $k \in \mathbb{N}$ be given and define

$$L := \alpha\left(\max\left\{\Phi(k+1)+2, \omega(K)\right\}\right),$$

where

$$K := \rho\left(\Phi(k+1)+2\right)+2.$$

We consider the set

$$S_k := \left\{ n \leq L : |(F(a_n))(K)| \leq 2^{-\rho(\Phi(k+1)+2)-1} \right\},$$

where here again, $(F(a_n))(K)$ is the 2^{-K}-rational approximation of $F(a_n)$ provided by $\widehat{h(a_n)}(K)$.
(i) : $S_k \neq \emptyset$. Let $z \in$ zer F. Then by the definition of α, there exists an $n \leq L$ such that

$$\|a_n - z\| < 2^{-\omega(K)}.$$

By the definition of ω and using that $F(z) = 0$, we get $|F(a_n)| < 2^{-K}$ and so

$$|(F(a_n))(K)| < 2^{-\rho(\Phi(k+1)+2)-1},$$

i.e. $n \in S_k$.
(ii) : The following implications holds for all $n \in \mathbb{N}$

$$n \in S_k \to \exists z \in \text{zer } F \left(\|z - a_n\| < 2^{-\Phi(k+1)-2} \right),$$

since by the assumption

$$|(F(a_n))(K)| \leq 2^{-\rho(\Phi(k+1)+2)-1}$$

and so

$$|F(a_n)| < 2^{-\rho(\Phi(k+1)+2)}$$

so that we can apply the definition of ρ.
Now compute (primitive recursively in k and the other data mentioned in the theorem) an $n_k \in S_k$ such that for all $m \in S_k$

$$(+) \ (\|a_n\|)(\Phi(k+1)+2) \leq (\|a_m\|)(\Phi(k+1)+2),$$

where $(\|a_i\|)(\Phi(k+1)+2)$ is the $2^{-\Phi(k+1)-2}$-rational approximation to $\|a_i\|$ provided by $\widehat{g(i)}(\Phi(k+1)+2)$.
By (ii), $\exists z \in \text{zer } F$ such that

$$\|a_{n_k} - z\| < 2^{-\Phi(k+1)-2}.$$

Let z_0 be the unique zero of F with minimal norm.
Claim: $\|z\| \leq \|z_0\| + 2^{-\Phi(k+1)}$.
Proof of Claim: Suppose that $\|z\| > \|z_0\| + 2^{-\Phi(k+1)}$. Then $\|a_{n_k}\| > \|z_0\| + 2^{-\Phi(k+1)} - 2^{-\Phi(k+1)-2}$ and so

$$(1) \ (\|a_{n_k}\|))(\Phi(k+1)+2) > \|z_0\| + 2^{-\Phi(k+1)} - 2^{-\Phi(k+1)-1} = \|z_0\| + 2^{-\Phi(k+1)-1}.$$

By the definition of L there exists an $l \leq L$ with

$$\|a_l - z_0\| < 2^{-\Phi(k+1)-2} \wedge \|a_l - z_0\| < 2^{-\omega(K)}.$$

By the second conjunct we get - reasoning as in (i) - that $l \in S_k$.
The first conjunct implies that

$$(2) \ (\|a_l\|)(\Phi(k+1)+2) < \|z_0\| + 2^{-\Phi(k+1)-1}.$$

(1) and (2) together yield that

$$(\|a_{n_k}\|)(\Phi(k+1)+2) > (\|a_l\|)(\Phi(k+1)+2)$$

which contradicts (+) and so finishes the proof of the claim.
Since Φ is a modulus of uniqueness, the claim implies that $\|z - z_0\| \le 2^{-k-1}$ and so (since $\Phi(k) \ge k$ which holds by the construction of Φ and which we, anyhow, may assume w.l.o.g. by taking $\max\{\Phi(k), k\}$, otherwise) $\|a_{n_k} - z_0\| < 2^{-k}$. This implies that $(n_k)_k$ is a name for z_0 in the sense of [16]. $\square$

The following axiom (formulated in the language of the system $\mathcal{A}^\omega[X, d, W]$ of classical analysis in all finite types augmented with an abstract type X for a bounded hyperbolic space defined in [14]) states that for an abstract bounded metric space (X, d) any nonexpansive function $F : X \to \mathbb{R}$ which has a zero possesses a modulus of regularity

$$\text{(NE-Reg)} : \forall F : X \to \mathbb{R},\, r > 0\, (F \text{ n.e.} \wedge \exists z \in \text{zer } F \to \exists \rho : \mathbb{N} \to \mathbb{N}(\rho \text{ mreg zer } F, r)).$$

We will show that this classically false axiom can - differently from the also classically false principle $\exists$-UB^X from [15] (see also [16]) which is admissible in logical metatheorems of proof mining - not be added to $\mathcal{A}^\omega[X, d, W]$ to obtain a formal system which admits the extraction of classically correct uniform bounds for (essentially) $\forall\exists$-theorems.[3] The argument is already implicit in the proof of [18][Theorem 3] but we include it here for completeness.

Proposition 2.5. *In* $\mathcal{A}^\omega[X, d, W] +$ (NE-Reg) *one can prove a sentence of the form*

$$A :\equiv \forall g \in \mathbb{N}^\mathbb{N}, k \in \mathbb{N},\, x^X, p^X, T^{X\to X}\, (T \textit{ nonexpansive } \to \exists n \in \mathbb{N}\, A_\exists(g, k, x, p, T)),$$

where $A_\exists(g, k, x, p, T)$ *is a provably extensional* Σ^0_1*-formula, such that* A *is not true in all bounded hyperbolic spaces* (X, d, W) *(in fact not even in all closed bounded convex subsets of* l_2*).*

Proof: Consider the sentence (abbreviating 'nonexpansive' by 'n.e.')

$$A :\equiv \forall g \in \mathbb{N}^\mathbb{N}, k \in \mathbb{N},\, x^X, p^X, T^{X\to X}\, \Big(T \text{ n.e.} \wedge p =_X Tp \to \exists n \in \mathbb{N}\, d_X(x_n, x_{g(n)}) <_\mathbb{R} 2^{-k}\Big),$$

where (x_n) is defined as $x_0 := x,\ x_{n+1} := W_X(x_n, Tx_n, 1/2)$, which is equivalent to a sentence of the required logical form noticing that $=_X \in \Pi^0_1$ and $<_\mathbb{R} \in \Sigma^0_1$.
$\mathcal{A}^\omega[X, d, W]$ proves that the Krasnoselski iteration of T is asymptotically regular, i.e. that $d(x_n, Tx_n) \to 0$ (see [19, 16]). Since (x_n) is Fejér monotone w.r.t. zer F where $F(x) := d(x, T(x))$, the assumption $F(p) =_\mathbb{R} 0$ by (NE-Reg) yields a modulus of regularity w.r.t. zer F and $\overline{B}(p, b)$ where $b^\mathbb{N}$ is the constant witnessing the (b-)boundedness of X in $\mathcal{A}^\omega[X, d, W]$. Hence by [21, Theorem 4.1], (x_n) is - reasoning in $\mathcal{A}^\omega[X, d, W] +$ (NE-Reg) - a Cauchy sequence and so for all $k \in \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$

$$\exists n \in \mathbb{N}\, d_X(x_n, x_{g(n)}) < 2^{-k}.$$

[3] So, in particular, (NE-Reg) cannot be derived from a combination of $\exists$-UB^X with comprehension over natural numbers, e.g. by an adaptation to the abstract type X of how the generalized uniform boundedness principles Π^0_k-UB$^-\!\restriction$ studied in [11] are formed.

Thus in total we get the provability of A. However, by [6], there exists a bounded closed and convex subset $C \subset l_2$ and a nonexpansive selfmapping $T : C \to C$ possessing a fixed point in C and a point $x_0 \in C$ such that (x_n) does not strongly converge. Hence (x_n) is not a Cauchy sequence and so $\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \in \mathbb{N}\ (d_X(x_n, x_{g(n)}) < 2^{-k})$ does not hold in this example, where $X := C$ and $W(x, y, \lambda) := (1-\lambda)x + \lambda y$. □

In the following we use the formal definition of the binary ('weak') Kőnig's lemma (in a language with function variables) as given in [27] (see also [28]; here $*, \bar{b}x, lth(n)$ refer to the primitive recursive coding of finite sequences from [26]):

Definition 2.6 ([27]).
$Tf :\equiv \forall n^0, m^0 (f(n * m) =_0 0 \to fn =_0 0) \wedge \forall n^0, x^0 (f(n * \langle x \rangle) =_0 0 \to x \leq_0 1)$
(i.e. $T(f)$ asserts that f represents (the characteristic function of) a binary tree)
$T^\infty(f) :\equiv T(f) \wedge \forall x^0 \exists n^0 (lth(n) = x \wedge fn = 0)$,
(i.e. $T^\infty(f)$ expresses that f represents an infinite binary tree),
$\mathrm{WKL} :\equiv \forall f^1 \big(T^\infty(f) \to \exists b^1 \forall x^0 (f(\bar{b}x) = 0)\big)$.

Definition 2.7. *We say that $\rho : \mathbb{N} \to \mathbb{N}$ is a modulus of regularity for an infinite binary tree (represented by) f w.r.t. infinite paths through f if*

$$\forall h \leq_1 1\, \forall k^0\ \left(f(\overline{h}(\rho(k))) = 0 \to \exists b \leq_1 1\, (\forall x^0 (f(\bar{b}x) = 0) \wedge \overline{h}k = \bar{b}k) \right).$$

Definition 2.7 can be seen as a special case of Definition 2.1 for $X := 2^{\mathbb{N}}$ with the Baire space metric

$$d(f, g) := \begin{cases} 2^{-\min k[f(k) \neq g(k)] - 1}, & \text{if } f \neq g \\ 0, & \text{otherwise} \end{cases}$$

and

$$F : 2^{\mathbb{N}} \to \mathbb{R},\ h \mapsto \sum_{i=0}^{\infty} \chi(h, i) \cdot 2^{-i},$$

where

$$\chi(h, i) = \begin{cases} 0, & \text{if } f(\overline{h}i) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Here we use that $d(f, g) < 2^{-k}$ iff $\overline{f}(k) = \overline{g}(k)$ and that

$$|F(h)| < 2^{-n} \to f(\overline{h}n) = 0 \to |F(h)| \leq 2^{-n}.$$

More precisely, if ρ satisfies Definition 2.7, then it also is a modulus of regularity in the sense of 2.1 for $\operatorname{zer} F$ and - conversely - if ρ is modulus in the sense of Definition 2.1, then $\tilde{\rho}(k) := \rho(k) + 1$ satisfies Definition 2.7.

As shown in [17], the existence of a modulus of regularity for continuous functions on compact spaces is equivalent to arithmetical comprehension ACA_0 (whereas the existence of the $\forall\varepsilon\exists\delta$-version is equivalent to WKL). The reason for this difference in strength is

that already the $\forall\varepsilon\exists\delta$-regularity implies Σ^0_1-LEM (see Theorem 2.11.2) below) which then, when strengthened into a modulus, becomes Σ^0_1-comprehension and so - by interation - arithmetical comprehension.
Using arithmetical comprehension, one can construct the leftmost infinite path in an infinite binary tree. The next result gives an explicit transformation of a modulus of regularity in the sense of Definition 2.7 into the leftmost branch:

Proposition 2.8. *There exists a Kalmar elementary functional φ of type 2 (more precisely φ is given by a closed term of G_3A^ω as defined in* [10]*) such that for any given infinite binary tree f with modulus of regularity ρ, $\varphi(f,\rho)$ is the leftmost infinite path in f.*

Proof: Define $\varphi(f,\rho)(k) :=$ k-th component $(\sigma)_k$ of the leftmost finite branch σ of length $\rho(k+1)$ in f which can be searched for by exponentially bounded search. We may assume that $k \leq \rho(k)$ for all k. Let k be fixed and consider the leftmost finite branch σ of length $\rho(k+1)$ in f. Then the infinite sequence $\sigma * 0^1$ (defined as continuing σ by 0's) satisfies $f(\overline{(\sigma * 0)}(\rho(k+1)) = f(\sigma) = 0$. By the definition of ρ we get the existence of an infinite path $b \leq 1$ with

$$\overline{b}(k+1) = \overline{(\sigma * 0)}(k+1).$$

Let $\tilde{b} \leq 1$ be the leftmost infinite path in f.
We show that $\forall i \leq k\,(\tilde{b}(i) = b(i))$ (and so, in particular, $\tilde{b}(k) = b(k) = (\sigma)_k$): suppose that $\exists i \leq k\,\left(\tilde{b}(i) \neq b(i)\right)$ and let i_0 be the smallest such i.

Case 1: $\tilde{b}(i_0) < b(i_0) = (\sigma)_{i_0}$. Then $\overline{\tilde{b}}(\rho(k+1))$ would be a finite branch in f of length $\rho(k+1)$ which is more to the left than σ (since $\tilde{b}(i) = b(i) = (\sigma)_i$ for all $i < i_0$) which contradicts the definition of σ.
Case 2: $b(i_0) < \tilde{b}(i_0)$. Then b would be a more to the left infinite path in f than $\tilde{b}$ contradicting the definition of $\tilde{b}$. □

As shown in [21, Theorem 4.1], a modulus of regularity always yields (given an approximate solution bound) a rate of convergence for Fejér monotone algorithms computing approximate solutions. The algorithm in the proof of Proposition 2.8, in fact, is an instance of this:

Proposition 2.9. *Let for a given infinite binary tree f, (x_k) be defined as follows: $x_k := \sigma_k * 0^1$, where σ_k is the lefmost finite branch of length k in f. Then (x_k) is Fejér monotone w.r.t. the set S of infinite paths through f.*

Proof: Let $b \in S$. We have to show that

$$\forall k \in \mathbb{N}\,(d(x_{k+1}, b) \leq d(x_k, b)).$$

This in turn follows from
Claim: $\forall k, m \in \mathbb{N}\,(\overline{x}_k m = \overline{b}m \to \overline{x}_{k+1} m = \overline{b}m)$.

Proof of Claim: let $k, m \in \mathbb{N}$ be fixed and assume that $\overline{x}_k m = \overline{b}m$. Suppose that for some $i < m$ we would have that $x_{k+1}(i) \neq b(i)$ and let i_0 be the least such i. Note that

$\forall j < i_0\,(x_k(j) = b(j) = x_{k+1}(j))$.
Case 1: $i_0 \le k$. Since σ_{k+1} is the leftmost branch of length $k+1$ while σ_k is leftmost of length k, it follows (using that $x_k(k) = 0$) that x_k is to the left of x_{k+1} so that $b(i_0) = x_k(i_0) \le x_{k+1}(i_0)$. Since $\overline{b}(k+1)$ is some finite branch of length $k+1$, σ_{k+1} is left of it and so $x_{k+1}(i_0) \le b(i_0)$. Hence in total $x_{k+1}(i_0) = b(i_0)$ which is a contradiction to the definition of i_0.
Case 2: $i_0 > k$. Then by definition $x_{k+1}(i_0) = 0 = x_k(i_0) = b(i_0)$ which again contradicts the definition of i_0. □

Definition 2.10. *1. The Σ^0_1-law-of-excluded-middle principle (with function parameters) is defined as*[4]

$$\Sigma^0_1\text{-LEM}:\ \forall f\ \left(\forall n^0\,(f(n)=0) \vee \exists n^0\,(f(n)\neq 0)\right).$$

2. The principle of binary choice for Π^0_1-formulas is defined as

$$\Pi^0_1\text{-AC}_{\le 1}:\ \forall f\ \left(\forall n^0\,\exists m \le_0 1\,\forall k^0\,(f(n,m,k)=0) \to \exists g \le_1 \lambda x.1\,\forall n,k\,(f(n,g(n),k)=0)\right).$$

In the following, EL denotes the system of elementary intuitionistic analysis from [26].

Theorem 2.11. *1.* EL+Σ^0_1-LEM+Π^0_1-AC$_{\le 1}$ *proves that for every compact metric space $X = \widehat{A}$ any continuous mapping $F: X \to \mathbb{R}$ having a zero is regular w.r.t. zer F.*

2. Already for Lipschitz continuous functions $F: [0,1] \to \mathbb{R}$ with zer $F \neq 0$, the regularity of F w.r.t. zer F implies Σ^0_1-LEM over EL.

Proof: 1) Inspection of the proof of [17, Theorem 4.2(1)] (see also [17, Remark 4.3]) shows that the claim can be established with induction, Σ^0_1-LEM and WKL. WKL in turn is provable in EL+LLPO+$\Pi^0_1-\text{AC}^{0,0}_{\le 1}$ as follows from the proof of [13, Theorem 3]. LLPO trivially follows from Π^0_1-LEM and hence - a fortiori - from Σ^0_1-LEM. In total the claim of the theorem follows.
2) We refine the proof of [17, Theorem 4.4(2)], which classically shows that the existence of a modulus of regularity implies the convergence of bounded monotone sequences of rationals in $[0,1]$ (and hence arithmetical comprehension ACA$_0$ by [24, Theorem III.2.2]), to get that intuitionistically the ε/δ-form of regularity implies the Cauchy property of bounded monotone sequences which is known to imply Σ^0_1-LEM over EL (see [25, Theorem 2.(ii)], which in turn refers to [29, 5.4.4], and the more recent [7]). Let (a_n) be a nondecreasing sequence of rational numbers in $[0,1]$ and $f, T: [0,1] \to [0,1]$ be nonexpansive as defined in the proof of [17, Theorem 4.4(2)] and $x_n := T^n 0$. Now suppose that the Lipschitz-2 function $F: [0,1] \to \mathbb{R},\ F(x) := |x - Tx|$ is $\forall\varepsilon\exists\delta$-regular and note that $1 \in \text{zer } F \neq \emptyset$. Since (x_n) is Fejér monotone w.r.t. zer $F = Fix(T)$ and asymptotically regular, i.e. $|x_n - Tx_n| \to 0$, and, in fact, with rate of convergence $n+3$ (see the proof of [17, Theorem 4.4(2)]) it follows

[4] For a proof-theoretic study of this principle (as a first-order principle without function variables) and its computational interpretation see [1] and [2] respectively.

that (x_n) is a Cauchy sequence: let $k \in \mathbb{N}$ be fixed and $n \in \mathbb{N}$ by the $\forall\varepsilon\exists\delta$-regularity be such that for all

$$\forall x \in [0,1] \left(|x - Tx| < 2^{-n} \to \exists p \in Fix(T)\,(|p-x| < 2^{-k-1}) \right).$$

Then $|x_{n+3} - p| < 2^{-k-1}$ for some $p \in Fix(T)$ and so by the Fejér monotonicity of (x_n)

$$\forall m \geq n+3\,(|x_m - p| < 2^{-k-1})$$

which implies - as k was arbitrary - that

$$(*)\ \forall k \in \mathbb{N}\,\exists n\,\forall m, \tilde{m} \geq n\,(|x_m - x_{\tilde{m}}| < 2^{-k}).$$

Let $k \in \mathbb{N}$ be fixed again and n_k be such that $(*)$ holds for k. For $C \in \mathbb{N}$, define $n_C := \max\{n_k, k+C+3\}$. Then

$$|x_{n_C} - Tx_{n_C}| < 2^{-k-C}$$

and so - by $T(x) = \frac{1}{2}(x + f(x))$ -

$$|x_{n_C} - f(x_{n_C})| < 2^{-k-C+1}$$

which in turn - by the f_n-definition used to define f - implies that

$$\forall l \leq C\,(a_l < x_{n_C} + 2^{-k})$$

and so - since $n_C \geq n_k$ -

$$\forall l \leq C\,(a_l < x_{n_k} + 2^{-k+1}).$$

Since $C \in \mathbb{N}$ was arbitrary, we get

$$(**)\ \forall l \in \mathbb{N}\,(a_l < x_{n_k} + 2^{-k+1}).$$

Let $l_k \in \mathbb{N}$ be so large that $2^{-l_k} \cdot n_k < 2^{-k}$. Then - reasoning as in [17], p.383, lines 3-7 -

$$(***)\ a_{l_k} \geq x_{n_k} - 2^{-k}.$$

Indeed, assume that $a_{l_k} < x_{n_k} - 2^{-k}$ and so - since (a_n) is nondecreasing

$$\forall l \leq l_k\,(a_l < x_{n_k} - 2^{-k}).$$

By induction on n we show that

$$(+)\ \forall n \in \mathbb{N}\,(x_n \leq x_{n_k} - 2^{-k} + n \cdot 2^{-l_k}):$$

$x_0 = 0 \leq a_0 \leq x_{n_k} - 2^{-k}$. For the induction step (using that $x_n, a_l \leq 1$):

$$\begin{aligned} x_{n+1} &= \tfrac{1}{2}\left(x_n + \sum_{l=0}^{\infty} 2^{-l-1} \max\{x_n, a_l\}\right) \\ &\overset{\text{I.H., assumption}}{\leq} \tfrac{1}{2}\left(x_{n_k} - 2^{-k} + n \cdot 2^{-l_k} + (x_{n_k} - 2^{-k} + n \cdot 2^{-l_k}) \sum_{l=0}^{\infty} 2^{-l-1} + \sum_{l=l_k+1}^{\infty} 2^{-l-1}\right) \\ &= x_{n_k} - 2^{-k} + n \cdot 2^{-l_k} + \tfrac{1}{2} 2^{-l_k-1} < x_{n_k} - 2^{-k} + (n+1) \cdot 2^{-l_k}. \end{aligned}$$

(+) applied to $n := n_k$ yields that

$$x_{n_k} \leq x_{n_k} - 2^{-k} + n_k \cdot 2^{-l_k} < x_{n_k}$$

which is a contradiction and, therefore, establishes $(***)$.
So by $(**)$ and $(***)$ together (using again that (a_n) is nondecreasing) we have shown that

$$\forall l \geq l_k \left(a_l \in [x_{n_k} - 2^{-k}, x_{n_k} + 2^{-k+1}] \right)$$

which yields that (a_n) is a Cauchy sequence. $\square$

As shown in [17], with classical logic (and WKL), Σ^0_1-IA suffices to prove the ε/δ-regularity in the compact case. However, then Σ^0_2-DNE

$$\forall f \ (\neg\neg\exists n \forall k \, (f(n,k) = 0) \rightarrow \exists n \forall k \, (f(n,k) = 0))$$

seems to be needed for the proof. To weaken the latter principle to Σ^0_1-LEM, one apparently needs a somewhat stronger induction in order to establish the principle of bounded Σ^0_1-comprehension

$$\Sigma^0_1\text{-BCA}: \ \forall f, k \, \exists \sigma \ (lth(\sigma) = k \wedge \forall i < k \, ((\sigma)_i = 0 \leftrightarrow \exists n \, (f(i,n) = 0)))$$

which was introduced under the name of AS^{Σ}_1 in [23] and studied with the name above in [7]. The situation is, therefore, analogous to that of the Cauchy property of bounded monotone sequences in $[0,1]$ studied under the name of PCM_{ar} in [12] and [25] and - under the name of MCT^- in [7]: Σ^0_1-BCA, which implies both Σ^0_1-LEM and Σ^0_1-IA over EL restricted to quantifier-free induction (and even weaker systems; see [7]), and which is implied by Σ^0_1-LEM and IA, suffices for the proof Theorem 2.11.1) but it is open whether here IA can be weakened to Σ^0_1-IA. This, of course, does not come as a surprise since - as the proof of Theorem 2.11.2) shows - metric regularity intuitionistically implies PCM_{ar} (the case of sequences of reals can easily be reduced to the one for rationals using rational approximations as in the proof of [12, Proposition 5.2(1)]).

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