Rate of metastability for Bruck’s iteration of pseudocontractive mappings in Hilbert space

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Abstract

We give a quantitative version of the strong convergence of Bruck’s iteration schema for Lipschitzian pseudocontractions in Hilbert space.

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1 Introduction

Let $X$ be a normed linear space and $S \subseteq X$ be a subset of $X$. In 1967, Browder introduced an important generalization of the class of nonexpansive mappings, namely the pseudocontractive mappings $T: S \to S$ defined by

$$\forall u, v \in S \forall \lambda > 1 \ ((\lambda - 1)\|u - v\| \leq \|(\lambda I - T)(u) - (\lambda I - T)(v)\|),$$

where $I$ denotes the identity mapping.

Apart from being a generalization of nonexpansive mappings, the pseudocontractive mappings are also closely related to accretive operators, where an operator $A$ is called accretive if for every $u, v \in S$ and for all $s > 0$,

$$\|u - v\| \leq \|u - v + s(Au - Av)\|.$$

Observe that $T$ is pseudocontractive if and only if $I - T$ is accretive. Therefore, any fixed point of $T$ is a root of the accretive operator $I - T$.

In a Hilbert space, $T$ is pseudocontractive iff

$$\forall u, v \in S \ (\langle Tu - Tv, u - v \rangle \leq \|u - v\|^2)$$

(see e.g. [5]).

In [4], Bruck introduced the following iteration schema for pseudocontractive mappings:

Definition 1.1 ([4]). Let $C$ be a nonempty convex subset of a real normed space and let $T: C \to C$ be a pseudocontraction. Let $(\lambda_n)$, $(\theta_n)$ be sequences in $[0, 1]$ with $\lambda_n(1 + \theta_n) \leq 1$ for all $n \in \mathbb{N}$. The Bruck iteration scheme with starting point $x_1 \in C$ is defined as

$$x_{n+1} = (1 - \lambda_n) x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1).$$

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Among many other things, Bruck showed that in Hilbert spaces and for bounded closed and convex subsets \( C \) this iteration strongly converges for so-called acceptably paired sequences \((\lambda_n), (\theta_n))\). Moreover the limit is a fixed point of \( T \) provided that \( T \) is demicontinuous (in addition to being pseudocontractive).

In \([6]\), it is shown that Bruck’s iteration (with more natural conditions on \((\lambda_n), (\theta_n))\) is asymptotically regular, i.e.

\[
\|x_n - T(x_n)\| \xrightarrow{n \to \infty} 0.
\]

in arbitrary Banach spaces provided that \( T \) is a Lipschitzian pseudocontractive mapping which still includes the important class of strictly pseudocontractive mappings in the sense of Browder and Petryshyn \([3]\) (see \([5]\)).

**Definition 1.2** (\([6]\)). The real sequences \((\lambda_n)\) and \((\theta_n)\) in \((0, 1]\) are said to satisfy the Chidume-Zegeye conditions if

1. \( \lim_{n \to \infty} \theta_n = 0; \)
2. \( \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty; \)
3. \( \forall \varepsilon > 0 \exists m \in \mathbb{N} \forall n \geq m \ (\lambda_n \leq \theta_n \varepsilon); \)
4. \( \forall \varepsilon > 0 \exists m \in \mathbb{N} \forall n \geq m \left( \frac{\|x_{n+1} - x_n\|}{\lambda_n \theta_n} \leq \varepsilon \right); \)
5. \( \lambda_n (1 + \theta_n) \leq 1 \) for all \( n \in \mathbb{N}. \)

**Notation:** For \( T : C \to C \) let \( F(T) \) be the set of fixed points of \( T \).

**Theorem 1.3** (\([6]\)). Let \( C \) be a nonempty closed convex subset of a real Banach space \( X \). Let \( T : C \to C \) be a Lipschitz pseudocontractive map with Lipschitz constant \( L \) and \( F(T) \neq \emptyset \). Let \((x_n)\) be the Bruck iteration with starting point \( x_1 \in C \), where the parameters \((\lambda_n)\) and \((\theta_n)\) satisfy the Chidume-Zegeye conditions. Then \( \|x_n - Tx_n\| \to 0 \) as \( n \to \infty \).

**Remark 1.4.** Instead of \( F(T) \neq \emptyset \) one can also assume that \( C \) is bounded.

In fact, Theorem 1.3 is shown as a consequence of the fact that \( \|x_n - z_{n-1}\| \to 0 \), where \( z_n \) is the unique point (whose existence is guaranteed by \([12]\)) satisfying

\[
z_n = t_n T(z_n) + (1 - t_n) x_1, \quad \text{where} \quad t_n := \frac{1}{1 + \theta_n}.
\]

In particular, \((x_n)\) strongly converges towards a fixed point of \( T \) provided that \((z_n)\) does. The latter is known to be the case e.g. for reflexive Banach spaces \( X \) with uniformly Gâteaux differentiable norm provided that \( T \) has a fixed point (or \( C \) being bounded) and every nonempty bounded closed convex subset of \( X \) has the fixed point property for nonexpansive self-mappings (see \([12, 13]\)). So, in particular, \((z_n)\) (and consequently \((x_n)\)) strongly converges to a fixed point of \( T \) if \( X \) is a uniformly smooth Banach space, \( T \) has a fixed point and \( C \) is closed and convex (see Corollary 11.8 in \([5]\)).

In \([10]\), we extracted from the proof in \([6]\) explicit and highly uniform rates of convergence for \( \|x_n - Tx_n\| \to 0 \) (asymptotic regularity) and \( \|x_n - z_{n-1}\| \to 0 \).

Effective uniform rates on the strong convergence of \((z_n)\), however, in general do not exist even in the special case of Hilbert spaces. Nevertheless, one can obtain effective uniform rates \( \Phi \) of so-called metastability in the sense of Tao, i.e. (here \( \{n; n + g(n)\} := \{n, n+1, n+2, \ldots, n+g(n)\} \))

\[
\forall \varepsilon > 0 \forall g : \mathbb{N} \to \mathbb{N} \exists n \leq \Phi(\varepsilon, g) \forall i,j \in [n; n + g(n)] (\|z_i - z_j\| < \varepsilon),
\]

which we extract for the Hilbert space case. We then combine this with our asymptotic regularity rate to obtain (again for Hilbert spaces) a rate of metastability \( \Omega \) for \((x_n)\), in fact we get

\[
(1) \forall \varepsilon > 0 \forall g : \mathbb{N} \to \mathbb{N} \exists n \leq \Omega(\varepsilon, g) \forall i,j \in [n; n + g(n)] \forall \ell \geq n (\|x_i - x_j\| < \varepsilon \land \|Tx_i - x_i\| < \varepsilon).
\]
Here $\Omega$ only depends (in addition to $\varepsilon, g$) on a Lipschitz constant $L$ for $T$, an upper bound $d \geq \|x_1 - p\|$ for some $T$-fixed point $p$ and some moduli related to the scalars $(\lambda_n), (\theta_n)$.

(1) trivially implies the finitary (in the sense that only a finite initial segment of $(x_n)$ is mentioned) statement

$$(2) \forall \varepsilon > 0 \forall g : \mathbb{N} \to \mathbb{N} \exists n \leq \Omega(\varepsilon, g) \forall i, j \in [n; n + g(n)] \left(\|x_i - x_j\| < \varepsilon \wedge \|T x_i - x_i\| < \varepsilon\right)$$

which - in turn - trivially implies that $(x_n)$ strongly converges to a fixed point of $T$ as metastability ineffectively is equivalent to the usual Cauchy property. In this sense, our quantitative form also constitutes a finitary version (in the sense of Tao [14, 15]) of that strong convergence theorem.

## 2 Quantitative Analysis

### 2.1 Resolvent Convergence

The following result is closely related to results of Browder [1] and Bruck [4]. It has been shown by Lan and Wu in [11] using techniques similar to those of Browder [2]. Although Browder’s proof (for the nonexpansive case) has been analyzed by Kohlenbach in [9], it is considerably more difficult to treat than our proof below which follows the ideas of [4] (which in turn is based on [7]).

**Theorem 2.1.** Let $H$ be a Hilbert space, $C \subseteq H$ be a nonempty bounded closed convex subset and $T : C \to C$ be a demicontinuous pseudocontraction. Then, for each $x \in H$ and $t \in [0, 1)$, there exists a unique $z_t \in H$ such that $z_t = tT z_t + (1 - t)x$. Moreover, the strong

$$\lim_{t \to 1^-} z_t = z,$$

exists and is the fixed point of $T$ closest to $x$.

**Proof.** For each $x \in H$ and nonnegative $t < 1$, the mapping $T_t : C \to C, z \mapsto tTz + (1 - t)x$ satisfies

$$(T_t x_1 - T_t x_2, x_1 - x_2) = (tT x_1 + (1 - t)x - tT x_2 - (1 - t)x, x_1 - x_2)$$

$$= t (T x_1 - T x_2, x_1 - x_2)$$

$$\leq t \|x_1 - x_2\|^2. \quad (1)$$

Therefore, $T_t$ is pseudocontractive. It is also demicontinuous: for any sequence $(x_n)$ in $H$ with $x_n \to x$, we have

$$(y, T_t x_n - T_t x) = t (y, T x_n - T x) \to 0 \quad \text{for all } y \in H$$

since $T$ was demicontinuous. We conclude by Corollary 4 of [4] that $T_t$ has a fixed point $z_t \in C$, i.e., a point satisfying the equation

$$z_t = tT z_t + (1 - t)x.$$

Moreover, by (1), $T_t$ is even strongly pseudocontractive, so $z_t$ is unique. To see this, suppose that $z_t$ and $z'_t$ are two fixed points of $T_t$. Then, by (1),

$$\|z_t - z'_t\|^2 = \langle z_t - z'_t, z_t - z'_t \rangle = \langle T_t z_t - T_t z'_t, z_t - z'_t \rangle \leq t \|z_t - z'_t\|^2.$$

Since $t < 1$, this implies $z_t = z'_t$.

Strong convergence of the path will be established in the course of the proof of Theorem 2.3. We now proceed to show that the strong limit is the fixed point of $T$ with minimal distance from $x$. Suppose that $y$ is a fixed point of $T$. Then $y = t Ty + (1 - t)x$ for $t = 1$. Repeating the calculations leading to (3) further below with $z_t = y$ and $t = 1$, we obtain

$$\|y - x\|^2 \geq \|z_s - x\|^2 + \|y - z_s\|^2, \quad \text{for all } 0 < s < 1.$$

Consequently $\|y - x\| \geq \|z_s - x\|$, so taking the strong limit $s \to 1$ implies the claim. \qed
In the following we present an effective rate of metastability for the strong convergence of \((z_t)\). Provided that we assume the existence of \((z_t)\) we not even need that \(T\) is demicontinuous (nor that \(X\) is complete or \(C\) closed).

**Notation:** Let \(f : \mathbb{N} \to \mathbb{N}\) and \(n, m \in \mathbb{N}\), then \(f^{(m)}(m)\) denotes the result of \(n\) times applying \(f\) starting from \(m\), i.e. \(f^{(0)}(m) := m, f^{(n+1)}(m) := f(f^{(n)}(m))\). 

\(f^M\) denotes the function \(f^M(n) := \max\{f(i) : i \leq n\}\).

We use the following

**Lemma 2.2** ([8]). Let \(D \in \mathbb{R}_+\) be a real number and \((a_n)\) be an increasing sequence in the interval \([0, D]\), i.e. \(0 \leq a_n \leq a_{n+1} \leq D\). Then the following holds

\[
\forall \varepsilon > 0 \ \forall g : \mathbb{N} \to \mathbb{N} \ \exists n \leq \tilde{g}((D/\varepsilon))(1) \ \forall i, j \in [n; n + g(n)] \ (|a_i - a_j| \leq \varepsilon),
\]

where \(\tilde{g}(n) := n + g(n)\). Moreover, \(n\) can be taken as \(\tilde{g}(1)\) for some suitable \(i \leq [D/\varepsilon]\).

**Theorem 2.3.** Let \(X\) be a real inner product space and \(C \subseteq X\) be a convex subset. Let \(T : C \to C\) be a pseudocontraction which possesses a fixed point \(v \in C\). Let \(x \in C\) and assume that there exists a path for \(x\) such that

\[
z_t = tTz_t + (1 - t)x, \quad t \in [0, 1).
\]

Let \((t_n)\) be a sequence in \((0, 1)\) that converges towards 1 and \(h : \mathbb{N} \to \mathbb{N}\) be such that \(t_n \leq 1 - \frac{1}{h(n)+1}\) for all \(n \in \mathbb{N}\). Set \(z_n := z_{t_n}\), where \(z_t\) is as in Theorem 2.1. Then, for all \(\varepsilon > 0\), all \(g : \mathbb{N} \to \mathbb{N}\) and all \(N \geq d \geq ||v - x||\)

\[
\exists n \leq \Phi(\varepsilon, g, \chi, h, d) \ \forall i, j \in [n; n + g(n)] \ (||z_i - z_j|| \leq \varepsilon),
\]

where

\[
\Phi(\varepsilon, g, \chi, h, d) := \chi_g^\bigg(\frac{[2D/\varepsilon]}{g_h(\chi_g)}(1)\bigg)
\]

with

\[
g_h(\chi_g)(n) := \max\{h(i) : i \leq \chi_g(n) + g(\chi_g(n))\}
\]

and \(\chi_g : \mathbb{N} \to \mathbb{N}\) is any function satisfying

\[
\forall n \in \mathbb{N} \ \forall i \in [\chi_g(n); \tilde{g}((\chi_g(n)))] \left(1 - s_i \leq \frac{1}{n + 1}\right).
\]

If \((t_n)\) is an increasing sequence in \((0, 1)\) (not necessarily converging towards 1), then the bound can be simplified to \(\Psi(\varepsilon, g, d) := \tilde{g}^\big([D/\varepsilon]\big)(1)\), where \(\tilde{g}(n) := n + g(n)\).

**Proof.** Assume that \(z_t \in C\) satisfies the equation

\[
z_t = tTz_t + (1 - t)x
\]

for all \(t \in [0, 1)\). For \(1 > t > s > 0\), we carry out a calculation similar to [9] and [7]; Since \(Tz_t = \frac{1}{t}z_t + \frac{1-t}{t}x\) and \(T\) is pseudocontractive,

\[
||z_t - z_s||^2 \geq \langle Tz_t - Tz_s, z_t - z_s \rangle = \left\langle \frac{1}{t}z_t - \frac{1-t}{t}x - \frac{1-s}{s}z_s + \frac{1-s}{s}x, z_t - z_s \right\rangle
\]

\[
= \left\langle \frac{1}{t}z_t - \frac{1-s}{s}z_s + \frac{1-s}{s}z_t - z_s, z_t - z_s \right\rangle + \frac{t-s}{ts} \langle x, z_t - z_s \rangle
\]

\[
= \frac{1}{t}||z_t - z_s||^2 + \left\langle \frac{s-t}{st}z_t, z_t - z_s \right\rangle + \frac{t-s}{ts} \langle x, z_t - z_s \rangle,
\]

and since \(0 < t < 1\),

\[
\left\langle \frac{t-s}{st}z_t, z_t - z_s \right\rangle \geq \left(\frac{1}{l} - 1\right) ||z_t - z_s||^2 + \frac{t-s}{ts} \langle x, z_t - z_s \rangle \geq \frac{t-s}{ts} \langle x, z_t - z_s \rangle
\]
Since $s < t$, we conclude
\[
\langle z_s - x, z_t - z_s \rangle \geq 0.
\]
Therefore,
\[
\|z_t - x\|^2 = \langle z_t - x, z_t - x \rangle = \langle z_s - x + (z_t - z_s), z_s - x + (z_t - z_s) \rangle \\
= \langle z_s - x, z_s - x \rangle + \langle z_t - z_s, z_t - z_s \rangle + 2 \langle z_s - x, z_t - z_s \rangle \\
\geq \|z_s - x\|^2 + \|z_t - z_s\|^2.
\]
(3)
Therefore, $(\|z_t - x\|^2)_t$ and, in turn, $(\|z_t - x\|)_t$ is nondecreasing.
The path is also bounded as follows from the existence of a fixed point $v \in C$ reasoning as in Proposition 2(iv) of [12]: If $v \in F(T)$, then
\[
\|z_t - v\|^2 = \langle tTz_t + (1 - t)x - v, z_t - v \rangle \\
= t \langle Tz_t - Tv, z_t - v \rangle + (1 - t) \langle x - v, z_t - v \rangle \\
\leq t \|z_t - v\|^2 + (1 - t) \|x - v, z_t - v\|
\]
which implies
\[
(1 - t) \|z_t - v\|^2 \leq (1 - t) \|x - v\| \cdot \|z_t - v\|.
\]
Since $t < 1$, this implies that $\|z_t - v\| \leq \|x - v\|$. Hence
\[
\|z_t - v\| + \|v - x\| \leq 2 \|v - x\| \leq d, \text{ i.e.}
\]
the path $(\|z_t - x\|)_t$ is bounded by $d$.
Together with Lemma 2.2, the theorem now follows in the case where $1 > t_{n+1} \geq t_n > 0$ for all $n \in \mathbb{N}$. For the case of a general sequence $(t_n)$ which is assumed to converge to 1 one reasons literally as in the proof of Theorem 4.2 in [9].

Remark 2.4. Theorem 4.2 of [9] establishes the same result for nonexpansive mappings.

Remark 2.5. It is not hard to show that Theorem 2.3 also holds with the assumption $F(T) \neq \emptyset$ being replaced by $\forall \varepsilon > 0 \exists v_\varepsilon \in C(\|x - v_\varepsilon\| \leq d \land \|Tv_\varepsilon - v_\varepsilon\| \leq \varepsilon)$.

2.2 Asymptotically Regularity of the Bruck Iteration

**Theorem 2.6** ([10]). Let $C$ be a nonempty, closed and convex subset of a real Banach space $X$.
Let $T : C \rightarrow C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant $L$ and for some $d > 0$ assume that $T$ possesses arbitrarily good $\varepsilon$-fixed points $x_\varepsilon \in K$ with $\|x_\varepsilon - x\| < d$.
Let $(x_n)$ be the Halpern iteration with starting point $x_1 \in C$. Let $z_n$ be the unique element in $C$ satisfying $z_n = t_nT(x_n) + (1 - t_n)x_1$ with $t_n := 1/(1 + \theta_n)$. Given rates of convergence/divergence
$R_i : \mathbb{R} \rightarrow \mathbb{N}$ for the Chidume-Zegeye conditions 1.2, we get
\[
\forall \varepsilon > 0 \forall n \geq \Psi(d, L, R_1, R_2, R_3, R_4, \varepsilon) \left(\|x_n - Tx_n\| < \varepsilon\right).
\]
and
\[
\forall \varepsilon > 0 \forall n \geq \chi(d, L, R_1, R_2, R_3, R_4, \varepsilon) \left(\|x_n - y_{n-1}\| < \varepsilon\right).
\]
where
\[
\Psi(d, L, R_1, R_2, R_3, R_4, \varepsilon) = \max\left\{N_2(C) + 1, R_1 \left(\frac{\varepsilon}{4r}\right) + 1\right\}
\]
and
\[
\chi(d, L, R_1, R_2, R_3, R_4, \varepsilon) = N_2(C) + 1
\]
Proof. The first claim is Theorem 1 in [10] and the second claim follows from formula (24) in the proof of that theorem (even with $\varepsilon$ being replaced by $2\varepsilon(1 + L)$ in the definition of $\chi$).

\[ \begin{align*}
N_1 (\varepsilon) &:= \max \left\{ R_3 \left( \frac{2\varepsilon s}{3r^2} \right), R_4 \left( \sqrt{\frac{\varepsilon}{\theta^2} + \frac{9}{4} - \frac{3}{2}} \right) \right\}, \\
N_2 (x) &:= R_2 \left( \frac{x}{2} \right) + 1,
\end{align*} \]

\[ C := \frac{18 (1 + L)^2 r^2}{\varepsilon^2} + 2 \left( \frac{N_1 \left( \frac{\varepsilon^2}{8 (1 + L)^2} \right) - 1}{1 + L} \right), \]

\[ r := \max \left\{ \frac{(2 + L)R_3(d) - 1}{1 + L}, d, 2d \right\}, \]

\[ s := \frac{1}{2 (\frac{1}{2} + L)(2 + L)}. \]

\[ \begin{align*}
\chi(d, L, R_1, R_2, R_3, R_4; \varepsilon) &:= N_2(C) + 1 \quad \text{and} \quad \Psi(d, L, R_1, R_2, R_3, R_4; \varepsilon) = \max \left\{ \chi(\varepsilon), \frac{\varepsilon}{2d} \right\} + 1
\end{align*} \]

\section{Strong Convergence of the Bruck Iteration}

\textbf{Theorem 2.8.} If, in the situation of Theorem 2.6, $X$ is a Hilbert space, then (assuming w.l.o.g. $L \geq 1$) for all $\varepsilon > 0$ and all $g : \mathbb{N} \to \mathbb{N}$

\[ \exists n \leq M \left( g_n, \chi \right) (1 + \Psi(\varepsilon) + 1) \quad \forall i, j \in \mathbb{N} \quad \forall n \geq n (\|x_i - x_j\| \leq \varepsilon \wedge \|Tx_i - x_i\| \leq \varepsilon) \]

where $h : \mathbb{N} \to \mathbb{N}$ is a function such that $h(n) \geq 1/\theta_n$ for all $n \in \mathbb{N}$ and $\chi(n) := R_1(1/n)$,

\[ g'(n) := g(n + 1 + \Psi(\varepsilon)) + \Psi(\varepsilon) + 1, \quad g_n, \chi(n) := \max \{ h(i) : i \leq \chi(n) + g'(\chi(n)) \}, \]

and $R_1$ and $\Psi$ as in Corollary 2.7.

Proof. In Theorem 2.6, the resolvent $z_t$ is instantiated with the sequence $t = t_n = \frac{1}{1 + \theta_n}$ and the starting point $x_1$. We now show how to apply Theorem 2.3 to this instantiation; if we set $\chi(n) := R_1(1/n)$, then $\theta_i \leq 1/n$ for all $k \geq \chi(n)$. Since $\theta_n \in (0, 1]$, this implies

\[ |1 - t_i| = 1 - \frac{1}{1 + \frac{1}{n}} \leq 1 - \frac{1}{1 + \frac{1}{n}} = \frac{1}{n + 1}, \quad \text{for all } i \geq \chi(n). \]

Since this holds for all $i \geq \chi(n)$, the function $\chi$ satisfies (2) independently of the counter-function $g$ and we may set $\chi_g := \chi$ in Theorem 2.3.
Moreover, for all \( n \in \mathbb{N} \), \( h(n) \geq 1/\theta_n \) implies \( 1 + h(n) \geq \frac{1+\theta_n}{\theta_n} \), whence

\[
\frac{1}{h(n) + 1} \leq \frac{\theta_n}{1 + \theta_n} = 1 - \frac{1}{1 + \theta_n}.
\]

Therefore,

\[
t_n = \frac{1}{1 + \theta_n} \leq 1 - \frac{1}{h(n) + 1}, \quad \text{for all } n \in \mathbb{N}.
\]

Now observe that, by Theorem 2.3 and Remark 2.5 applied to the counter-function \( g' \) and error \( \varepsilon/2 \), there exists an \( n \leq \chi^M \left( \frac{4d}{\varepsilon} \right) \) such that, for all \( i, j \in [n; n + g'(n)] \)

\[
\|z_i - z_j\| \leq \frac{\varepsilon}{2}, \quad \text{for all } i, j \in [n; n + g'(n)] \tag{4}
\]

Since \([n; n + g'(n)] = [n; n + 1 + \Psi(\varepsilon) + g(n + 1 + \Psi(\varepsilon))] \supseteq [n + \Psi(\varepsilon); n + 1 + \Psi(\varepsilon) + g(n + 1 + \Psi(\varepsilon))]\), we conclude that if we set \( n_0 := n + 1 + \Psi(\varepsilon) \), then

\[
\|z_{i-1} - z_{j-1}\| \leq \frac{\varepsilon}{2}, \quad \text{for all } i, j \in [n_0; n_0 + g(n_0)].
\]

Since \( n_0 \geq \Psi(\varepsilon) \), we conclude from (24) of [10] for all \( n \geq n_0 \), \( \|x_n - z_{n-1}\| \leq \frac{\varepsilon}{2(1 + L)} \leq \varepsilon/4 \), since we may w.l.o.g. assume \( L \geq 1 \). Thus,

\[
\|x_i - x_j\| \leq \|x_i - z_{i-1}\| + \|z_{i-1} - z_{j-1}\| + \|z_{j-1} - x_j\| \leq \varepsilon, \quad \text{for all } i, j \in [n_0; n_0 + g(n_0)]
\]

Moreover, we get from the last equation of the proof of Theorem 1 of [10]

\[
\|x_n - Tx_n\| \leq \varepsilon, \quad \text{for all } n \geq \Psi(\varepsilon).
\]

This completes the proof. \( \square \)

**Corollary 2.9.** If \((\lambda_n)\) and \((\theta_n)\) are nondecreasing, then for all \( \varepsilon > 0 \) and \( g : \mathbb{N} \to \mathbb{N} \)

\[
\exists n \leq \frac{\chi^M(4d/\varepsilon)}{1 + \Psi(\varepsilon) + 1} \forall i, j \in [n; n + g(n)] \left( \|x_i - x_j\| \leq \varepsilon \land \|Tx_i - x_i\| \leq \varepsilon \right)
\]

where \( \bar{g}'(n) = g'(n) + n \) and \( g'(n) = g(n + 1 + \Psi(\varepsilon)) + \Psi(\varepsilon) + 1 \).

**Proof.** Since \((\theta_n)\) is increasing, the second part of Theorem 2.3 implies that there exists an \( n \leq \bar{g}'(4d/\varepsilon) \) such that

\[
\|z_i - z_j\| \leq \frac{\varepsilon}{2}, \quad \text{for all } i, j \in [n; n + g'(n)],
\]

which is the analog to equation (4). The remainder of the proof is then the same. \( \square \)

As a corollary to the proof of Theorem 2.8 we get the following transformation of an assumed rate of metastability for \((z_n)\) into one for \((x_n)\) in general Banach spaces:

**Corollary 2.10.** In the Situation of Theorem 2.6 (so \( X \) is not necessarily a Hilbert space), suppose that for all \( g : \mathbb{N} \to \mathbb{N} \) and \( \varepsilon > 0 \),

\[
\exists n \leq \Omega(d, g, \varepsilon) \forall i, j \in [n; n + g(n)] \left( \|z_i - z_j\| \leq \varepsilon \right)
\]

and let \( \chi^M(n) := R_t(1/n) \). Then, for all \( \varepsilon > 0 \) and \( g : \mathbb{N} \to \mathbb{N} \),

\[
\exists n \leq \chi^M(\Omega(d, g, \varepsilon/2) + \Psi(\varepsilon) + 1) \forall i, j \in [n; n + g(n)] \left( \|x_i - x_j\| \leq \varepsilon \land \|Tx_i - x_i\| \leq \varepsilon \right).
\]

**Remark 2.11.** For the canonical choice \( \lambda_n = \frac{1}{(n+1)^2} \) and \( \theta_n = \frac{1}{(n+1)^2} \), where \( 0 < b < a \) and \( a + b < 1 \), the bound is as stated in Corollary 2.9.
References


