FOUNDATIONAL AND MATHEMATICAL USES OF HIGHER TYPES

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§1. Introduction. A central theme of proof theory is expressed by the following question:

'What parts of ordinary mathematics (in particular of analysis) can be carried out in certain restricted formal systems?'

The relevance of this question is twofold:

1) **Foundational relevance:** suppose a formal system $\mathcal{T}_{\mathbf{PA}}$ allows one to formalize a great amount of mathematics but can be shown (by restricted means) to be a conservative extension of first order Peano Arithmetic **PA**, then that part of mathematics has an arithmetical foundation (partial realization of H. Weyl's program, see S. Feferman's discussion in [8]).
   If we work in a system $\mathcal{T}_{\mathbf{PRA}}$ which can be shown (finitistically) even to be conservative over Primitive Recursive Arithmetic **PRA** and identify (following [36]) **PRA** with finitism, then the parts of mathematics which can be carried out in $\mathcal{T}_{\mathbf{PRA}}$ have a finitistic foundation (partial realization of D. Hilbert's program, see e.g. [34]).
2) **Mathematical relevance:** here the guiding question is

   'What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?' (G. Kreisel)

   The aim is to get additional mathematical information out of the fact that a certain theorem **S** has been proved by certain restricted means. Such additional information may be the extractability of a realizing construction for an existential statement or of an algorithm or a numerical bound for a $\forall\exists$-theorem by unwinding the given proof.

Both motivations are of course closely related and research on them has mutually influenced each other: e.g. a proof of a Π^0_2-theorem carried out in a

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system which can be (effectively) reduced to **PRA** allows one to extract at least a primitive recursive algorithm. In the other direction, e.g. our analysis of proofs in approximation theory (which used the principle of the attainment of the maximum of $f \in C[0,1]$, see [20]) led us to an elimination procedure of weak König's lemma WKL over a variety of subsystems of arithmetic in all finite types thereby contributing to '1)' above (see [19]). Likewise our treatment of e.g. the Bolzano-Weierstraß principle in [26] via an elimination technique of Skolem functions yielded also new conservation results for comprehension principles ([27]).

However, there are also important differences due to the different points emphasized in 1) and 2):

Whereas there are hardly foundational (understood in the sense of Hilbert) reasons to study systems weaker than **PRA**, merely primitive recursive algorithms and bounds are in most cases much too complex to be of any mathematical value. So on the one hand further restrictions are needed to guarantee the extractability of mathematically more interesting data whereas on the other hand e.g. proofs of large classes of lemmas (having a certain logical form) can be shown not to contribute to the complexity or growth of algorithms or bounds extracted from proofs of theorems using these lemmas. Hence such lemmas can be treated simply as axioms (no matter how non-constructive their proofs might be) in the course of the analysis of a given proof. Also, for successful unwindings the complexity of the proof transformations used is critical. It has turned out that methods using functionals of finite type like appropriate versions of Gödel's functional interpretation or modified realizability combined with tools like negative translation and/or the Friedman-Dragalin translation are most useful (in particular compared to techniques which try to avoid any passage through higher types, see [28]).

Whereas we have focused on '2)' in several publications (see [21],[20],[24] among others), this paper addresses '1),' to which S. Feferman has contributed so profoundly. We study mathematical strong, but nevertheless **PRA**-reducible, systems in all finite types, emphasizing the need of third order variables already for a faithful formalization of continuous functions between Polish spaces. We investigate the relationship between the direct representation of continuous functions (which is possible in the presence of third order variables but not in systems based on the language of second order arithmetic) with their representation via certain codes used in the second order context of reverse mathematics (short: r.m.-codes). It turns out that not even a finite type extension of a second order system like RCA together with the axiom of quantifier-free choice suffices to prove that every continuous function

$\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ has an r.m.-code. So the encoding used in reverse mathematics tacitly yields a constructively enriched representation of continuous functions. More precisely, already for continuous functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, the representation used in reverse mathematics entails the existence of a (continuous) modulus of pointwise continuity functional. In the presence of arithmetical comprehension, the difference between both representations disappears, since the existence of such a modulus of pointwise continuity can be proved using arithmetical comprehension and QF-AC^{1,0}. For the **restriction** of continuous functionals $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ to the **Cantor space** one can show using WKL that such a code exists. This follows from a construction due to D. Normann which recently was communicated to the author. So in the presence of WKL, the constructively enriched encoding of continuous functions used in reverse mathematics can be shown to be faithful (i.e. not to be a genuine enrichment) for functions on the Cantor space (and more generally on **compact** Polish spaces). It remains open whether this is true also without WKL. The higher order context of our systems also allows one to investigate the relation between ε-δ-continuity and sequential continuity of functions (in reverse mathematics, the use of r.m.-codes by definition enforces ϵ-δ-continuity). It turns out that a form of quantifier-free choice (available in our systems) suffices to prove even the local pointwise equivalence between both concepts.
These types of results can not even be formulated in a second order setting. The use of higher order (at least third order) variables is therefore necessary to address the question of how closely various representations of anaytical objects correspond to their ordinary mathematical definitions and to develop a general theory of representations.

Let us recall very briefly some of the history of research on '1)'. As Feferman pointed out in [7], 'Hermann Weyl initiated a program for the arithmetical foundation of mathematics' in his book 'Das Kontinuum' ([40]). In this book, Weyl observed that large parts of analysis can be developed on the basis of **arithmetical** comprehension. This theme was further developed in the 50's by P. Lorenzen among others. In the late 70's Feferman [5] and G. Takeuti [37] independently designed formal systems based on arithmetical comprehension in the framework of higher order arithmetic which are conservative over **PA**. For this property it is important that the schema of induction is restricted to arithmetical formulas only.[1] Work on the program of so-called reverse mathematics by H. Friedman, S. Simpson and others has shown that almost all of

[1]As was shown by Feferman in [5], the corresponding system with full induction is proof-theoretically stronger than **PA**. In [37], Takeuti considers in addition the variant where no parameters (except arithmetical parameters) are permitted in the schema of arithmetical

the mathematics that can be developed based on arithmetical comprehension at all can also be carried out if induction is restricted in this way. This work uses a second order fragment $\mathbf{ACA}_0$ (formulated in the language of second order arithmetic) of the system from [5] (which is formulated in the language of functionals of all finite types). Via appropriate representations and codings of higher objects (like continuous functions between Polish spaces) a great deal of mathematics can be developed already in $\mathbf{ACA}_0$ (see [35] for a comprehensive treatment).

Feferman's system, however, allows a more **direct** treatment of such objects and their mathematics and also contains a strong uniform ('explicit') version of arithmetical comprehension via a non-constructive μ-operator. These features hold in an even stronger form for theories with flexible (variable) types which were developed successively by Feferman in his framework of explicit mathematics in [4],[6],[7] culminating in a formal system called **W** (where 'W' stands for 'Weyl') which was shown to be proof-theoretically reducible to and conservative over **PA** in [11]. The enormous mathematical power and flexibility of the system **W** led Feferman in [9] to the formulation of the thesis that all (or almost all) scientifically applicable mathematics can be developed in **W**.

In the late 70's, H. Friedman observed that large parts of the mathematics that can be carried out in $\mathbf{ACA}_0$ are already formalizable in a subsystem $\mathbf{WKL}_0$ which instead of the schema of arithmetical comprehension is based on the binary König's lemma (for quantifier-free trees) WKL and Σ^0_1-induction only (see again [35] for a comprehensive treatment of ordinary mathematics in $\mathbf{WKL}_0$). This fact is of foundational relevance since $\mathbf{WKL}_0$ can be proof-theoretically reduced to and is Π^0_2-conservative over **PRA** (H. Friedman (1976, unpublished) and [33]; for a historical discussion which in particular points out various errors in the literature on WKL see [23] (p.69)).

In [19] we introduced an extension (in the spirit of Feferman's **PA**-conservative system from [5] mentioned above) of $\mathbf{WKL}_0$ to all finite types and proved among other things that this extension still can be proof-theoretically reduced to **PRA** and is Π^0_2-conservative over **PRA**.

Although this extension is already much more flexible than the system $\mathbf{WKL}_0$, the use of WKL still requires a complicated encoding of analytical objects. While working on '2)' mentioned above and investigating what parts of analysis produce only provable recursive function(al)s which can be bounded by polynomials (see [24] for a survey) we faced the problem that already the

comprehension. In this case the resulting system is conservative over **PA** even in the presence of the full schema of induction.

formulation of WKL involves coding devices of exponential growth. That is why we introduced a non-standard axiom F which together with some form of quantifier-free choice proves a strong principle of uniform boundedness Σ^0_1-UB which allows one to give short proofs of the usual WKL-applications in analysis relative to very weak (polynomially bounded) systems (see [23],[25]) but does not contribute to the growth of provably recursive functionals. This axiom as well as the principle of uniform boundedness is 'non-standard' in the sense that it is not true in the full set-theoretic type structure. Nevertheless all of its analytic (i.e. second order) consequences are true. In [23] we also studied a restricted version F^- of F which yields a correspondingly restricted version of uniform boundedness which is sufficient for many applications (although more complicated to use, see [25]) but which allows a very easy proof-theoretical elimination. In section 3 of this paper we show that in the presence of the axiom of extensionality and a form of quantifier-free choice, F actually is implied by F^- so that in this context (which we use throughout this paper) the F^--elimination applies to proofs based on F as well. The proof of this fact uses an argument due to Grilliot [14]. The result allows one to construct a **PRA**-reducible finite type system $\mathcal{T}^*$ which is based on Σ^0_1-UB. The relevance of this is due to fact that $\mathcal{T}^*$ allows one to develop the analysis of continuous functions between Polish spaces treating such functions directly as certain type-2-functionals and to prove all the usual WKL-consequences known from reverse mathematics without the passage through the encoding of such objects used in reverse mathematics and without formulating explicitly any continuity assumptions.

In section 5-7 we show that Σ^0_1-UB not only allows one to give shorter (coding-free) proofs for usual WKL-applications but also allows one to prove new – classically valid – third order principles which are not derivable from WKL. We develop a non-collapsing hierarchy Φ_n-WKL$_+$ of extensions of WKL. Basically, Φ_n-WKL$_+$ extends WKL from binary trees which are given by quantifier-free predicates to binary trees which are given by formulas belonging to a larger class Φ_n (see section 5 below for details). Φ_0-WKL$_+$ is equivalent to WKL, but for $n \geq 1$, Φ_n-WKL$_+$ is not even provable in E-PA$^\omega$+QF-AC$^{1,0} + \mu$ (here μ is Feferman's non-constructive μ-operator mentioned above). Nevertheless, Φ_n-WKL$_+$ is provable in $\mathcal{T}^*$ for all $n \in \mathbb{N}$ so that by the results mentioned before the whole hierarchy can be reduced proof-theoretically to **PRA**.

One might also ask for an explicit version (with flexible types) of such systems based on (extensions of) WKL. However, things are quite delicate in this case as for the uniform ('explicit') version UWKL of WKL (analogously to the uniform version of arithmetical comprehension given by μ), the strength of the resulting system crucially depends on the amount of extensionality available (see [30]).

§2. Description of the theories E-G$_n$A$^\omega$, E-PRA$^\omega$ and E-PA$^\omega$. The set $\mathbf{T}$ of all finite types is defined inductively by

$$(i)\ 0 \in \mathbf{T} \text{ and } (ii)\ \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type ρ to objects of type τ.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

$$(i)\ 0 \in \mathbf{P} \text{ and } (ii)\ \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}.$$

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write $0(00)$ instead of $0(0(0))$. Furthermore we write for short $\tau\rho_k \ldots \rho_1$ instead of $\tau(\rho_k)\ldots(\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n+1$.

Our theories $\mathcal{T}$ used in this paper are based on many–sorted classical logic formulated in the language of functionals of all finite types plus the combinators $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$ which allow the definition of λ–abstraction.
The systems E-G$_n$A$^\omega$ (for all $n \geq 1$) are introduced in [23] to which we refer for details. E-G$_n$A$^\omega$ has as primitive relations $=_0, \leq_0$ for objects of type 0, the constant 0^0, functions $\min_0, \max_0, S$ (successor), $A_0, \ldots, A_n$, where A_i is the i–th branch of the Ackermann function (i.e. $A_0(x,y) = y', A_1(x,y) = x+y, A_2(x,y) = x \cdot y, A_3(x,y) = x^y, \ldots$), functionals of degree 2: $\Phi_1, \ldots, \Phi_n$, where $\Phi_1 fx = \max_0(f0, \ldots, fx)$ and Φ_i is the iteration of A_{i-1} on the f–values for $i \geq 2$, i.e. $\Phi_2 fx = \sum\limits_{i=0}^{x} fi, \Phi_3 fx = \prod\limits_{i=0}^{x} fi, \ldots$. We also have a bounded search functional μ_b and bounded predicative recursion provided by recursor constants $\tilde{R}_\rho$ (where 'predicative' means that recursion is possible only at the type 0 as in the case of the (unbounded) Kleene-Feferman recursors $\widehat{R}_\rho$).
In this paper our systems always contain the axioms of extensionality

$$(E):\ \forall x^\rho, y^\rho, z^{\tau\rho}(x =_\rho y \to zx =_\tau zy)$$

for all finite types ($x =_\rho y$ is defined as $\forall z_1^{\rho_1}, \ldots, z_k^{\rho_k}(xz_1 \ldots z_k =_0 yz_1 \ldots z_k)$ where $\rho = 0\rho_k \ldots \rho_1$).
In [23] we had in addition to the defining axioms for the constants of our theories all true sentences having the form $\forall x^\rho A_0(x)$, where A_0 is quantifier–free

and $deg(\rho) \leq 2$, added as axioms.[2]
By 'true' we refer to the full set–theoretic model $\mathcal{S}^\omega$. In given proofs of course only very special universal axioms are used which can be proved in suitable extensions of our theories. Nevertheless one can include them all as axioms if one is only interested in the applied aspect '2)' discussed above, since they (more precisely their proofs) do not contribute to the provable recursive function(al)s of the system. In particular this covers all instances of the schema of quantifier-free induction. In this paper, however, we include only the schema of quantifier-free induction to E-$\mathrm{G}_n\mathrm{A}^\omega$ instead of taking arbitrary universal axioms, since we are interested in proof-theoretical reductions.

E-PRA^ω results if we add the functional

$$\Phi_{it}0yf =_0 y,\ \Phi_{it}x'yf =_0 f(x, \Phi_{it}xyf)$$

to E-$\mathrm{G}_\infty\mathrm{A}^\omega := \bigcup_{n\in\omega}$ E-$\mathrm{G}_n\mathrm{A}^\omega$. The system E-$\mathrm{PRA}^\omega$ is equivalent to Feferman's system E-$\widehat{\mathrm{PA}}^\omega\restriction$ from [5] since Φ_{it} allows (relative to E-$\mathrm{G}_\infty\mathrm{A}^\omega$) to define the predicative recursor constants $\widehat{R}_\rho$ (see [23]).
E-PA^ω is the extension of E-PRA^ω obtained by the addition of the schema of full induction and all (impredicative) primitive recursive functionals in the sense of [13].

The schema of full choice is given by

$$\mathrm{AC}^{\rho,\tau}\ :\ \forall x^\rho \exists y^\tau A(x,y) \to \exists Y^{\tau(\rho)} \forall x^\rho A(x, Yx),\quad \mathrm{AC}\ := \bigcup_{\rho,\tau\in\mathbf{T}} \{\, \mathrm{AC}^{\rho,\tau}\}.$$

The schema of **quantifier-free choice** QF-$\mathrm{AC}^{\rho,\tau}$ is defined as the restriction of $\mathrm{AC}^{\rho,\tau}$ to quantifier-free formulas A_0.[3]

REMARK 2.1.

E-PRA^ω+QF-$\mathrm{AC}^{0,0} \vdash \Sigma^0_1$-IA, Δ^0_1-CA, *where*

$$\Sigma^0_1\text{-IA: } \exists y^0 A_0(0,y) \wedge \forall x^0 \big(\exists y^0 A_0(x,y) \to \exists y^0 A_0(x',y)\big) \to \forall x^0 \exists y^0 A_0(x,y),$$

and

$$\Delta^0_1\text{-CA: } \forall x^0 \big(\exists y^0 A_0(x,y) \leftrightarrow \forall y^0 B_0(x,y)\big) \to \exists f^1 \forall x^0 \big(fx = 0 \leftrightarrow \exists y^0 A_0(x,y)\big),$$

with A_0, B_0 quantifier-free (parameters of arbitrary types allowed).
Hence the system RCA_0 *from reverse mathematics (see* [35]*) can be viewed as a subsystem of* E-PRA^ω+QF-$\mathrm{AC}^{0,0}$ *by identifying sets $X \subseteq \mathbb{N}$ with their characteristic function.*

[2]The restriction $deg(\rho) \leq 2$ has a technical reason discussed in [23].
[3]Throughout this paper $A_0, B_0, C_0, \ldots$ denote quantifier-free formulas.

The theory $\mathcal{T}+\mu$ results from $\mathcal{T}$ if we add the non-constructive μ-operator μ^2 to $\mathcal{T}$ together with the characterizing axiom

$$\mu(f) = \begin{cases} \text{the least } x \text{ such that } f(x) =_0 0, \text{ if } \exists x^0(f(x) =_0 0) \\ 0, \text{ otherwise.} \end{cases}$$

Notation: For $\rho = 0\rho_k \ldots \rho_1$, we define $1^\rho := \lambda x_1^{\rho_1} \ldots x_k^{\rho_k}.1^0$, where $1^0 := S0$.

Definition 2.2. 1) *Between functionals of type ρ we define the relation $\leq_\rho$:*

$$\begin{cases} x_1 \leq_0 x_2 :\equiv x_1 \leq x_2, \\ x_1 \leq_{\tau\rho} x_2 :\equiv \forall y^\rho (x_1 y \leq_\tau x_2 y); \end{cases}$$

2) $\min_{\rho\tau}(x_1^{\rho\tau}, x_2^{\rho\tau}) := \lambda y^\tau . \min_\rho(x_1 y, x_2 y)$, *with min_0 from above.*

In the following we will need the definition of the binary ('weak') König's lemma as given in [39]:

Definition 2.3 (Troelstra(74)).

$$\text{WKL}:\equiv \begin{cases} \forall f^1 \big(T(f) \wedge \forall x^0 \exists n^0 (lth\ n =_0 x \wedge fn =_0 0) \\ \qquad \to \exists b \leq_1 \lambda k.1 \forall x^0 (f(\overline{b}x) =_0 0)\big), \ \textit{where} \end{cases}$$

$Tf :\equiv \forall n^0, m^0 (f(n * m) =_0 0 \to fn =_0 0) \wedge \forall n^0, x^0 (f(n * \langle x \rangle) =_0 0 \to x \leq_0 1)$ *(i.e. $T(f)$ asserts that f represents a 0,1–tree).*

§3. On two non-standard principles. In this section we in particular prove a new conservation result for the non-standard axiom F which was introduced first in [23][4] (and has been applied e.g. in [25]):

$$F :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0 \forall z \leq_1 yk \big(\Phi k z \leq_0 \Phi k (y_0 k)\big).$$

We call this axiom 'non-standard' since it does not hold in the full set-theoretic type structure $\mathcal{S}^\omega$. Nevertheless its use can be eliminated from certain proofs thereby yielding classically true results. This has been discussed extensively in [23] to which we refer for further information. In that paper we mainly made use of a weaker version F^- of F which allows a direct proof-theoretic elimination whereas the elimination of F was based on a model-theoretic argument. In this paper however we need the full version F. We show – using an argument known as Grilliot's trick in the context of recursion theory for the countable functionals (see [14])[5] – that in the fully extensional context of

[4] A special case of F was studied already in [21] and called also F in that paper but F_0 in [23].

[5] This argument recently has had a further proof-theoretic applications in [30] and [31].

theories like E-PRA$^\omega$+QF-AC^{1,0}, F^- actually implies F. This allows one to apply the proof-theoretic elimination of F^- to F thereby strengthening results in [23].
We apply F via one of its consequences, the following principle of uniform Σ^0_1-boundedness:

DEFINITION 3.1 ([23]). *The schema*[6] *of* **uniform Σ^0_1–boundedness** *is defined as*

$$\Sigma^0_1\text{–UB}: \begin{cases} \forall y^{1(0)}\big(\forall k^0 \forall x \leq_1 yk \exists z^0\ A(x,y,k,z) \\ \qquad\qquad \to \exists \chi^1 \forall k^0 \forall x \leq_1 yk \exists z \leq_0 \chi k\ A(x,y,k,z)\big), \end{cases}$$

where $A \equiv \exists \underline{l} A_0(\underline{l})$ and $\underline{l}$ is a tuple of variables of type 0 and A_0 is a quantifier–free formula (which may contain parameters of arbitrary types).

PROPOSITION 3.2 ([23]). *Let $\mathcal{T}$:=E-G$_n$A$^\omega$ ($n \geq 2$), E-PRA$^\omega$ or E-PA$^\omega$. Then* $\mathcal{T}$+QF-AC^{1,0} + $F \vdash \Sigma^0_1$-UB.

PROPOSITION 3.3 ([23]). E-G$_3$A$^\omega$ + Σ^0_1-UB $\vdash$ WKL.

Σ^0_1-UB implies the existence of a modulus of uniform continuity for each extensional $\Phi^{1(1)}$ on $\{z^1 : z \leq_1 y\}$ (where 'continuity' refers to the usual metric on the Baire space $\mathbb{N}^\mathbb{N}$):

PROPOSITION 3.4 ([23]).

$$\begin{aligned} &\text{E-G}_2\text{A}^\omega + \Sigma^0_1\text{-UB}\ \vdash \\ &\forall \Phi^{1(1)} \forall y^1 \exists \chi^1 \forall k^0 \forall z_1, z_2 \leq_1 y\Big(\bigwedge_{i \leq_0 \chi k}(z_1 i =_0 z_2 i) \to \bigwedge_{j \leq_0 k}(\Phi z_1 j =_0 \Phi z_2 j)\Big). \end{aligned}$$

REMARK 3.5. *The argument above can actually be used to show that a sequence of functionals $\Phi_i^{1(1)}$ has a sequence of moduli of uniform continuity on a sequence of sets $\{z : z \leq_1 y_i\}$.*

As mentioned above, in [23] we mainly studied a weaker version

$$\begin{aligned} &F^- :\equiv \\ &\forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0, z^1, n^0\Big(\bigwedge_{i <_0 n}(zi \leq_0 yki) \to \Phi k(\overline{z,n}) \leq_0 \Phi k(y_0 k)\Big) \end{aligned}$$

(where, for $z^{\rho 0}$, $(\overline{z,n})(k^0) :=_\rho zk$, if $k <_0 n$ and $:= 0^\rho$, otherwise) of F and gave a proof-theoretic elimination procedure for the use of F^- which – relative

[6] Σ^0_1-UB can be written as a single axiom. However the schematic version is easier to apply.

to so-called weakly extensional variants WE-$\mathrm{G}_n\mathrm{A}^\omega$+QF-AC of our systems E-$\mathrm{G}_n\mathrm{A}^\omega$+QF-$\mathrm{AC}^{1,0}$+QF-$\mathrm{AC}^{0,1}$ – applies for quite general classes of formulas. In the presence of the full extensionality axiom (E) we got corresponding results if the types involved were somewhat restricted. We now show that in the presence of (E), F is already implied by F^- and so that these results extend to F as well.

Proposition 3.6. E-$\mathrm{G}_2\mathrm{A}^\omega$+QF-$\mathrm{AC}^{1,0} + F^- \vdash F$.

Proof: We argue in E-$\mathrm{G}_2\mathrm{A}^\omega$+QF-$\mathrm{AC}^{1,0} + F^-$. It is clear that F follows from F^- if

$$(1)\ \forall \Phi^2 \forall f^1 \exists n^0 \big(\Phi(f) = \Phi(\overline{f,n})\big).$$

So let's suppose that on the contrary there exist Φ^2 and f such that

$$(2)\ \forall n^0 \big(\Phi(f) \neq \Phi(\overline{f,n})\big).$$

Then for $f_i := \overline{f, i+1}$ we have

$$(3)\ \forall i^0 \forall j \geq i \big(f_j(i) =_0 f(i)\big)$$

and

$$(4)\ \forall i^0 \big(\Phi(f) \neq \Phi(f_i)\big).$$

Define $\Psi g^1 :=_0 \begin{cases} 1, \text{ if } \Phi(g) \neq \Phi(f) \\ 0, \text{ if } \Phi(g) = \Phi(f). \end{cases}$ Then

$$(5)\ \forall i, j \big(\Psi(f_i) =_0 \Psi(f_j) \neq \Psi(f)\big).$$

Now one can apply an argument from [14], which can be formalized in E-$\mathrm{G}_2\mathrm{A}^\omega$ (see [30] for details on this and a further proof-theoretic application of that argument), to derive

$$(6)\ \exists \varphi^2 \forall g^1 \big(\varphi(g) = 0 \leftrightarrow \exists x (gx = 0)\big)$$

from (3) and (5). (6), however, contradicts F^- (relative to E-$\mathrm{G}_2\mathrm{A}^\omega$+QF-$\mathrm{AC}^{1,0}$), since F^- implies that every Φ^2 is bounded on the set of all functions $\overline{g,n}$ with $g \leq_1 1, n \in \mathbb{N}$, whereas QF-$\mathrm{AC}^{1,0}$ together with (6) yields the existence of a functional μ such that

$$(7)\ \forall g^1 \big(\exists x^0 (gx = 0) \to g(\mu(g)) = 0\big),$$

which obviously is unbounded on this set. □

THEOREM 3.7. *Let* $\forall f^1, x^0 \exists y^0 A_0(f,x,y)$ *be a sentence of the language of* $\mathcal{T}$ *where* $\mathcal{T}$:=E-G$_n$A$^\omega$ $(n \geq 2)$, E-PRA$^\omega$ *or* E-PA$^\omega$. *Then the following rule holds*

$$\begin{cases} \mathcal{T} + \text{QF-AC}^{1,0} + \text{QF-AC}^{0,1} + F \vdash \forall f^1, x^0 \exists y^0 A_0(f,x,y) \\ \Rightarrow \text{ one can extract a closed term } \Psi^{001} \text{ of } \mathcal{T} \text{ such that} \\ \mathcal{T}^* \vdash \forall f^1, x^0 A_0(f,x,\Psi f x), \end{cases}$$

where $\mathcal{T}^*$:=E-G$_3$A$^\omega$ *if* $\mathcal{T}$ =E-G$_2$A$^\omega$ *and* := $\mathcal{T}$*, otherwise.*
In particular, if $\mathcal{T}$ =E-G$_2$A$^\omega$ (E-G$_3$A$^\omega$ *resp.* E-PRA$^\omega$) *and if 'f' is not present, then* Ψx *is bounded by a polynomial in* x *(is an elementary recursive resp. primitive recursive function in* x*).*

Proof: The theorem follows from proposition 3.6 together with theorem 4.21 from [23]. □

§4. Continuous functions: direct representations versus codes. A functional $\Phi^{1(1)}$ is continuous at x^1 if

$$\forall k^0 \exists n^0 \forall y^1 \big(\bigwedge_{i=0}^{n} (xi =_0 yi) \to \bigwedge_{j=0}^{k} (\Phi x j =_0 \Phi y j)\big).$$

Φ is continuous if it is continuous at every x.
Using a suitable so-called standard representation of complete separable metric ('Polish') spaces X (which in turn relies on a representation of real numbers as Cauchy sequences of rational numbers with fixed rate of convergence), elements of X can be represented by number-theoretic functions x^1 and, moreover, every such function can be considered as a representative of a uniquely determined element of X (see [2] and [20] for details). On these representatives we have a pseudo metric d_X. The elements of X can be identified with the equivalence classes w.r.t. $x =_Y x :\equiv (d_X(x,y) =_{\mathbb{R}} 0)$. Functions $G : X \to Y$ between Polish spaces therefore are just given by functionals $\Phi_G^{1(1)}$ which respect $=_X$, $=_Y$, i.e.

$$\forall x^1, y^1 (x =_X y \to \Phi_G x =_Y \Phi_G y).$$

Φ_G represents a continuous function $G : X \to Y$ if

$$\forall x^1 \forall k^0 \exists n^0 \forall y^1 \big(d_X(x,y) \leq_{\mathbb{R}} \frac{1}{n+1} \to d_Y(\Phi_G x, \Phi_G y) \leq_{\mathbb{R}} \frac{1}{k+1} \big).$$

This definition is just the usual ε-δ–definition of continuous functions. One could also consider to define continuity as sequential continuity. In the presence

of QF-AC^{0,1} (which is included in all the systems we consider in this paper) both definitions are equivalent as we will show now.
As usual $G : X \to Y$ is called **sequentially continuous** in x iff

$$\forall x^{1(0)}_{(\cdot)}\big(\lim_{n\to\infty} x_n =_X x \to \lim_{n\to\infty} \Phi_G(x_n) =_Y \Phi_G(x)\big),$$

where $(\lim_{n\to\infty} x_n =_X x) :\equiv \forall k^0 \exists n^0 \forall m \geq_0 n(d_X(x_m, x) \leq \frac{1}{k+1})$.

Proposition 4.1. *The theory* E-G$_3$A$^\omega$+QF-AC^{0,1} *proves that for all functions $G : X \to Y$ and points $x \in X$:*

G is sequentially continuous at $x \leftrightarrow G$ is ε–δ–continuous at x.

Proof: '$\leftarrow$': Obvious!
'$\to$': Suppose that G is not ε–δ–continuous at x, i.e.

$$(*)\ \exists k^0 \forall n^0 \exists y^1 \big(\underbrace{d_X(x,y) <_{\mathbb{R}} \frac{1}{n+1} \wedge d_Y(\Phi_G(x), \Phi_G(y)) >_{\mathbb{R}} \frac{1}{k+1}}_{\equiv: A \in \Sigma^0_1} \big).$$

By coding pairs of natural numbers and numbers into functions one can express $\exists y^1 A$ in the form $\exists y^1 A_0$. Hence QF-AC^{0,1} applied to $(*)$ yields

$$\exists k^0, \xi^{1(0)} \forall n^0 \big(d_X(x, \xi n) <_{\mathbb{R}} \frac{1}{n+1} \wedge d_Y(\Phi_G(x), \Phi_G(\xi n)) >_{\mathbb{R}} \frac{1}{k+1}\big),$$

i.e. $(\xi n)_{n \in \mathbb{N}}$ represents a sequence of elements of X which converges to x. But $\lim_{n\to\infty} \Phi_G(\xi n) \neq_{\mathbb{R}} \Phi_G(x)$ and thus G is not sequentially continuous at the point represented by x. $\Box$

Remark 4.2. The use of QF-AC^{0,1} in the proof of '$\to$' in the proposition above is unavoidable already for $X = Y = \mathbb{R}$ since in this case the implication is known to be unprovable even in Zermelo–Fraenkel set theory ZF, see [16],[15] and [12].

We now discuss the indirect representation of continuous functions $G : X \to Y$ between Polish spaces X, Y via codes g as used in the context of reverse mathematics (see definition II.6.1 in [35]). Since reverse mathematics takes place in the language of second-order arithmetic (instead of a language with higher types), the direct representation of such continuous function which is available in our systems is not possible. We will show that provably in E-G$_3$A$^\omega$+QF-AC^{1,0}, for every such code g there exists a direct representation in our sense of the function coded by g, but that the reverse direction in general is not even provable in E-PA$^\omega$+QF-AC. The latter phenomenon is due to the fact that the indirect representation of continuous functions G via codes g tacitly yields a constructive enrichment of the direct representation of G by a modulus of pointwise continuity. To be more specific, let us consider the special case $X =$ Baire space, $Y = \mathbb{N}$ (with the usual metrics). Then the existence of a code g for a continuous functional Φ^2 is (relative to E-G$_3$A$^\omega$+QF-AC^{1,0}) equivalent to the existence of a continuous modulus of pointwise continuity functional Ψ^2 for Φ^2 which in turn is equivalent to the existence of an associate of Φ in the sense of the Kleene/Kreisel countable functionals.

Definition 4.3. 1) α^1 *is a neighborhood function if*

(a) $\forall\beta^1\exists n^0\big(\alpha(\overline{\beta}n) > 0\big)$ *and*

(b) $\forall m, n\big(m \sqsubseteq n \wedge \alpha(m) > 0 \to \alpha(m) = \alpha(n)\big)$, *where '$m \sqsubseteq n$' expresses the (elementary recursive) predicate that the sequence encoded by m is an initial segment of the one encoded by n.*

2) α^1 *is an associate of* Φ^2 *if*

(a) $\forall\beta^1\exists n^0\big(\alpha(\overline{\beta}n) > 0\big)$ *and*

(b) $\forall\beta, n\big(n$ *least s.t.* $\alpha(\overline{\beta}n) > 0 \to \alpha(\overline{\beta}n) = \Phi\beta + 1\big)$.

Without loss of generality we may assume that an associate of Φ^2 is a neighborhood function, since otherwise we define

$$\tilde{\alpha}(n) := \begin{cases} \alpha(m), \ \text{for } m \text{ shortest initial segment of } n \text{ s.t. } \alpha(m) > 0, \text{ if } \exists. \\ 0, \ \text{otherwise.} \end{cases}$$

Proposition 4.4. E-G$_3$A$^\omega$+QF-AC^{1,0} *proves (uniformly in* Φ^2*) that the following properties are pairwise equivalent:*

1) $\exists f(f$ is an r.m.-code of $\Phi)$, where 'r.m.' abbreviates 'reverse math'.[7]
2) $\exists\alpha^1(\alpha$ is an associate of $\Phi)$,
3) $\exists\omega^2_\Phi\big(\omega_\Phi$ is a continuous modulus of pointwise continuity for $\Phi\big)$.

Proof: '1) $\to$ 3)': Let f be a r.m.-code of Φ^2. Since Φ is total, we have[8]

$$\forall\beta^1\exists a^0, r^0, b^0, s^0\big(d(\beta, \lambda i.(a)_i) <_{\mathbb{R}} 2^{-r} \wedge (a,r)f(b,s) \wedge 2^{-s} <_{\mathbb{Q}} 1\big)$$

and hence

$$\forall\beta^1\exists a^0, r^0, b^0, s^0, l^0\big(\underbrace{d(\beta, \lambda i.(a)_i) + 2^{-l} <_{\mathbb{R}} 2^{-r} \wedge (a,r)f(b,s) \wedge 2^{-s} <_{\mathbb{Q}} 1}_{\equiv:\exists v^0 A_0(f,\beta,a,r,b,s,l,v)}\big),$$

[7] By 'r.m.-code' we here refer to definition II.6.1 in [35] specialized to $\widehat{A} := \mathbb{N}^{\mathbb{N}}$ and $\widehat{B} := \mathbb{N}$. We identify the set Φ in that definition with its characteristic function f.

[8] As in reverse mathematics we represent real numbers as Cauchy sequences with fixed rate of convergence. As a consequence of this, $<_{\mathbb{R}} \in \Sigma^0_1$. Analogously to [35](II.6.1) we write $(a,r)f(b,s)$ as abbreviation for the Σ^0_1-formula $\exists n^0(f(a,r,b,s,n) =_0 0)$.

where A_0 is quantifier-free. By quantifier-free induction and QF-AC^{1,0} we obtain a functional X^2 such that

$$\forall\beta\big(X\beta \text{ minimal s.t. } A_0(f,\beta,\nu_1^6(X\beta),\ldots,\nu_6^6(X\beta))\big).$$

It is clear that X is continuous[9] and that $\Phi\beta = \nu_3^6(X\beta)$. With X, also

$$\omega_\Phi\beta :=_{\mathbb{Q}} 2^{-\nu_5^6(X\beta)}$$

is continuous. One easily verifies that ω_Φ is a modulus of pointwise continuity for Φ.

'3) $\rightarrow$ 2)': Let ω_Φ be a continuous modulus of pointwise continuity for Φ^2. Then

$$(1)\ \forall\beta,\gamma\big(\overline{\beta}(\omega_\Phi\beta) =_0 \overline{\gamma}(\omega_\Phi\beta) \rightarrow \Phi\beta =_0 \Phi\gamma\big)$$

and

$$(2)\ \forall\beta\exists n^0\big(\omega_\Phi(\overline{\beta,n}) \leq n\big)$$

(where $\overline{\beta,n}$ is the continuation of $\overline{\beta}n$ with 0).

Define

$$\alpha(n) := \begin{cases} \Phi(\lambda i.(n)_i)+1, \text{ if } \omega_\Phi(\lambda i.(n)_i) \leq lth(n) \\ 0, \text{ otherwise.} \end{cases}$$

(2) yields

$$\forall\beta\exists k\big(\alpha(\overline{\beta}k) > 0\big).$$

Assume that $\alpha(\overline{\beta}k) > 0$, then – by (1) and the definition of α – $\omega_\Phi(\overline{\beta,k}) \leq k \wedge \Phi(\overline{\beta,k}) = \Phi\beta$ and therefore $\alpha(\overline{\beta}k) = \Phi\beta + 1$.

'2) $\rightarrow$ 1)': Let α be an associate for Φ. By the remark above we may assume that α is a neighborhood function. Define an r.m.-code f for Φ by

$$(a,r)f(b,s) :\equiv \alpha((\overline{\lambda i.(a)_i})r) > 0 \wedge |(\alpha((\overline{\lambda i.(a)_i})r)-1)-b| < 2^{-s}.$$

This is a quantifier-free (and hence Σ_1^0-)predicate (which we identify with its characteristic function). It is straightforward to verify that f satisfies the properties of an r.m.-code and that f is a code for Φ. We omit the tedious details. □

Remark 4.5. *For the equivalence between 2) and 3), see also* [2] (p.143, E.8).

[9]Here we use the fact that $A_0(f,\beta,a,r,b,s,l,v)$ can be written as $t_{A_0}(f,\beta,a,r,b,s,l,v) =_0 0$ for a suitable closed term t_{A_0} of E-G$_3$A$^\omega$ and that every closed term t^2 of E-G$_3$A$^\omega$ is provably pointwise continuous.

THEOREM 4.6. E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} *does not prove that every continuous functional* Φ^2 *has an r.m.-code (i.e. that* Φ *is continuous in the sense of reverse mathematics).*

Proof: In [32](6.4) a type-structure $A = \langle A_k \rangle_{k\in\mathbb{N}}$ over ω is constructed with the following properties:
(i) $E_2 \restriction A_1 \notin A_2$, where $E_2(f^1) = 0 \leftrightarrow \exists x(fx = 0)$;
(ii) A is closed under computation in the sense of Kleene's schemata S1-S9.
(iii) there exists a $\Phi \in A_2$ such that Φ has no associate in A_1. By (ii), A is a model of the restriction of E-PA$^\omega$+QF-AC^{1,0} to the fragment with pure types only. Modulo the well-known reduction to pure types (see [38](1.8.5-1.8.8)), E-PA$^\omega$+QF-AC^{1,0} therefore has a model in which there exists a functional Φ^2 which has no associate and therefore – by the previous proposition – no r.m.-code f. Nevertheless, all functionals Φ^2 of type 2 are continuous: one could use here an argument due to [14] to show that the existence of a non-continuous functional in A_2 would contradict (i). However, it requires some care to verify that this argument (which usually is formulated for the full type-structure) relativises to A. We therefore use directly the construction of A which is based on a certain type-2 functional $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ (constructed by L. Harrington using a complicated priority construction, see [32](4.21)) which has the following properties

(i) F is continuous (and therefore has an associate in $\mathbb{N}^{\mathbb{N}}$),
(ii) $F\restriction$REC is not computable (in the sense of S1-S9) and therefore has no recursive associate,
(iii) 1-sc(F) =REC, where 1-sc(F) is the set of functions computable in F.

A_1 :=REC, $A_{k+1} := \{\Phi : A_k \to \mathbb{N} : \Phi$ computable in $F\restriction$REC.$\}$
It is clear that every $\Phi \in A_2$ is continuous.
As a further consequence of this, QF-AC^{0,1} reduces in A to QF-AC^{0,0} since $\forall x^0 \exists f^1 A_0(x, f) \to \forall x^0 \exists y^0 A_0(\lambda i.(y)_i)$. So $A \models$ QF-AC^{0,1}. $\Box$

The next proposition shows that in the presence of arithmetical comprehension Π^0_∞-CA : $\exists f^1 \forall x^0 \big(f(x) = 0 \leftrightarrow A(x)\big)$, where A is arithmetical with arbitrary parameters, every continuous function on $\mathbb{N}^{\mathbb{N}}$ has a code in the sense of reverse mathematics:

PROPOSITION 4.7. E-PRA$^\omega$+QF-AC^{1,0} + Π^0_∞-CA *proves that every continuous functional* $\Phi^{1(1)}$ *has an r.m.-code (i.e. that* Φ *is continuous in the sense of reverse mathematics).*

Proof: Let $\Phi^{1(1)}$ be pointwise continuous. Then

$$(+)\forall f^1, x^0 \exists y^0 \forall i^0, j^0 \big(\Phi(x, \overline{f}y * \lambda k.(i)_k) =_0 \Phi(x, \overline{f}y * \lambda k.(j)_k)\big),$$

where $(a^0 * f^1)(k) := (a)_k$ for $k < lth(a)$ and $:= f(k - lth(a))$ otherwise. By Π^0_∞-CA there exists a function χ such that

$$\forall x^0, a^0 \big(\chi(x,a) =_0 0 \leftrightarrow \forall i, j \big(\Phi(x, a * \lambda k.(i)_k) =_0 \Phi(x, a * \lambda k.(j)_k)\big)\big).$$

Hence $(+)$ can be written as

$$\forall f, x \exists y \big(\chi(x, \overline{f}y) =_0 0\big).$$

By QF-AC^{1,0} and QF-IA we obtain a functional $\tilde{\omega}(f,x)$ such that

$$\forall f, x \big(\chi(x, \overline{f}(\tilde{\omega}(f,x)) =_0 0 \wedge \forall z < \tilde{\omega}(f,x) \big(\chi(x, \overline{f}z) \neq 0\big)\big).$$

By the pointwise continuity of Φ one easily verifies that

$$\omega(f,x) := \max_{i \leq x}(\tilde{\omega}(f,i))$$

is a (pointwise continuous) modulus of (pointwise) continuity of Φ. The claim now follows with proposition 4.4. □

PROPOSITION 4.8. E-PA$^\omega$+QF-AC+'all continuous functionals $\Phi^{1(1)}$ have an r.m.-code' *is* Π^1_∞*-conservative over* E-PA$^\omega$+QF-AC^{0,0}.

Proof: Formalizing the fact that the extensional continuous functionals ECF form a model of the first theory and the proof for the faithfulness of this model for the analytical fragment (see [38](2.6.5-2.6.12)), a proof of $A \in \Pi^1_\infty$ in the first theory translates into a proof of $[A]_{\mathrm{ECF}}$ (and hence of A) in E-PA$^\omega$+QF-AC^{0,0}. □

COROLLARY 4.9. E-PA$^\omega$+QF-AC+'all continuous functionals $\Phi^{1(1)}$ have an r.m.-code' *proves neither* Π^0_∞-CA *nor* WKL.

Proof: The corollary follows from proposition 4.8 and the fact that the hereditarily effective operations **HEO** form a model of E-PA$^\omega$+QF-AC^{0,0} (see [38](2.4.11,2.6.13,2.6.20)) but not of WKL (and hence a-fortiori not of Π^0_∞-CA). □

The fact that the representation of continuous functions in reverse mathematics via codes goes together with a constructive enrichment is used in many proofs of basic properties of continuous functions in the system WKL$_0$. So the question arises whether WKL is sufficient to prove the same results for our direct representation. We discuss this for simplicity again for the case of continuous functions $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. As we have seen above, reverse mathematics treats Φ via an associate α^1. This representation allows one to prove the uniform continuity of Φ on the Cantor space of all 0-1-functions by WKL as follows. Define a binary tree by

$$f(n) := \begin{cases} 1, \text{ if } \forall i < lth(n)\big((n)_i \leq 1\big) \wedge \alpha(n) > 0 \\ 0, \text{ otherwise.} \end{cases}$$

Since we may assume that α is a neighborhood function, f satisfies $T(f)$. The contraposition of WKL applied to f yields

$$\forall \beta \leq_1 1 \exists x^0 (\alpha(\overline{\beta}x) > 0) \to \exists x \forall \beta \leq_1 1 (\alpha(\overline{\beta}x) > 0),$$

i.e. $\Phi\beta = \alpha(\overline{\beta} \min n[\alpha(\overline{\beta}n) > 0]) - 1$ is uniformly continuous on $\{\beta : \beta \leq_1 1\}$. This argument can be adopted to real functions encoded as in reverse mathematics to show in that context that e.g. every continuous function $f : [0,1] \to \mathbb{R}$ is uniformly continuous. Together with QF-AC^{0,0} one even gets a modulus of uniform continuity (see proposition 4.10).

In our direct type-2-treatment of continuous functions $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ as functionals Φ^2 satisfying

$$\forall f^1 \exists n^0 \forall g^1 (\overline{f}n = \overline{g}n \to \Phi f = \Phi g),$$

the binary tree to which we have to apply König's lemma in order to prove the uniform continuity of Φ on $\{f : f \leq_1 1\}$ is given by

$$Tree(n) :\equiv \exists g, h \leq_1 1\Big(\bigwedge_{i<lth(n)} (g(i) = (n)_i = h(i)) \wedge \Phi g \neq \Phi h\Big)$$

which no longer is quantifier-free and apparently does not possess a characteristic function in E-PA$^\omega$+QF-AC^{1,0} which would be necessary to apply WKL right away. However, a construction due to D. Normann[10], which easily can be formalized in our setting, shows that WKL allows one to prove the existence of a characteristic function χ_T for $Tree(n)$ (for pointwise continuous functionals Φ). Applying then WKL to '$\chi_T(n) = 0$' yields the uniform continuity (and – together with QF-AC^{0,0} – even a modulus of uniform continuity) of the restriction of $\Phi^{1(1)}$ to the Cantor space. In particular this implies that this restriction has an r.m.-code, but note that even the proof of this corollary, which doesn't mention the uniform continuity of Φ, uses WKL.

[10]We are grateful to Professor Dag Normann for communicating this construction to us.

PROPOSITION 4.10. E-PRA$^\omega$+QF-AC^{0,0}+WKL *proves that if the restriction of a functional* $\Phi^{1(1)}$ *to the Cantor space* 2^ω *is pointwise continuous then it has a modulus of uniform continuity.*

Proof: We prove the proposition in two steps. We first show (using QF-AC^{0,0} and WKL) that the modulus of uniform continuity can be constructed if we have a characteristic function χ_T for

$$(1)\ T(k^0,n^0) :\equiv \exists g,h \leq_1 1\big(\bigwedge_{i<lth(n)} (g(i) = (n)_i = h(i)) \wedge \Phi(g,k) \neq \Phi(h,k)\big)$$

and then show the existence of χ_T using again WKL.
Let χ_T be such that

$$(2)\ \forall k,n\big(\chi_T(k,n) = 0 \leftrightarrow T(k,n)\big).$$

By WKL and (2), we have

$$(3)\ \exists k\forall n\exists f \leq_1 1\, T(k,\overline{f}n) \rightarrow \exists k\exists f \leq_1 1\forall n\, T(k,\overline{f}n).$$

In E-PRA$^\omega$, '$\exists f \leq_1 1(\chi_T(k,\overline{f}n) = 0)$' can be written as a quantifier-free formula $A_0(k,n)$. Hence using QF-AC^{0,0} and (2), (3) implies

$$(4)\ \forall h^1\exists k\exists f \leq_1 1\, T(k,\overline{f}(hk)) \rightarrow \exists k\exists f \leq_1 1\forall n\, T(k,\overline{f}n).$$

Contraposition of (4) together with the assumption of the pointwise continuity of $\Phi|_{2^\omega}$ yields

$$(5)\ \exists h^1\forall k\forall f \leq_1 1\neg T(k,\overline{f}(hk)).$$

Hence $\tilde{h}(k) := \max_{i\leq k}(hi)$ is a modulus of uniform continuity of the restriction of Φ to the Cantor space.
We now show (applying again WKL) the existence of χ satisfying (2) using a construction due to D. Normann. We first notice that because of the pointwise continuity of $\Phi|_{2^\omega}$ it suffices to show the existence of χ such that[11]

$$(6)\ \forall k\forall n \in 2^{<\omega}\big(\chi(k,n) = 0 \leftrightarrow \forall i \in 2^{<\omega}(\Phi(n * \lambda j.(i)_j, k) = \Phi(n * 0^1, k))\big),$$

[11] Here $n * f$ is defined as in the proof of proposition 4.7 above.

where '$n \in 2^{<\omega}$' denotes the primitive recursive predicate 'n is the code of a finite 0-1-sequence'. Primitive recursively in k, n, Φ we define a tree as follows:

$$(7)\ T_{k,n}(q) := \begin{cases} 0, \text{ if } q \in 2^{<\omega} \wedge \\ \qquad \big(\forall i \in 2^{<\omega}(lth(i) \leq lth(q) \to \Phi(n * q * 0^1, k) = \Phi(n * i * 0^1, k)) \\ \qquad\quad \vee \exists \tilde{q} \in 2^{<\omega} \exists l \leq lth(q)[q = \tilde{q} * \overline{0}(l) \text{ with } lth(\tilde{q}) \text{ minimal s.t.} \\ \qquad\qquad\qquad \Phi(n * \tilde{q} * 0^1, k) \neq \Phi(n * 0^1, k)]\big) \\ 1, \text{ otherwise.} \end{cases}$$

For every $k \in \mathbb{N}, n \in 2^{<\omega}$, $T_{k,n}$ is an infinite binary tree. So WKL yields the existence of an infinite path $f \leq_1 1$ in that tree. One easily verifies (using again the pointwise continuity of $\Phi|_{2^\omega}$) that

$$(8)\ \forall i \in 2^{<\omega}(\Phi(n * \lambda j.(i)_j, k) = \Phi(n * 0^1, k)) \leftrightarrow \Phi(n * f, k) = \Phi(n * 0^1, k).$$

In E-PRA$^\omega$ one can code the sequence of trees $T_{k,n}$ into a single infinite binary tree T such that any infinite path g of that tree (which by WKL exists) yields a whole sequence $\lambda k, n.f_{k,n}$ of infinite paths in $T_{k,n}$ (for $n \in 2^{<\omega}$; see [30] for details) and hence – in view of (8) – the characteristic function χ satisfying (6) which concludes the proof. □

Corollary 4.11. E-PRA$^\omega$+QF-AC^{0,0}+WKL *proves that the restriction of every continuous functional* $\Phi^{1(1)}$ *to the Cantor space has an r.m.-code.*

Open problems:

1) Does corollary 4.11 hold without WKL?
2) Does theorem 4.6 hold with WKL added?

§5. Generalization of WKL to more complex trees: Φ_∞-WKL$_+$. The discussion before proposition 4.10 above as well as the proof of that proposition, showed that a direct application of a König's lemma-based argument to prove the uniform continuity of a pointwise continuous functional $\Phi^{1(1)}$ on 2^ω would require a form of the binary König's lemma for trees given by predicates of the form $T(n) \equiv \exists g \leq_1 1 A_0(g, n)$ with quantifier-free A_0 (allowing arbitrary parameters of higher type). In the particular application in 4.10, WKL plus QF-AC was sufficient to reduce this to a quantifier-free tree predicate (so that WKL could then again be applied to that tree). But this was possible only because $A_0(g, n)$ was pointwise continuous in g. However, without this continuity assumption, the corresponding extension of WKL to trees of the form above still results in a classically valid form of König's lemma. Since Σ^0_1-UB allows one to prove the (uniform) continuity of every functional $\Phi^{1(1)}$ on 2^ω, this more general form of WKL can be derived from Σ^0_1-UB and hence is **PRA**-reducible. This obervation can be largely extended to develop a whole strict hierarchy of extensions of WKL which are not derivable from WKL but which all can be shown to be proof-theoretically reducible to **PRA**.

DEFINITION 5.1. 1) *$A \in \Phi_n$ if*

$$A \equiv \forall f_1 \leq_1 s_1[\underline{a}]\exists f_2 \leq_1 s_2[\underline{a}] \ldots \forall^{(d)} f_n \leq_1 s_n[\underline{a}]\forall x^0 A_0(\underline{a}, f_1, \ldots, f_n, x),$$

where A_0 is quantifier-free and $\underline{a}$ contains all free variables of A and s_i (which may have arbitrary types). The f_i must not occur in $\underline{a}$.[12]

2) *$A \in \Psi_n$ if*

$$A \equiv \exists f_1 \leq_1 s_1[\underline{a}]\forall f_2 \leq_1 s_2[\underline{a}] \ldots \exists^{(d)} f_n \leq_1 s_n[\underline{a}]\forall x^0 A_0(\underline{a}, f_1, \ldots, f_n, x),$$

where A_0 and s_i as above.

3) *The classes Φ^-_n and Ψ^-_n result if we restrict ourselves to parameters $\underline{a}$ of type level ≤ 1 in A_0 and s_i.*

REMARK 5.2. *One could also allow further universal number quantifiers $\forall x^0$ (but no existential quantifiers) to occur in between the bounded function quantifiers in the definition of Φ_n. The results of this paper easily extend to this slightly generalized case. However, for notational simplicity we restrict ourselves to the definition of Φ_n as stated above.*

DEFINITION 5.3. *The generalization of* WKL *to Φ_n-trees is given by*

$$\Phi_n\text{-WKL}: \ \forall n^0 \exists f \leq_1 1 \forall \tilde{n} \leq n\, A(\overline{f}\tilde{n}) \rightarrow \exists f \leq_1 1 \forall n^0 A(\overline{f}n),$$

where $A(k^0) \in \Phi_n$ (with arbitrary further parameters of arbitrary types). Ψ_n-WKL *is defined analogously.* Φ_∞-WKL$:= \bigcup_{n\in\omega}\{\Phi_n\text{-WKL}\}$.

The next proposition shows that in the absence of parameters of types ≥ 2 (and so in particular in a second-order context) there is no point in considering Φ_n-WKL instead of WKL.[13] For its proof we need the following

LEMMA 5.4. *Let $A_0(\underline{a}, g^1, y^0)$ be a quantifier-free formula of $\mathcal{T}$:=E-G$_n$A$^\omega$ ($n \geq 3$),* E-PRA$^\omega$ *or* E-PA$^\omega$ *containing (in addition to g, y) only parameters $\underline{a}$ of type levels ≤ 1 and let s be a term of $\mathcal{T}$ containing at most $\underline{a}$ as free*

[12] Here $\forall^d = \exists$, $\exists^d = \forall$.

[13] This is in sharp contrast to the case where arbitrary parameters are allowed as we will see below.

variables. Then one can construct a Π^0_1*-formula* $B(\underline{a})$ *of* $\mathcal{T}$ *(containing only* $\underline{a}$ *free) such that*

$$\mathcal{T} + \text{WKL} \vdash \forall \underline{a}\big(B(\underline{a}) \leftrightarrow \exists g \leq_1 s[\underline{a}]\forall y^0 A_0(\underline{a}, g, y)\big).$$

Proof: For $\mathcal{T}$ =E-PRA$^\omega$ and $\mathcal{T}$ = E-PA$^\omega$ this follows from (the proofs of) proposition 4.14 and corollary 4.15 in [19]. The use of the modulus $\tilde{t}xyk$ of pointwise continuity in y used in the proof of proposition 4.14 in [19] can easily be replaced by a modulus $\widehat{t}xk$ of uniform continuity on $\{y : y \leq_1 sx\}$. For closed t ∈E-G$_n$A$^\omega$ such a modulus $\widehat{t}$ can be constructed in E-G$_n$A$^\omega$ by the method of [18] since the majorization argument used there is available in E-G$_n$A$^\omega$ as was shown in [23]. □

PROPOSITION 5.5. *Let* $m, n \geq 0$*. Over* $\mathcal{T}$:=E-G$_k$A$^\omega$ $(k \geq 3)$, E-PRA$^\omega$ *or* E-PA$^\omega$ *the following principles are equivalent:*

(i) WKL, (ii) Φ_0-WKL, (iii) Ψ_0-WKL, (iv) Φ^-_m-WKL, (v) Ψ^-_n-WKL.

Proof: We first show the following
Claim: Let $A(\underline{a})$ be a Φ^-_n (or Ψ^-_n) formula containing only parameters $\underline{a}$ of type degree ≤ 1. Then one can construct a Π^0_1-formula $B(\underline{a})$ such that

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow B(\underline{a}).$$

Proof of the claim: We proceed by meta-induction on n:
$n = 0$: In this case $A \in \Pi^0_1$ and so $B := A$ suffices.
$n \to n{+}1$: Case 1: $A \in \Phi_{n+1}$. Then $A(\underline{a}) \equiv \forall f \leq_1 s[\underline{a}]\, \tilde{A}(\underline{a}, f)$, where $\tilde{A} \in \Psi_n$. By the induction hypothesis there exists a formula $\tilde{B}(\underline{a}, f) \equiv \forall y^0 \tilde{B}_0(\underline{a}, f, y) \in \Pi^0_1$ with

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow \forall f \leq_1 s[\underline{a}]\forall y^0 \tilde{B}_0(\underline{a}, f, y).$$

Let $t_{\tilde{B}_0}$ be a closed term of $\mathcal{T}$ such that

$$\mathcal{T} \vdash \forall \underline{a}, f, y\big(t_{\tilde{B}_0}(\underline{a}, f, y) =_0 0 \leftrightarrow \tilde{B}_0(\underline{a}, f, y)\big).$$

From results in [18] (using for the case of E-G$_k$A$^\omega$ also [23]) it follows that one can construct a closed term $\widehat{t}_{\tilde{B}_0}$ of $\mathcal{T}$ such that $\widehat{t}_{\tilde{B}_0}(\underline{a}, y)$ is (provably in $\mathcal{T}$) a modulus of uniform continuity for $\lambda f.t_{\tilde{B}_0}(\underline{a}, f, y)$ on $\{f : f \leq_1 s[\underline{a}]\}$. Using this modulus, $\forall f \leq_1 s[\underline{a}]\, \tilde{B}_0(\underline{a}, f, y)$ can be written as a quantifier-free formula and hence $\forall f \leq_1 s[\underline{a}]\forall y\, \tilde{B}_0(\underline{a}, f, y)$ as a Π^0_1-formula $\widehat{B}(\underline{a})$. So

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow \widehat{B}(\underline{a}).$$

Case 2: $A(\underline{a}) \in \Psi_{n+1}$. Then $A(\underline{a}) \equiv \exists f \leq_1 s[\underline{a}]\, \tilde{A}(\underline{a}, f)$ with $\tilde{A}(\underline{a}, f) \in \Phi_n$. By I.H. there exists a formula $\tilde{B}(\underline{a}, f) \equiv \forall y^0 \tilde{B}_0(\underline{a}, f, y) \in \Pi^0_1$ with

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow \exists f \leq_1 s[\underline{a}] \forall y^0 \tilde{B}_0(\underline{a}, f, y).$$

By the lemma, there exists a Π^0_1-formula $\widehat{B}(\underline{a})$ such that

$$\mathcal{T} + \text{WKL} \vdash \widehat{B}(\underline{a}) \leftrightarrow \exists f \leq_1 s[\underline{a}] \forall y^0 \tilde{B}_0(\underline{a}, f, y).$$

So again

$$\mathcal{T} + \text{WKL} \vdash A(\underline{a}) \leftrightarrow \widehat{B}(\underline{a})$$

with $\widehat{B} \in \Pi^0_1$. This finishes the proof of the claim.
The claim implies that

$$\mathcal{T} + \text{WKL} \vdash \Phi^-_m\text{-WKL} \leftrightarrow \Psi^-_n\text{-WKL}$$

for all $m, n \geq 0$. Since trivially Φ^-_0-WKL $\leftrightarrow$ Φ_0-WKL, it therefore remains to show that

$$\mathcal{T} \vdash \Phi_0\text{-WKL} \leftrightarrow \Psi_0\text{-WKL} \leftrightarrow \text{WKL}.$$

Φ_0-WKL$\equiv$ Ψ_0-WKL holds by definition. It is an easy exercise to show that WKL$\leftrightarrow$ Φ_0-WKL which we leave to the reader. □

In the presence of higher type parameters, however, we get non-collapsing hierarchies of principles Φ_n-WKL and Ψ_n-WKL, as we will show now.

Definition 5.6. *We define the classes of formulas $\Pi^{1,b}_n$ and $\Psi^{1,b}_n$ simultaneously by induction on n:*

(i) *$A \in \Pi^{1,b}_0 = \Sigma^{1,b}_0$, if A is quantifier-free;*
(ii) *if $A(f) \in \Pi^{1,b}_n$, then $\exists f \leq_1 1\, A(f) \in \Sigma^{1,b}_{n+1}$;*
(iii) *if $A(f) \in \Sigma^{1,b}_n$, then $\forall f \leq_1 1\, A(f) \in \Pi^{1,b}_{n+1}$.*

A may contain arbitrary parameters (of arbitrary types).

Remark 5.7. $\Pi^{1,b}_n \subseteq \Phi_n$ *and* $\Sigma^{1,b}_n \subseteq \Psi_n$.

Definition 5.8. 1) *The schema of $\Pi^{1,b}_n$-comprehension is given by*

$$\Pi^{1,b}_n\text{-CA}: \ \exists g^1 \forall x^0 (gx = 0 \leftrightarrow A(x)),$$

where $A(x) \in \Pi^{1,b}_n$ and may contain arbitrary parameters (of arbitrary types) in addition to x. $\Sigma^{1,b}_n$-CA is defined analogously but with $\Sigma^{1,b}_n$ instead of $\Pi^{1,b}_n$.

2) *The schema of* $\Pi^{1,b}_n$*-choice for numbers is given by*

$$\Pi^{1,b}_n\text{-AC}^{0,0}:\ \forall x^0 \exists y \leq_0 1\, A(x,y) \to \exists g \leq_1 1 \forall x\, A(x,gx),$$

where $A(x,y) \in \Pi^{1,b}_n$ *and may contain arbitrary parameters.*

PROPOSITION 5.9. *Let* $\mathcal{T}$:=E-PA$^\omega$. *Then*

$$\mathcal{T} + \Phi_{n+1}\text{-WKL} \vdash \Pi^{1,b}_n\text{-CA}$$

(Likewise for Ψ_{n+1}-WKL*).*

Proof: We use the following tree-predicate from [39]:

$$\tilde{A}(k) :\equiv \begin{cases} (k)_{lth(k)\dot{-}1} \leq 1 \wedge ((k)_{lth(k)\dot{-}1} = 0 \to A(lth(k)\dot{-}1)) \wedge \\ \qquad ((k)_{lth(k)\dot{-}1} = 1 \to \neg A(lth(k)\dot{-}1))\big),\ \text{if } lth(k) > 0 \\ true,\ \text{otherwise.} \end{cases}$$

For $A \in \Pi^{1,b}_n$, $\tilde{A}(k)$ can be written as a Φ_{n+1}-formula. By induction on n we can prove in E-PA$^\omega$ that

$$\forall n^0 \exists f \leq_1 1 \forall \tilde{n} \leq n \tilde{A}(\overline{f}\tilde{n}).$$

Φ_{n+1}-WKL therefore yields the characteristic function for $A(n)$. □

PROPOSITION 5.10. E-PA$^\omega$ + $\Pi^{1,b}_n$-CA+μ *contains (modulo a canonical embedding which doesn't change the first order part) the second order system* (Π^1_n-CA) *known from reverse mathematics.*[14]

Proof: Systems formulated in the language of second-order arithmetic with set variables like (Π^1_n-CA) can be embedded in (suitable) systems formulated in the language of functionals of all finite types by representing sets X by their characteristic functions χ_X and replacing formulas '$t \in X$' by '$\chi_X(t) =_0 0$'. In doing so and using the fact that the presence of μ allows one to absorb an arbitrary arithmetical quantifier-prefix in front of a quantifier-free formula with arbitrary parameters uniformly in these parameters, the comprehension schema of (Π^1_n-CA) reduces to $\Pi^{1,b}_n$-CA above. □

The two propositions above show that the systems E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + μ + Φ_n-WKL (and similar with Ψ_n-WKL) form a non-collapsing hierarchy which as n increases eventually exhausts full second-order arithmetic. Together with the result due to Feferman that E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+ μ can be reduced proof-theoretically to $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$[15] and hence is proof-theoretically much weaker than (Π^1_1-CA), it in particular follows that for $n \geq 2$,

[14] In the notation of [35], (Π^1_n-CA) is the system Π^1_n-CA$_0$+full induction.

[15] This follows from [5] together with elimination of extensionality (see also [1]).

Φ_n-WKL and Ψ_n-WKL are underivable in E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + μ. The next proposition sharpens this further:

PROPOSITION 5.11. E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + $\mu \nvdash \Phi_1$-WKL.

Proof: One easily verifies that E-PA$^\omega$ + Φ_1-WKL proves $\Pi^{1,b}_1$-AC^{0,0} which in the presence of μ yields the so-called Σ^1_1-separation principle (see [35]). Hence (again by [35]) the subsystem ATR of second order arithmetic, whose proof-theoretic strength is much higher than that of $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$, is contained in E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + μ + Φ_1-WKL. □

In '(4)' in the proof of proposition 4.10 we used a slightly more general form of WKL which, however, could be derived from WKL by the use of QF-AC^{0,0}. For our extensions Φ_n-WKL ($n > 0$), quantifier-free choice does not seem to be sufficient to do the same trick. This suggests to generalize Φ_n-WKL as follows:

DEFINITION 5.12. *Let* $A(a^0, k^0) \in \Phi_n$ *(with arbitrary parameters).*

$$\Phi_n\text{-WKL}_+ : \forall h^1 \exists a^0 \exists f \leq_1 1 \forall \tilde{n} \leq h(a)\, A(a, \overline{f}\tilde{n}) \rightarrow \exists a \exists f \leq_1 1 \forall n^0 A(a, \overline{f}n)$$

(Ψ_n-WKL$_+$ is defined analogously with $A \in \Psi_n$.)

It is an easy exercise to show

PROPOSITION 5.13. E-G$_3$A$^\omega$+QF-AC^{0,0} $\vdash \Phi_0$-WKL$\leftrightarrow \Phi_0$-WKL$_+$.

§6. PRA-reducible theories. We now show that F (and in fact Σ^0_1-UB) suffices to prove the whole hierarchy Φ_∞-WKL$_+$:

PROPOSITION 6.1. *Let* $\mathcal{T}$:=E-G$_k$A$^\omega$ ($k \geq 3$), E-PRA$^\omega$ *or* E-PA$^\omega$. *Then*

$$\mathcal{T} + \text{QF-AC}^{1,0} + F^- \vdash \Phi_\infty\text{-WKL}_+.$$

Proof: Because of proposition 3.6 it suffices to show that

$$\mathcal{T} + \text{QF-AC}^{1,0} + F \vdash \Phi_\infty\text{-WKL}_+.$$

The idea of the proof is to use proposition 3.4 (together with propositions 3.2 and 3.3) to show similarly to the argument in the proof of proposition 5.5 that every $A \in \Phi_n$ (or $\in \Psi_n$) can be written as a Π^0_1-formula B. Whereas in the proof of proposition 5.5 we could use the fact that for every term $t^2[\underline{a}]$ of $\mathcal{T}$ containing only variables $\underline{a}$ of type ≤ 1 one can construct a modulus of uniform continuity on $\{x : x \leq_1 b\}$ (uniformly in $\underline{a}$ and b), we have to use proposition 3.4 in the presence of arbitrary parameters. The latter provides such a modulus of uniform continuity only uniformly in number parameters but not uniformly in function parameters f unless the latter are themselves restricted to a compact set $\{f : f \le_1 b\}$ (in which case a modulus that is independent of f does exist). However this is just the case in the situation at hand since all function variables $f_1, \ldots, f_n$ of $A \in \Phi_n$ which are not parameters are bounded. So all we need is

$$(*)\begin{cases} \forall \Phi, \underline{a} \exists \alpha^1 \forall x^0, z^0 \big(\lambda \underline{f}.(\Phi x z \underline{f} \underline{a})^0 \text{ is uniformly continuous for all} \\ f_1 \le_1 s_1[x, \underline{a}], \ldots, f_n \le_1 s_n[x, \underline{a}] \text{ with modulus } \alpha x z\big), \end{cases}$$

where $\underline{a}$ are all the remaining free variables of s_i (which may have arbitrary types).[16]
$(*)$ is implied by

$$(**)\begin{cases} \forall \Phi, \underline{a}, \underline{b}^{1(0)} \exists \alpha^1 \forall x^0, z^0 \big(\lambda \underline{f}.(\Phi x z \underline{f} \underline{a})^0 \text{ is uniformly continuous for all} \\ f_1 \le_1 b_1 x, \ldots, f_n \le_1 b_n x \text{ with modulus } \alpha x z\big). \end{cases}$$

But this follows in $\mathcal{T} + \Sigma^0_1$-UB (and therefore in $\mathcal{T}$+QF-AC^{1,0} $+ F$ by proposition 3.2) similarly to the proof of proposition 3.4. Since by proposition 3.3 also WKL is available in this theory, we can argue as in the proof of the claim in the proof of proposition 5.5 and show that for $A(x) \in \Phi_n$ (with arbitrary additional parameters)

$$\mathcal{T} + \Sigma^0_1\text{-UB} \vdash \exists \Phi \forall x^0 \big(A(x) \leftrightarrow \forall z^0 (\Phi x z =_0 0)\big).$$

Hence for all $n \in \mathbb{N}$

$$(***) \mathcal{T} + \Sigma^0_1\text{-UB} \vdash \Phi_0\text{-WKL}_+ \to \Phi_n\text{-WKL}_+$$

and therefore (using propositions 3.3,5.5 and 5.13)

$$\mathcal{T} + \text{QF-AC}^{0,0} + \Sigma^0_1\text{-UB} \vdash \Phi_n\text{-WKL}_+$$

and therefore by proposition 3.2

$$\mathcal{T} + \text{QF-AC}^{1,0} + F \vdash \Phi_n\text{-WKL}_+,$$

which concludes the proof. □

Corollary to the proof of proposition 6.1:

$$\mathcal{T} + \text{QF-AC}^{0,0} + \Sigma^0_1\text{-UB} \vdash \Phi_\infty\text{-WKL}_+$$

THEOREM 6.2.
1) E-G$_3$A$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} $+ \Sigma^0_1$-UB *is* Π^0_2*-conservative over* EA,
2) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} $+ \Sigma^0_1$-UB *is* Π^0_2*-conservative over* PRA,

[16]Here 'z' is the variable from the Π^0_1-kernel of A (which of course can be merged together with x).

3) E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Σ^0_1-UB *is conservative over* PA.

Proof: We first prove 3): Let A be a sentence of PA which is provable in E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Σ^0_1-UB and hence (using proposition 3.2) in E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + F. Then the Herbrand normal form $A^H \equiv \forall \underline{f} \exists \underline{y} A_0(\underline{f}, \underline{y})$ of A (we may assume that A is in prenex normal form) is provable there a-fortiori. Hence by theorem 3.7

$$\text{E-PA}^\omega \vdash \forall \underline{f}\, A_0(\underline{f}, \underline{\Psi}(\underline{f}))$$

for suitable closed terms $\underline{\Psi}$ of E-PA$^\omega$. Thus

$$\text{E-PA}^\omega \vdash A^H.$$

By [17](thm.4.1) we can conclude that[17]

$$\text{PA} \vdash A.$$

1) and 2): For Π^0_2-sentences A the argument above applies equally to E-G$_3$A$^\omega$ (resp. E-PRA$^\omega$) yielding E-G$_3$A$^\omega \vdash A$ (resp. E-PRA$^\omega \vdash A$). The conclusion now follows from the fact that E-G$_3$A$^\omega$ (resp. E-PRA$^\omega$) is Π^0_2-conservative over EA (resp. PRA). □

THEOREM 6.3.

1) E-G$_3$A$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+Φ_∞-WKL$_+$ *is* Π^0_2*-conservative over* EA*,*
2) E-PRA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Φ_∞-WKL$_+$ *is* Π^0_2*-conservative over* PRA*,*
3) E-PA$^\omega$+QF-AC^{1,0}+QF-AC^{0,1} + Φ_∞-WKL$_+$ *is conservative over* PA

Proof: The theorem follows from theorem 6.2 and the corollary to the proof of proposition 6.1. □

REMARK 6.4. *The purely proof-theoretic proofs of theorems 6.2 and 6.3 also yield corresponding proof-theoretic reductions.*

Theorems 6.2.2) and 6.3.2) yield two new mathematically strong PRA-reducible and Π^0_2-conservative extensions of PRA. One of these systems

$$\mathcal{T}^* := \text{E-PRA}^\omega\text{+QF-AC}^{1,0}\text{+QF-AC}^{0,1} + \Sigma^0_1\text{-UB}$$

is a non-standard system in the sense that the full set-theoretic type structure $\mathcal{S}^\omega$ is not a model of $\mathcal{T}^*$.

[17]Warning: this argument does not apply to the subsystems E-PRA$^\omega$, PRA; see [17] for a counterexample to this.

Analysing the greater mathematical strength of $\mathcal{T}^*$ (w.r.t. to derivable consequences which **are** true in $\mathcal{S}^\omega$) in terms of generalizations of WKL to logically more complex binary trees, we developed the subsystem

$$\mathcal{T} := \text{E-PRA}^\omega + \text{QF-AC}^{1,0} + \text{QF-AC}^{0,1} + \Phi_\infty\text{-WKL}_+$$

which has $\mathcal{S}^\omega$ as a model.

There is also a different route to design PRA-reducible systems which is based on E-$G_\infty A^\omega$ instead of E-PRA$^\omega$. Although E-$G_\infty A^\omega$ contains all primitive recursive functions and primitive recursive functionals of every Grzegorczyk level n, it does not contain all ordinary Kleene-primitive recursive functionals of type 2, in particular it does not contain Φ_{it}. As a consequence of this, E-$G_\infty A^\omega$+QF-AC^{0,0} does not prove the schema of Σ^0_1-induction. As we have shown in [26],[27] and [29], one can add to E-$G_\infty A^\omega$+QF-AC^{1,0}+ QF-AC^{0,1} function parameter-free schematic forms of e.g. Π^0_1-comprehension, the Bolzano-Weierstraß principle for sequences in $[0,1]^d$, the Arzela-Ascoli lemma etc. and still obtain a PRA-reducible system (whereas the addition of any of these principles to E-PRA$^\omega$ would make the Ackermann function provably total). This result was obtained via a certain Σ^0_2-generalization of the principle Σ^0_1-UB$^-$ mentioned in the proof of proposition 3.6. Using the results of this paper we can even allow a corresponding generalization of the principle Σ^0_1-UB instead. As a consequence of this and the fact that Φ_∞-WKL$_+$ follows from Σ^0_1-UB already relative to E-$G_\infty A^\omega$, we may add Φ_∞-WKL$_+$ to the principles listed above without losing PRA-conservation. This results in a mathematically fairly strong system (note that E-$G_\infty A^\omega$+QF-AC^{0,0} contains – identifying sets $X \subseteq \mathbb{N}$ with their characteristic function – the weak base system RCA$^*_0$ from reverse mathematics and see remark X.4.3 in [35]) which is incompatible with the systems studied in this paper. A detailed treatment of this theme, however, has to be postponed for another paper.

The results of this paper and [30] suggest to propose the following extension of the program of reverse mathematics to finite types: Replace the base system RCA$_0$ by its finite type extension RCA$^\omega_0$:= E-PRA$^\omega$+QF-AC^{1,0}. This system can be shown to be conservative over (an inessential variant with function variables instead of set variable of) RCA$_0$. So for second order statements A, B (i.e. the type of statements which can be discussed in the framework of currently existing reverse mathematics) **nothing is lost** if we prove an equivalence between A and B relative to RCA$^\omega_0$ instead of RCA$_0$. However, the richer language allows one to consider new statements (in their direct formulation) which can not even be expressed in RCA_0 and to apply reverse mathematics to them as well. As an example, we can recast a result from [30] as a result in reverse mathematics in this extended sense:

'Relative to RCA_0^ω, the uniform weak König's lemma UWKL and the existence of Feferman's μ-operator are equivalent'.

Likewise, the equivalence between μ and strong uniform versions of analytical theorems like the attainment of the maximum of $f \in C[0,1]$ can be obtained. This theme is devloped further in [31].

REFERENCES

[1] Avigad, J., Feferman, S., Gödel's functional ('Dialectica') interpretation. In: [3], pp. 337-405 (1998).

[2] Beeson, M.J., Foundations of Constructive Mathematics. Springer Ergebnisse der Mathematik und ihrer Grenzgebiete 3.Folge, Bd.6., Berlin Heidelberg New York Tokyo 1985.

[3] Buss, S.R. (editor), Handbook of Proof Theory. Studies in Logic and the Foundations of Mathematics Vol 137, Elsevier, vii+811 pp. (1998).

[4] Feferman, S., A language and axioms for explicit mathematics. In: Crossley, J.N. (ed.), Algebra and Logic, pp. 87-139. Springer Lecture Notes in Mathematics 450 (1975).

[5] Feferman, S., Theories of finite type related to mathematical practice. In: Barwise, J. (ed.), Handbook of Mathematical Logic, pp. 913-972, North-Holland, Amsterdam (1977).

[6] Feferman, S., Working foundations. Synthese **62**, pp. 229-254 (1985).

[7] Feferman, S., Weyl vindicated: Das Kontinuum seventy years later, in: Cellucci, C., Sambin, G. (eds.), Temi e prospettive della logica e della filosofia della scienza contemporance, vol. I, pp. 59-93 (1988), CLUEB, Bologna. Reprinted (with minor additions) in [10].

[8] Feferman, S., Infinity in mathematics: Is Cantor necessary?. In: G. Toraldo di Francia (ed.), L'infinito nella scienza, Istituto della Enciclopedia Italiana, Rome, pp. 151-209 (1987). Reprinted (with minor additions) in [10].

[9] Feferman S., Why a little bit goes a long way: Logical foundations of scientifically applicable mathematics. In PSA 1992, Vol.2 (Philosophy of Science Association, East Lansing), pp. 442-455. Reprinted (with minor additions) in [10].

[10] Feferman, S., In the Light of Logic. Oxford University Press, 340 pp. (1998).

[11] Feferman, S., Jäger, G., Systems of explicit mathematics with non-constructive μ-operator. Ann. Pure Applied Logic, Part I. **65**, pp. 243-263 (1993), Part II. **79**, pp. 37-52 (1996).

[12] Felgner, U., Models of ZF–Set Theory. Lecture Notes in Mathematics **223**, Springer, Berlin 1971.

[13] Gödel, K., Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica **12**, pp. 280–287 (1958).

[14] Grilliot, T.J., On effectively discontinuous type-2 objects. J. Symbolic Logic **36**, pp. 245–248 (1971).

[15] Halpern, J.D.–Levy, A., The Boolean Prime Ideal theorem does not imply the axiom of choice. In: Proc. of Symp. Pure Math. vol.XIII, pp.83–134, AMS, Providence (1971).

[16] Jägermann, M., The axiom of choice and two definitions of continuity. Bull. Acad. Polon. Sci. **13**, pp.699–704 (1965).

[17] Kohlenbach, U., Remarks on Herbrand normal forms and Herbrand realizations. Arch. Math. Logic **31**, pp. 305-317 (1992).

[18] Kohlenbach, U., Pointwise hereditary majorization and some applications. Arch. Math. Logic **31**, pp. 227-241 (1992).

[19] Kohlenbach, U., Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. J. Symbolic Logic **57**, pp. 1239–1273 (1992).

[20] Kohlenbach, U., Effective moduli from ineffective uniqueness proofs. An unwinding of de La Vallée Poussin's proof for Chebycheff approximation. Ann. Pure Appl. Logic **64**, pp. 27–94 (1993).

[21] Kohlenbach, U., Analysing proofs in analysis. In: W. Hodges, M. Hyland, C. Steinhorn, J. Truss, editors, *Logic: from Foundations to Applications. European Logic Colloquium* (Keele, 1993), pp. 225–260, Oxford University Press (1996).

[22] Kohlenbach, U., Real growth in standard parts of analysis. Habilitationsschrift, pp. xv+166, Frankfurt (1995).

[23] Kohlenbach, U., Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals. Arch. Math. Logic **36**, pp. 31–71 (1996).

[24] Kohlenbach, U., Proof theory and computational analysis.
Electronic Notes in Theoretical Computer Science **13**, Elsevier
(http://www.elsevier.nl/locate/entcs/volume13.html), 1998.

[25] Kohlenbach, U., The use of a logical principle of uniform boundedness in analysis. In: Cantini, A., Casari, E., Minari, P. (eds.), Logic and Foundations of Mathematics. Synthese Library 280, pp. 93–106. Kluwer Academic Publishers (1999).

[26] Kohlenbach, U., Arithmetizing proofs in analysis. In: Larrazabal, J.M. et al. (eds.), Proceedings Logic Colloquium 96 (San Sebastian), Springer Lecture Notes in Logic **12**, pp. 115–158 (1998).

[27] Kohlenbach, U., On the arithmetical content of restricted forms of comprehension, choice and general uniform boundedness. Ann. Pure and Applied Logic **95**, pp. 257–285 (1998).

[28] Kohlenbach, U., On the no-counterexample interpretation. J. Symbolic Logic **64**, pp. 1491–1511 (1999).

[29] Kohlenbach, U., Things that can and things that can't be done in PRA. Ann. Pure Applied Logic **102**, pp. 223–245 (2000).

[30] Kohlenbach, U., On the uniform weak König's lemma. To appear in: Ann. Pure Applied Logic.

[31] Kohlenbach, U., Higher order reverse mathematics. Preprint 14pp., submitted.

[32] Normann, D., Recursion on the Countable Functionals. Springer Lecture Notes in Mathematics **811** (1980).

[33] Sieg, W., Fragments of arithmetic. Ann. Pure Appl. Logic **28**,pp. 33–71 (1985).

[34] Simpson, S.G., Partial realizations of Hilbert's program. J. Symbolic Logic **53**, 349–363 (1988).

[35] Simpson, S.G., Subsystems of Second Order Arithmetic. Perspectives in Mathematical Logic. Springer-Verlag. xiv+445 pp. 1999.

[36] Tait, W.W., Finitism. Journal of Philosophy **78**, pp. 524–546 (1981).

[37] Takeuti, G., A conservative extension of Peano arithmetic. Part II of 'Two applications of logic to mathematics', Publ. Math. Soc. Japan **13**. 1978

[38] Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes in Mathematics **344** (1973).

[39] Troelstra, A.S., Note on the fan theorem. J. Symbolic Logic **39**, pp. 584–596 (1974).

[40] Weyl, H., Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis. Veit, Leipzig 1918. English translation: Weyl, H., The Continuum: A critical Examination of the Foundation of Analysis. New York, Dover 1994.

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