Rate of metastability for Bruck's iteration of pseudocontractive mappings in Hilbert space

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Abstract

We give a quantitative version of the strong convergence of Bruck's iteration schema for Lipschitzian pseudocontractions in Hilbert space.

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1 Introduction

Let X be a normed linear space and $S \subseteq X$ be a subset of X. In 1967, Browder introduced an important generalization of the class of nonexpansive mappings, namely the *pseudocontractive* mappings $T : S \to S$ defined by

$$\forall u, v \in S \, \forall \lambda > 1 \; ((\lambda - 1)\|u - v\| \leq \|(\lambda I - T)(u) - (\lambda I - T)(v)\|),$$

where I denotes the identity mapping.

Apart from being a generalization of nonexpansive mappings, the pseudocontractive mappings are also closely related to accretive operators, where an operator A is called accretive if for every $u, v \in S$ and for all $s > 0$,

$$\|u - v\| \leq \|u - v + s\,(Au - Av)\|\,.$$

Observe that T is pseudocontractive if and only if $I - T$ is accretive. Therefore, any fixed point of T is a root of the accretive operator $I - T$.
In a Hilbert space, T is pseudocontractive iff

$$\forall u, v \in S \, (\langle Tu - Tv, u - v\rangle \leq \|u - v\|^2)$$

(see e.g. [5]).

In [4], Bruck introduced the following iteration schema for pseudocontractive mappings:

Definition 1.1 ([4]). Let C be a nonempty convex subset of a real normed space and let $T : C \to C$ be a pseudocontraction. Let $(\lambda_n), (\theta_n)$ be sequences in $[0, 1]$ with $\lambda_n(1 + \theta_n) \leq 1$ for all $n \in \mathbb{N}$. The **Bruck iteration scheme** with starting point $x_1 \in C$ is defined as

$$x_{n+1} = (1 - \lambda_n)\,x_n + \lambda_n T x_n - \lambda_n \theta_n\,(x_n - x_1)\,.$$

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Among many other things, Bruck showed that in Hilbert spaces and for bounded closed and convex subsets C this iteration strongly converges for so-called acceptably paired sequences $(\lambda_n), (\theta_n)$. Moreover the limit is a fixed point of T provided that T is demicontinuous (in addition to being pseudocontractive).

In [6], it is shown that Bruck's iteration (with more natural conditions on $(\lambda_n), (\theta_n)$) is asymptotically regular, i.e.

$$\|x_n - T(x_n)\| \overset{n\to\infty}{\to} 0,$$

in **arbitrary** Banach spaces provided that T is a Lipschitzian pseudocontractive mapping which still includes the important class of strictly pseudocontrative mappings in the sense of Browder and Petryshyn [3] (see [5]).

Definition 1.2 ([6])**.** The real sequences (λ_n) and (θ_n) in $(0, 1]$ are said to satisfy the Chidume-Zegeye conditions if

1. $\lim_{n\to\infty} \theta_n = 0$;
2. $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$;
3. $\forall \varepsilon > 0 \exists m \in \mathbb{N} \forall n \geq m \, (\lambda_n \leq \theta_n\varepsilon)$;
4. $\forall \varepsilon > 0 \exists m \in \mathbb{N} \forall n \geq m \left(\frac{\left|\frac{\theta_{n-1}}{\theta_n} - 1\right|}{\lambda_n\theta_n} \leq \varepsilon \right)$;
5. $\lambda_n (1 + \theta_n) \leq 1$ for all $n \in \mathbb{N}$.

Notation: For $T : C \to C$ let $F(T)$ be the set of fixed points of T.

Theorem 1.3 ([6])**.** *Let C be a nonempty closed convex subset of a real Banach space X. Let $T : C \to C$ be a Lipschitz pseudocontractive map with Lipschitz constant L and $F(T) \neq \emptyset$. Let (x_n) be the Bruck iteration with starting point $x_1 \in C$, where the parameters (λ_n) and (θ_n) satisfy the Chidume-Zegeye conditions. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Remark 1.4. Instead of $F(T) \neq \emptyset$ one can also assume that C is bounded.

In fact, Theorem 1.3 is shown as a consequence of the fact that $\|x_n - z_{n-1}\| \to 0$, where z_n is the unique point (whose existence is guaranteed by [12]) satisfying

$$z_n = t_nT(z_n) + (1 - t_n)x_1, \quad \text{where } t_n := \tfrac{1}{1+\theta_n}.$$

In particular, (x_n) strongly converges towards a fixed point of T provided that (z_n) does. The latter is known to be the case e.g. for reflexive Banach spaces X with uniformly Gâteaux differentiable norm provided that T has a fixed point (or C being bounded) and every nonempty bounded closed convex subset of X has the fixed point property for nonexpansive self-mappings (see [12, 13]). So, in particular, (z_n) (and consequently (x_n)) strongly converges to a fixed point of T if X is a uniformly smooth Banach space, T has a fixed point and C is closed and convex (see Corollary 11.8 in [5]).

In [10], we extracted from the proof in [6] explicit and highly uniform rates of convergence for $\|x_n - Tx_n\| \to 0$ (asymptotic regularity) and $\|x_n - z_{n-1}\| \to 0$.

Effective uniform rates on the strong convergence of (z_n), however, in general do not exist even in the special case of Hilbert spaces. Nevertheless, one can obtain effective uniform rates Φ of so-called metastability in the sense of Tao, i.e. (here $[n; n+g(n)] := \{n, n+1, n+2, \ldots, n+g(n)\}$)

$$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \leq \Phi(\varepsilon, g) \, \forall i, j \in [n; n + g(n)] \left(\|z_i - z_j\| < \varepsilon\right),$$

which we extract for the Hilbert space case. We then combine this with our asymptotic regularity rate to obtain (again for Hilbert spaces) a rate of metastability Ω for (x_n), in fact we get

(1) $\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \leq \Omega(\varepsilon, g) \, \forall i, j \in [n; n + g(n)] \forall l \geq n \left(\|x_i - x_j\| < \varepsilon \wedge \|Tx_l - x_l\| < \varepsilon\right).$

Here Ω only depends (in addition to ε, g) on a Lipschitz constant L for T, an upper bound $d \geq \|x_1 - p\|$ for some T-fixed point p and some moduli related to the scalars $(\lambda_n), (\theta_n)$.

(1) trivially implies the finitary (in the sense that only a finite initial segment of (x_n) is mentioned) statement

$$(2)\ \forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Omega(\varepsilon, g)\, \forall i, j \in [n; n + g(n)] \left(\|x_i - x_j\| < \varepsilon \wedge \|Tx_i - x_i\| < \varepsilon\right)$$

which - in turn - trivially implies that (x_n) strongly converges to a fixed point of T as metastability ineffectively is equivalent to the usual Cauchy property. In this sense, our quantitative form also constitutes a finitary version (in the sense of Tao [14, 15]) of that strong convergence theorem.

2 Quantitative Analysis

2.1 Resolvent Convergence

The following result is closely related to results of Browder [1] and Bruck [4]. It has been shown by Lan and Wu in [11] using techniques similar to those of Browder [2]. Although Browder's proof (for the nonexpansive case) has been analyzed by Kohlenbach in [9], it is considerably more difficult to treat than our proof below which follows the ideas of [4] (which in turn is based on [7]).

Theorem 2.1. *Let H be a Hilbert space, $C \subseteq H$ be a nonempty bounded closed convex subset and $T : C \to C$ be a demicontinuous pseudocontraction. Then, for each $x \in C$ and $t \in [0, 1)$, there exists a unique path (z_t) in C such that $z_t = tTz_t + (1 - t)\,x$. Moreover, the strong*

$$\lim_{t \to 1^-} z_t = z,$$

exists and is the fixed point of T closest to x.

Proof. For each $x \in C$ and nonnegative $t < 1$, the mapping $T_t : C \to C$, $z \mapsto tTz + (1 - t)\,x$ satisfies

$$\begin{aligned} \langle T_t x_1 - T_t x_2, x_1 - x_2 \rangle &= \langle tTx_1 + (1 - t)\,x - tTx_2 - (1 - t)\,x, x_1 - x_2 \rangle \\ &= t\,\langle Tx_1 - Tx_2, x_1 - x_2 \rangle \\ &\leq t\,\|x_1 - x_2\|^2 . \end{aligned} \tag{1}$$

Therefore, T_t is pseudocontractive. It is also demicontinuous: for any sequence (x_n) in C with $x_n \to x$, we have

$$\langle y, T_t x_n - T_t x \rangle = t\,\langle y, Tx_n - Tx \rangle \to 0 \quad \text{for all } y \in H$$

since T was demicontinuous. We conclude by Corollary 4 of [4] that T_t has a fixed point $z_t \in C$, *i.e.*, a point satisfying the equation

$$z_t = tTz_t + (1 - t)\,x.$$

Moreover, by (1), T_t is even strongly pseudocontractive, so z_t is unique. To see this, suppose that z_t and z_t' are two fixed points of T_t. Then, by (1),

$$\|z_t - z_t'\|^2 = \langle z_t - z_t', z_t - z_t' \rangle = \langle T_t z_t - T_t z_t', z_t - z_t' \rangle \leq t\,\|z_t - z_t'\|^2 .$$

Since $t < 1$, this implies $z_t = z_t'$. That (z_t) is continuous in t follows as in [12].

Strong convergence of (z_t) will be established in the course of the proof of Theorem 2.3. That the strong limit is a fixed point of T follows from (here we use that C is bounded)

$$|\langle Tz_t - z_t, Tz - z \rangle| \leq \|Tz_t - z_t\| \cdot \|Tz - z\| \stackrel{t \to 1^-}{\to} 0$$

and that (using that T is demicontinuous)

$$\langle Tz_t - z_t, Tz - z\rangle \overset{t\to 1^-}{\to} \langle Tz - z, Tz - z\rangle.$$

We now proceed to show that the strong limit is the fixed point of T with minimal distance from x. Suppose that y is a fixed point of T. Then $y = tTy + (1-t)\,x$ for $t = 1$. Repeating the calculations leading to (3) further below with $z_t = y$ and $t = 1$, we obtain

$$\|y - x\|^2 \geq \|z_s - x\|^2 + \|y - z_s\|^2, \quad \text{for all } 0 < s < 1.$$

Taking the strong limit $s \to 1$ implies

$$\|y - x\|^2 \geq \|z - x\|^2 + \|y - z\|^2$$

showing that z is the (unique) fixed point of T that is closest to x. □

In the following we present an effective rate of metastability for the strong convergence of (z_t). Provided that we assume the existence of (z_t) we not even need that T is demicontinuous (nor that X is complete or C closed).

Notation: Let $f : \mathbb{N} \to \mathbb{N}$ and $n, m \in \mathbb{N}$, then $f^{(n)}(m)$ denotes the result of n-times applying f starting from m, i.e. $f^{(0)}(m) := m, f^{(n+1)}(m) := f(f^{(n)}(m))$.
f^M denotes the function $f^M(n) := \max\{f(i) : i \leq n\}$.

We use the following

Lemma 2.2 ([8])**.** Let $D \in \mathbb{R}_+$ be a nonnegative real number and (a_n) be a nondecreasing sequence in the interval $[0, D]$, i.e. $0 \leq a_n \leq a_{n+1} \leq D$. Then the following holds

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \tilde{g}^{(\lceil D/\varepsilon \rceil)}(1)\, \forall i, j \in [n; n + g(n)]\, (|a_i - a_j| \leq \varepsilon),$$

where $\tilde{g}(n) := n + g(n)$. Moreover, n can be taken as $\tilde{g}^{(i)}(1)$ for some suitable $i \leq \lceil D/\varepsilon \rceil$.

Theorem 2.3. *Let X be a real inner product space and $C \subseteq X$ be a convex subset. Let $T : C \to C$ be a pseudocontraction which possesses a fixed point $v \in C$. Let $x \in C$ and assume that there exists (z_t) for x such that*

$$z_t = tTz_t + (1-t)x, \quad t \in [0, 1).$$

Let (t_n) be a sequence in $(0, 1)$ that converges towards 1 and $h : \mathbb{N} \to \mathbb{N}$ be such that $t_n \leq 1 - \frac{1}{h(n)+1}$ for all $n \in \mathbb{N}$. Set $z_n := z_{t_n}$. Then, for all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$ and all $\mathbb{N} \ni d \geq \|v - x\|$

$$\exists n \leq \Phi\left(\varepsilon, g, \chi_g, h, d\right) \forall i, j \in [n; n + g\left(n\right)] \left(\|z_i - z_j\| \leq \varepsilon\right),$$

where

$$\Phi\left(\varepsilon, g, \chi_g, h, d\right) := \chi_g^M\left(g_{h,\chi_g}^{(\lceil 16d^2/\varepsilon^2 \rceil)}\left(1\right)\right)$$

with

$$g_{h,\chi_g}\left(n\right) := \max\left\{h\left(i\right) : i \leq \chi_g\left(n\right) + g\left(\chi_g(n)\right)\right\}$$

and $\chi_g : \mathbb{N} \to \mathbb{N}$ is any function satisfying

$$\forall n \in \mathbb{N} \forall i \in [\chi_g(n); \tilde{g}\left(\chi_g(n)\right)] \left(|1 - t_i| \leq \frac{1}{n+1}\right). \tag{2}$$

If (t_n) is a nondecreasing sequence in $(0, 1)$ (not necessarily converging towards 1), then the bound can be simplified to $\Psi\left(\varepsilon, g, d\right) := \tilde{g}^{(\lceil 4d^2/\varepsilon^2 \rceil)}(1)$, where $\tilde{g}(n) := n + g(n)$.

Proof. Assume that $z_t \in C$ satisfies the equation

$$z_t = tTz_t + (1-t)\,x$$

for all $t \in [0,1)$. For $1 > t > s > 0$, we carry out a calculation similar to [9] and [7]; Since $Tz_t = \frac{1}{t}z_t - \frac{1-t}{t}x$ and T is pseudocontractive,

$$\begin{aligned}\|z_t - z_s\|^2 \geq \langle Tz_t - Tz_s, z_t - z_s\rangle &= \left\langle \frac{1}{t}z_t - \frac{1-t}{t}x - \frac{1}{s}z_s + \frac{1-s}{s}x, z_t - z_s \right\rangle \\ &= \left\langle \frac{1}{t}z_t - \frac{1}{t}z_s + \frac{1}{t}z_s - \frac{1}{s}z_s, z_t - z_s \right\rangle + \frac{t-s}{ts}\langle x, z_t - z_s\rangle \\ &= \frac{1}{t}\|z_t - z_s\|^2 + \left\langle \frac{s-t}{st}z_s, z_t - z_s \right\rangle + \frac{t-s}{ts}\langle x, z_t - z_s\rangle,\end{aligned}$$

and since $0 < t < 1$,

$$\left\langle \frac{t-s}{st}z_s, z_t - z_s \right\rangle \geq \left(\frac{1}{t} - 1\right)\|z_t - z_s\|^2 + \frac{t-s}{ts}\langle x, z_t - z_s\rangle \geq \frac{t-s}{ts}\langle x, z_t - z_s\rangle.$$

Since $s < t$, we conclude

$$\langle z_s - x, z_t - z_s\rangle \geq 0.$$

Therefore,

$$\begin{aligned}\|z_t - x\|^2 = \langle z_t - x, z_t - x\rangle &= \langle z_s - x + (z_t - z_s), z_s - x + (z_t - z_s)\rangle \\ &= \langle z_s - x, z_s - x\rangle + \langle z_t - z_s, z_t - z_s\rangle + 2\langle z_s - x, z_t - z_s\rangle \\ &\geq \|z_s - x\|^2 + \|z_t - z_s\|^2. \qquad (3)\end{aligned}$$

Therefore, $(\|z_t - x\|^2)_t$ is nondecreasing (as $t \nearrow 1^-$) and

$$\|z_t - z_s\|^2 \leq |\,\|z_s - x\|^2 - \|z_t - x\|^2\,|. \qquad (4)$$

(z_t) is also bounded as follows from the existence of a fixed point $v \in C$ reasoning as in Proposition 2(iv) of [12]: If $v \in F(T)$, then

$$\begin{aligned}\|z_t - v\|^2 &= \langle tTz_t + (1-t)\,x - v, z_t - v\rangle \\ &= t\,\langle Tz_t - Tv, z_t - v\rangle + (1-t)\,\langle x - v, z_t - v\rangle \\ &\leq t\|z_t - v\|^2 + (1-t)\,\langle x - v, z_t - v\rangle,\end{aligned}$$

which implies

$$(1-t)\,\|z_t - v\|^2 \leq (1-t)\,\|x - v\| \cdot \|z_t - v\|.$$

Since $t < 1$, this implies that $\|z_t - v\| \leq \|x - v\|$. Hence

$$\|z_t - x\| \leq \|z_t - v\| + \|v - x\| \leq 2\,\|v - x\| \leq 2d, \text{ i.e.}$$

$(\|z_t - x\|^2)_t$ is bounded by $4d^2$.

Together with Lemma 2.2 applied to $(\|z_{t_n} - x\|^2)_n, 4d^2$ and ε^2 and (4) above the theorem now follows in the case where $1 > t_{n+1} \geq t_n > 0$ for all $n \in \mathbb{N}$. For the case of a general sequence (t_n) which is assumed to converge to 1 one reasons literally as in the proof of Theorem 4.2 in [9]. □

Remark 2.4. Theorem 4.2 of [9] establishes the same result for nonexpansive mappings.

Remark 2.5. It is not hard to show that Theorem 2.3 also holds with the assumption $F(T) \neq \emptyset$ being replaced by $\forall \varepsilon > 0 \exists v_\varepsilon \in C(\|x - v_\varepsilon\| \leq d \wedge \|Tv_\varepsilon - v_\varepsilon\| \leq \varepsilon)$.

2.2 Asymptotic Regularity of the Bruck Iteration

Theorem 2.6 ([10]). *Let C be a nonempty, closed and convex subset of a real Banach space X and $x \in C$. Let $T : C \to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant L and for some $d > 0$ assume that T possesses arbitrarily good ε-fixed points $p_\varepsilon \in C$ with $\|x - p_\varepsilon\| < d$. Let (x_n) be the Bruck iteration (Definition 1.1) with starting point $x_1 := x$. Let z_n be the unique element in C satisfying $z_n = t_n T(z_n) + (1 - t_n)x_1$ with $t_n := 1/(1 + \theta_n)$. Given rates of convergence/divergence $R_i : (0, \infty) \to \mathbb{N}$ for the Chidume-Zegeye conditions 1.2, we get*

$$\forall \varepsilon > 0 \forall n \geq \Psi\left(d, L, R_1, R_2, R_3, R_4, \varepsilon\right)\left(\|x_n - Tx_n\| < \varepsilon\right)$$

and

$$\forall \varepsilon > 0 \forall n \geq \chi\left(d, L, R_1, R_2, R_3, R_4, \varepsilon\right)\left(\|x_n - z_{n-1}\| < \varepsilon\right),$$

where

$$\Psi\left(d, L, R_1, R_2, R_3, R_4, \varepsilon\right) = \max\Big\{N_2\left(C\right) + 1, R_1\left(\frac{\varepsilon}{4r}\right) + 1\Big\}$$

and

$$\chi\left(d, L, R_1, R_2, R_3, R_4, \varepsilon\right) = N_2\left(C\right) + 1$$

with

$$\begin{aligned}
&N_1\left(\varepsilon\right) := \max\left\{R_3\left(\frac{2\varepsilon s}{3r^2}\right), R_4\left(\sqrt{\frac{\varepsilon}{r^2} + \frac{9}{4}} - \frac{3}{2}\right)\right\},\\
&N_2\left(x\right) := R_2\left(\frac{x}{2}\right) + 1,\\
&C := \frac{8\left(1 + L\right)^2 r^2}{\varepsilon^2} + 2\left(N_1\left(\frac{\varepsilon^2}{8\left(1 + L\right)^2}\right) - 1\right),\\
&r := \max\left\{\frac{\left(2 + L\right)^{R_3(d)} - 1}{1 + L}d, 2d\right\},\\
&s := \frac{1}{2\left(\frac{5}{2} + L\right)\left(2 + L\right)}.
\end{aligned}$$

Proof. The first claim is Theorem 1 in [10] and the second claim follows from formula (24) in the proof of that theorem (even with ε being replaced by $\varepsilon/(2(1 + L))$ in the definition of χ). $\square$

Corollary 2.7 ([10]). *In the situation of Theorem 2.6, one may drop the condition that T has arbitrarily good approximate fixed points and instead require* $\operatorname{diam}(C) \leq d$. *In this case,*

$$\chi(d, L, R_1, R_2, R_3, R_4, \varepsilon) := N_2(C) + 1 \text{ and } \Psi\left(d, L, R_1, R_2, R_3, R_4, \varepsilon\right) = \max\Big\{\chi(\varepsilon), R_1\left(\frac{\varepsilon}{2d}\right) + 1\Big\}$$

and

$$\begin{aligned}
&N_1\left(\varepsilon\right) := \max\left\{R_3\left(\frac{\varepsilon}{4d^2\left(2 + L\right)}\right), R_4\left(\sqrt{\frac{\varepsilon}{d^2} + 1} - 1\right)\right\},\\
&N_2\left(x\right) := R_2\left(\frac{x}{2}\right) + 1,\\
&C := \frac{8\left(1 + L\right)^2 d^2}{\varepsilon^2} + 2\left(N_1\left(\frac{\varepsilon^2}{8\left(1 + L\right)^2}\right) - 1\right).
\end{aligned}$$

2.3 Strong Convergence of the Bruck Iteration

Theorem 2.8. *If, in the situation of Theorem 2.6, X is a Hilbert space, then (assuming w.l.o.g. $L \geq 1$) for all $\varepsilon > 0$ and all $g : \mathbb{N} \to \mathbb{N}$*

$$\exists n \leq \chi^M \left(g_{h,\chi}^{(\lceil 64d^2/\varepsilon^2 \rceil)}(1) \right) + \Psi(\varepsilon) + 1 \, \forall i, j \in [n; n + g\,(n)] \, \forall l \geq n \, (\|x_i - x_j\| \leq \varepsilon \wedge \|Tx_l - x_l\| \leq \varepsilon)$$

where $h : \mathbb{N} \to \mathbb{N}$ is a function such that $h(n) \geq 1/\theta_n$ for all $n \in \mathbb{N}$ and $\chi(n) := R_1(1/n)$,

$$g'(n) := g\,(n + 1 + \Psi\,(\varepsilon)) + \Psi\,(\varepsilon) + 1, \quad g_{h,\chi}(n) := \max\left\{h\,(i) : i \leq \chi(n) + g'\,(\chi(n))\right\},$$

and R_1 and Ψ as in Corollary 2.7.

Proof. In Theorem 2.6, the resolvent z_t is instantiated with the sequence $t = t_n = \frac{1}{1+\theta_n}$ and the starting point x_1. We now show how to apply Theorem 2.3 to this instantiation; if we set $\chi(n) := R_1\,(1/n)$, then $\theta_i \leq 1/n$ for all $i \geq \chi(n)$. Since $\theta_n \in (0, 1]$, this implies

$$|1 - t_i| = 1 - \frac{1}{1 + \theta_i} \leq 1 - \frac{1}{1 + \frac{1}{n}} = \frac{1}{n + 1}, \quad \text{for all } i \geq \chi(n).$$

Since this holds for all $i \geq \chi(n)$, the function χ satisfies (2) independently of the counter-function g and we may set $\chi_g := \chi$ in Theorem 2.3.
Moreover, for all $n \in \mathbb{N}$, $h(n) \geq 1/\theta_n$ implies $1 + h(n) \geq \frac{1+\theta_n}{\theta_n}$, whence

$$\frac{1}{h(n) + 1} \leq \frac{\theta_n}{1 + \theta_n} = 1 - \frac{1}{1 + \theta_n}.$$

Therefore,

$$t_n = \frac{1}{1 + \theta_n} \leq 1 - \frac{1}{h(n) + 1}, \quad \text{for all } n \in \mathbb{N}.$$

Now observe that, by Theorem 2.3 and Remark 2.5 applied to the counter-function g' and error $\varepsilon/2$, there exists an $n \leq \chi^M \left(g_{h,\chi}^{(\lceil 64d^2/\varepsilon^2 \rceil)}\,(1) \right)$ such that

$$\|z_i - z_j\| \leq \frac{\varepsilon}{2}, \quad \text{for all } i, j \in [n; n + g'(n)]\,. \tag{5}$$

Since $[n; n + g'(n)] = [n; n + 1 + \Psi(\varepsilon) + g\,(n + 1 + \Psi(\varepsilon))] \supseteq [n + \Psi(\varepsilon); n + 1 + \Psi(\varepsilon) + g\,(n + 1 + \Psi(\varepsilon))]$, we conclude that if we set $n_0 := n + 1 + \Psi(\varepsilon)$, then

$$\|z_{i-1} - z_{j-1}\| \leq \frac{\varepsilon}{2}, \quad \text{for all } i, j \in [n_0; n_0 + g(n_0)]\,.$$

Since $n_0 \geq \Psi(\varepsilon)$, we conclude from (24) of [10] for all $n \geq n_0$, $\|x_n - z_{n-1}\| \leq \frac{\varepsilon}{2(1+L)} \leq \varepsilon/4$, since we may w.l.o.g. assume $L \geq 1$. Thus,

$$\|x_i - x_j\| \leq \|x_i - z_{i-1}\| + \|z_{i-1} - z_{j-1}\| + \|z_{j-1} - x_j\| \leq \varepsilon, \quad \text{for all } i, j \in [n_0; n_0 + g(n_0)]\,.$$

Moreover, we get from Theorem 2.6

$$\|x_n - Tx_n\| \leq \varepsilon, \quad \text{for all } n \geq \Psi(\varepsilon).$$

This completes the proof. □

Corollary 2.9. *If (θ_n) is nondecreasing, then for all $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$*

$$\exists n \leq \tilde{g}'^{(\lceil 16d^2/\varepsilon^2 \rceil)}(1) + \Psi(\varepsilon) + 1 \, \forall i, j \in [n; n + g\,(n)] \, \forall l \geq n \, (\|x_i - x_j\| \leq \varepsilon \wedge \|Tx_l - x_l\| \leq \varepsilon)$$

where $\tilde{g}'(n) = g'(n) + n$ and $g'(n) = g(n + 1 + \Psi(\varepsilon)) + \Psi\,(\varepsilon) + 1$.

Proof. Since (θ_n) is nondecreasing, the second part of Theorem 2.3 implies that there exists an $n \leq \tilde{g}'^{(\lceil 16d^2/\varepsilon^2 \rceil)}(1)$ such that

$$\|z_i - z_j\| \leq \frac{\varepsilon}{2}, \quad \text{for all } i, j \in [n; n + g'(n)],$$

which is the analog to equation (5). The remainder of the proof is then the same. □

As a corollary to the proof of Theorem 2.8 we get the following transformation of an assumed rate of metastability for (z_n) into one for (x_n) in general Banach spaces:

Corollary 2.10. *In the situation of Theorem 2.6 (so X is not necessarily a Hilbert space), suppose that for all $g : \mathbb{N} \to \mathbb{N}$ and $\varepsilon > 0$,*

$$\exists n \leq \Omega\,(d, g, \varepsilon)\, \forall i, j \in [n; n + g(n)]\, (\|z_i - z_j\| \leq \varepsilon)\,,$$

and let $\chi^M(n) := R_1(1/n)$. Then, for all $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists n \leq \chi^M\,(\Omega(d, g, \varepsilon/2)) + \Psi(\varepsilon) + 1\, \forall i, j \in [n; n + g\,(n)]\, \forall l \geq n\, (\|x_i - x_j\| \leq \varepsilon \wedge \|Tx_l - x_l\| \leq \varepsilon)\,.$$

Remark 2.11. For the canonical choice $\lambda_n = \frac{1}{(n+1)^a}$ and $\theta_n = \frac{1}{(n+1)^b}$, where $0 < b < a$ and $a + b < 1$, the bound is as stated in Corollary 2.9.

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