On the asymptotic behavior of odd operators

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Abstract

We give quantitative versions of strong convergence results due to Baillon, Bruck and Reich for iterations of nonexpansive odd (and more general) operators in uniformly convex Banach spaces.

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1 Introduction

Let X be a uniformly convex Banach space and $C \subseteq X$ a closed convex subset satisfying $C = -C$. In [2], Baillon, Bruck and Reich showed (among many other things) that the iteration $(T^n x)$ of an odd nonexpansive mapping $T : C \to C$ that is asymptotically regular at $x \in C$ strongly converges to a fixed point of T. By a famous result due to Ishikawa [4] the averaged mapping $T_\lambda x := (1-\lambda)x + \lambda T(x)$ with $\lambda \in (0,1)$ of a nonexpansive mapping $T : C \to C$ always is asympotically regular provided that $(T_\lambda^n x)$ is bounded (in fact – by another result from [2] – it suffices that $(\|T_\lambda^n x\|/n)$ converges to 0).
With T also T_λ is nonexpansive and odd and so the sequence (x_n) defined by $x_n := T_\lambda^n x$ (which trivially is bounded) converges strongly towards a fixed point $p \in C$ of T.

We first observe that the condition of T being nonexpansive and odd can be weakened to the condition

$$(W):\ \forall x, y \in C\ \big(\|Tx + Ty\| \leq \|x + y\|\big)$$

studied in [14] which also makes the assumption $C = -C$ superfluous.

It is easy to show that there is no computable (in the data at hand) rate of convergence even for $X := \mathbb{R}, C := [0,1], \lambda := 1/2, x := 1$ in the sense that there is a computable sequence

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(f_l) of odd nonexpansive functions $f_l : [-1,1] \to [-1,1]$ such that there is no computable function $\delta : \mathbb{N} \to \mathbb{N}$ such that for $x^l_n := (f_l)^n_{\frac{1}{2}}(1)$

$$(+)\ \forall m \geq \delta(l)\, (|x^l_m - x^l_{\delta(l)}| \leq \frac{1}{2}).$$

Define $f_l(x) := a_l \cdot x$, where $a_l := \sum\limits_{i=0}^{\infty} g(l,i) \cdot 2^{-i-1} \in [0,1]$ with

$$g(l,n) := \begin{cases} 1, & \text{if } \neg T(l,l,n) \\ 0, & \text{otherwise,} \end{cases}$$

Here T denotes the Kleene T-predicate.

Now observe that

$$(++)\ a_l = 1 \to x^l_{\delta(l)} = 1 \text{ and } a_l < 1 \to x^l_{\delta(l)} \in [0, 1/2].$$

While the first implication is immediate from the definition of x^l_n, the second follows using $(+)$ and the fact that (by – an essentially trivial use of – Ishikawa's theorem [4]) (x^l_n) converges towards the unique fixed point 0 of f_l.

By $(++)$ the computability of δ would allow us to decide whether $a_l = 1$ or $a_l < 1$ and so whether or not $\exists n \in \mathbb{N}\, T(l,l,n)$ contradicting the undecidability of the (special) Halting problem.

While we do not know whether for single computable operators $T : C \to C$ in effective uniformly convex spaces, the iteration $x_n := T^n_\lambda x$ (for computable $x \in C, \lambda \in (0,1)$) might have no computable rate of convergence, we show that the rate is computable iff the norm $\|p\|$ of the strong limit p of (x_n) is computable.

Things are much better for a reformulation of the convergence property known in logic as the no-counterexample interpretation of the former ([7, 8], see also [5]) which recently has been popularized under the name of 'metastability' by T. Tao (see [11, 12]). Here one considers the statement

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists k \in \mathbb{N}\, \forall i,j \in [k; k+g(k)]\, \big(\|T^i x - T^j x\| < \varepsilon\big)$$

which, ineffectively, is equivalent to the strong convergence of $(T^n x)$. Here $[k; k+m] := \{k, k+1, k+2, \ldots, k+m\}$.

We then give an explicit effective (in fact even primitive recursive) and highly uniform rate $\Phi(b, \alpha, \varepsilon, g)$ of metastability of $(T^n x)$

$$\begin{aligned} \forall \varepsilon \in (0,2]\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall b \in \mathbb{N}^*\, \forall x \in C_b\, \exists n \leq \Phi(b, \alpha, \varepsilon, g) \\ \forall i,j \in [n; n+g(n)]\, \big(\|T^i x - T^j x\| < \varepsilon\big) \end{aligned}$$

that (in addition to ε and g) only depends on a norm upper bound $b \geq \|x\|$ of x and a uniform rate α of asymptotic regularity of T on $C_b := \{x \in C : \|x\| \leq b\}$, i.e.

$$\forall \varepsilon > 0\, \forall b \in \mathbb{N}^*\, \forall x \in C_b\, \forall n \geq \alpha(b, \varepsilon)\, \big(\|T^{n+1} x - T^n x\| < \varepsilon\big).$$

In fact, instead of α a (uniform) rate on the metastable version of asymptotic regularity, i.e. a φ such that

$$\forall \varepsilon > 0\, \forall f : \mathbb{N} \to \mathbb{N}\, \forall b \in \mathbb{N}^*\, \forall x \in C_b\, \exists k \leq \varphi(b, f, \varepsilon)\, \forall i \in [k; k + f(k)]\, \big(\|T^{i+1}x - T^i x\| < \varepsilon\big),$$

is sufficient.

The bound Φ is independent of X (and C) except for a modulus of uniform convexity η of X (and an upper bound b on $\|x\|$). The extraction of this bound is an instance of a general logical metatheorem which not only guarantees the extractability of such bounds for large classes of proofs but also provides an algorithm for the actual construction of the bound from a given proof. This then results again in an ordinary proof that no longer relies on any facts from logic (see [5], in particular Chapters 17 and 18, for all this).

Using the optimal rate of asymptotic regularity α for T_λ from [1] this gives an effective (and even primitive recursive) rate of metastability for the strong convergence of (x_n) (as defined above) that only depends on ε, g and b.

A primitive recursive rate on the metastability of the **Cesàro means** (i.e. ergodic averages) of operators in Hilbert space satisfying Wittmann's condition was recently extracted from Wittmann's [14] proof of strong convergence of these means by Safarik [10]. For another quantitative strong nonlinear ergodic theorem see [6]. Again, these results have been obtained using the aforementioned proof-theoretic approach.

2 Results

In the following, let X be a uniformly convex Banach space with a modulus of convexity $\eta : (0, 2] \to (0, 1]$, i.e.

$$\forall x, y \in B_1(0)\, \forall \varepsilon \in (0, 2]\, \left(\left\| \frac{x + y}{2} \right\| > 1 - \eta(\varepsilon) \to \|x - y\| < \varepsilon \right),$$

where $B_d(0)$ denotes the closed ball with center 0 and radius d in X.

Lemma 2.1. *Let $x, y \in B_d(0) \subset X$ with $0 < d \leq b \in \mathbb{N}$. Then*

$$\forall \varepsilon \in (0, 2]\, \left(\left\| \frac{x + y}{2} \right\| > d(1 - \eta(\varepsilon/b)) \to \|x - y\| < \varepsilon \right).$$

Proof: Define $\tilde{x} := x/d, \tilde{y} := y/d$ so that $\tilde{x}, \tilde{y} \in B_1(0)$.
Assume that $\left\|\frac{x+y}{2}\right\| > d(1 - \eta(\varepsilon/b))$. Then

$$\left\| \frac{\tilde{x} + \tilde{y}}{2} \right\| = \frac{1}{d} \left\| \frac{x + y}{2} \right\| > 1 - \eta(\varepsilon/b)$$

and so $\frac{1}{d}\|x - y\| = \|\tilde{x} - \tilde{y}\| < \frac{\varepsilon}{b}$. Hence $\|x - y\| < \frac{d \cdot \varepsilon}{b} \leq \varepsilon$. □

Notation: For $b \in \mathbb{N}^*$ define $C_b := \{x \in C : \|x\| \leq b\}$.
For $n, m \in \mathbb{N}$ we define $n \dot{-} m := n - m$ if $n \geq m$ and $:= 0$, otherwise.

Theorem 2.2. *Let $C \subseteq X$ be any nonempty subset of X and $T : C \to C$ a selfmapping of C that satisfies Wittmann's [14] condition*

$$(W): \ \forall x, y \in C \left(\|Tx + Ty\| \leq \|x + y\| \right).$$

Moreover, assume that for each $0 < b \in \mathbb{N}$ the mapping T is (uniformly on C_b) asymptotically regular with a rate $\alpha : \mathbb{N} \times \mathbb{R}^*_+ \to \mathbb{N}$, i.e.

$$\forall \varepsilon > 0 \, \forall b \in \mathbb{N}^* \, \forall x \in C_b \, \forall n \geq \alpha(b, \varepsilon) \, \left(\|T^{n+1}x - T^n x\| < \varepsilon \right).$$

Then $(T^n x)_{n \in \mathbb{N}}$ converges strongly with the following rate of metastability

$$\begin{array}{l} \forall \varepsilon \in (0, 2] \, \forall g : \mathbb{N} \to \mathbb{N} \, \forall b \in \mathbb{N}^* \, \forall x \in C_b \, \exists n \leq \Phi(b, \alpha, \varepsilon, g) \\ \qquad \qquad \qquad \forall i, j \in [n; n + g(n)] \, \left(\|T^i x - T^j x\| < \varepsilon \right), \end{array}$$

where

$$\begin{array}{l} \Phi(b, \alpha, \varepsilon, g) := \Psi(b, h_{b,\alpha,\varepsilon,g}, \frac{\delta_b(\varepsilon)}{2}) \text{ with} \\ h_{b,\alpha,\varepsilon,g}(n) := h(n) := \max \left\{ \alpha \left(b, \frac{\delta_b(\varepsilon)}{\max\{g(n),1\}} \right) \dot{-} n, g(n) \right\} \text{ and} \\ \Psi(b, f, \delta) := \tilde{f}^{(\lceil b/\delta \rceil)}(0) \text{ with } \tilde{f}(n) := n + f(n) \text{ for } f : \mathbb{N} \to \mathbb{N}, \\ \delta_b(\varepsilon) := \frac{\varepsilon}{2} \cdot \eta(\varepsilon/b). \end{array}$$

If T is continuous and C closed, then the strong limit of $(T^n x)_{n \in \mathbb{N}}$ is a fixed point of T. For the metastability statement the completeness of X is not needed.

Proof: It suffices to prove the metastability statement which (ineffectively) implies the strong Cauchy property of the sequence (and so using the completeness of X its convergence). That for continuous T (and closed C) the limit is a fixed point of T then trivially follows from the asymptotic regularity of T.
Let $\varepsilon \in (0, 2], b \in \mathbb{N}^*, g : \mathbb{N} \to \mathbb{N}$ and C, T, x be as in the theorem. By the condition (W) the sequence $(\|T^n x\|)_{n \in \mathbb{N}}$ is nonincreasing and hence convergent. By [5] (Proposition 2.27, Remark 2.29) it follows that Ψ is a rate of metastability for this sequence, i.e.

$$\forall \delta > 0 \, \forall f : \mathbb{N} \to \mathbb{N} \, \exists n \leq \Psi(b, f, \delta) \, \forall i, j \in [n; n + f(n)] \left(\big| \|T^i x\| - \|T^j x\| \big| < \delta \right).$$

For $\delta := \frac{\delta_b(\varepsilon)}{2}$ and $f := h := h_{b,\alpha,\varepsilon,g}$ let $n \in \mathbb{N}$ be such a number.
Define $d := \|T^n x\| = \max\{\|T^k x\| : k \in [n; n + h(n)]\} \leq b$. Then

$$(1) \ \forall k \in [n; n + h(n)] \, \left(d - \frac{\delta_b(\varepsilon)}{2} < \|T^k x\| \leq d \right).$$

From the assumption on α we get

$$(2) \ \forall i \in \mathbb{N}^* \, \forall \varepsilon > 0 \, \forall k \geq \alpha(b, \varepsilon/i) \, \forall j \leq i \, \left(\|T^k x - T^{k+j} x\| < \varepsilon \right),$$

since

$$\|T^k x - T^{k+j} x\| \leq \sum_{l=0}^{j-1} \|T^{k+l} x - T^{k+l+1} x\| < \sum_{l=0}^{j-1} \frac{\varepsilon}{i} \leq \varepsilon$$

for all $0 < j \le i$ and $k \ge \alpha(b, \varepsilon/i)$.
For $k := n + h(n) \ge \alpha\left(b, \frac{\delta_b(\varepsilon)}{\max\{g(n),1\}}\right)$ we get from (1) and (2) that for all $i \in [n; n+g(n)] \subseteq [n;k]$:

$$\begin{aligned}\forall j \le g(n)\ \big(2(d - \tfrac{\delta_b(\varepsilon)}{2}) \le 2\|T^k x\| &\le \|T^{k+j}x + T^k x\| + \|T^k x - T^{k+j}x\| \\ &< \|T^{k+j}x + T^k x\| + \delta_b(\varepsilon) \\ &\overset{(W)}{\le} \|T^{i+j}x + T^i x\| + \delta_b(\varepsilon)\big).\end{aligned}$$

Hence

$$(3)\ \forall i,j \in [n; n+g(n)]\ \left(d - \delta_b(\varepsilon) < \left\|\frac{T^i x + T^j x}{2}\right\|\right).$$

Case 1: $d := \|T^n x\| < \frac{\varepsilon}{2}$. Then

$$\forall i,j \in [n; n+g(n)]\ \big(\|T^i x - T^j x\| \le \|T^i x\| + \|T^j x\| \le 2\|T^n x\| < \varepsilon\big)$$

and so we are done.
Case 2: $d \ge \frac{\varepsilon}{2}$. Then by the definition of $\delta_b(\varepsilon)$ and (3) we have

$$(4)\ \forall i,j \in [n; n+g(n)]\ \left(d(1 - \eta(\varepsilon/b)) < \left\|\frac{T^i x + T^j x}{2}\right\|\right).$$

Using (1), (4) and lemma 2.1 yields that

$$\forall i,j \in [n; n+g(n)]\ \big(\|T^i x - T^j x\| < \varepsilon\big).$$

□

Remark 2.3. *If $\eta(\varepsilon)$ can be written as $\varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $0 < \varepsilon_1 \le \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \le \tilde{\eta}(\varepsilon_2)$, then we can replace $\delta_b(\varepsilon)$ in the bound in theorem 2.2 by $\delta_b(\varepsilon) := \varepsilon \cdot \tilde{\eta}(\varepsilon/b)$. In particular, in the case of a Hilbert space X (where one can take $\eta(\varepsilon) := \varepsilon^2/8$, see e.g. [6]), this yields $\delta_b(\varepsilon) := \frac{\varepsilon^2}{8b}$.*

Proof: With $\delta_b(\varepsilon) := \varepsilon \cdot \tilde{\eta}(\varepsilon/b)$ one gets instead of (4) in the proof of theorem 2.2

$$(4)'\ \left\{\begin{array}{l}\forall i,j \in [n; n+g(n)] \\ \left(d(1 - \eta(\varepsilon/d)) = d(1 - \frac{\varepsilon}{d}\cdot\tilde{\eta}(\varepsilon/d)) \le d(1 - \frac{\varepsilon}{d}\cdot\tilde{\eta}(\varepsilon/b)) < \left\|\frac{T^i x + T^j x}{2}\right\|\right).\end{array}\right.$$

The claim now follows using lemma 2.1 since $T^i x, T^j x \in B_d(0)$ for $i,j \in [n; n+g(n)]$. □

The above extraction of the rate of metastability Φ from the proof given in [2] (and also the fact that Φ only depends on the arguments $b, \alpha, \varepsilon, g$) is an instance of a general logical metatheorem (see [3] Theorem 6.3.2 or [5] Theorem 17.69.2 and note that the condition (W) is purely universal and implies that T is majorized by the identity function). In fact, that metatheorem even guarantees such a bound when the rate of asymptotic regularity α is replaced by a weaker rate of metastability φ instead, i.e.

$(*)$ $\forall \varepsilon > 0 \, \forall f : \mathbb{N} \to \mathbb{N} \, \forall b \in \mathbb{N}^* \, \forall x \in C_b \, \exists k \leq \varphi(b, f, \varepsilon) \, \forall i \in [k; k+f(k)] \, \big(\|T^{i+1}x - T^i x\| < \varepsilon\big)$.

We will briefly demonstrate this now. In fact, one only needs φ for constant-c functions ($c \in \mathbb{N}$) that we also denote by c. Modifying φ to $\varphi'(b, c, l, \varepsilon) := \varphi(b, c+l, \varepsilon) + l$ one gets for each $l \in \mathbb{N}$

$$(**) \ \exists k \leq \varphi'(b, c, l, \varepsilon) \, \forall i \in [k; k+c] \, \big(k \geq l \wedge \|T^{i+1}x - T^i x\| < \varepsilon\big).$$

Now define (using $(**)$) $\alpha_{n,g}(b, \varepsilon)$ as the least $k \leq \varphi'(b, g(n), n+g(n), \varepsilon)$ such that

$$\forall i \in [k; k+g(n)] \, \big(k \geq n + g(n) \wedge \|T^{i+1}x - T^i x\| < \varepsilon\big).$$

Then theorem 2.2 holds with α and $h_{b,\alpha,\varepsilon,g}$ being replaced by $\alpha_{n,g}$ and $h_{b,\varphi,\varepsilon,g}(n) := \alpha_{n,g}\left(b, \frac{\delta_b(\varepsilon)}{\max\{g(n),1\}}\right) - n$ respectively. Replacing $h_{b,\varphi,\varepsilon,g}$ by the monotone upper bound

$$h^*_{b,\varphi,\varepsilon,g}(n) := \max\{\varphi'(b, g(m), m+g(m), \delta_b(\varepsilon)/\max\{g(m),1\}) - m \, : \, m \leq n\}$$

yields an upper bound

$$\Phi(b, \varphi, \varepsilon, g) := \Psi(b, h^*_{b,\varphi,\varepsilon,g}, \frac{\delta_b(\varepsilon)}{2}) \geq \Psi(b, h_{b,\varphi,\varepsilon,g}, \frac{\delta_b(\varepsilon)}{2})$$

satisfying theorem 2.2. This yields the following qualitative improvement of theorem 2.2

Corollary 2.4. *For the strong convergence of $(T^n x)$ in theorem 2.2 one can weaken the asymptotic regularity assumption to*

$$\forall \varepsilon > 0 \, \forall c \in \mathbb{N} \, \forall x \in C \, \exists k \in \mathbb{N} \, \forall i \in [k; k+c] \, \big(\|T^{i+1}x - T^i x\| < \varepsilon\big).$$

If T is continuous and C is closed, then the limit of $(T^n x)$ is a fixed point of T.

Proof: By the reasoning above, the sequence $(T^n x)$ is metastable (note that for metastability in the point x we also only need the above weak form of asymptotic regularity in x) and hence is strongly Cauchy. For closed C the limit is in C and – for continuous T – a fixed point of T as the condition in the corollary implies that

$$\forall \varepsilon > 0 \, \forall n \in \mathbb{N} \, \exists k \geq n \, (\|T^{k+1}x - T^k x\| < \varepsilon).$$

$\square$

In the following, we apply theorem 2.2 to averages mappings for which effective (full) rates of asymptotical regularity are known (here 'π' denotes the constant π):

Theorem 2.5. *Let X be a uniformly convex Banach space and $C \subseteq X$ a closed and convex subset. Assume that $T : C \to C$ satisfies (W) and is nonexpansive. Let $\lambda \in (0,1)$ and define $T_\lambda x := (1-\lambda)x + \lambda T x, x_n := T^n_\lambda x$ for $x \in C$. Then $(x_n)_{n \in \mathbb{N}}$ strongly converges to a fixed point $p \in C$ of T and the following rate of metastability holds:*

$$\begin{aligned}\forall \varepsilon \in (0,2] \, \forall g : \mathbb{N} \to \mathbb{N} \, \forall b \in \mathbb{N}^* \, \forall x \in C_b \, \exists n \leq \Phi(b, \alpha, \varepsilon, g) \\ \forall i, j \in [n; n+g(n)] \, \big(\|x_i - x_j\| < \varepsilon\big),\end{aligned}$$

where Φ is as in theorem 2.2 and $\alpha(b,\varepsilon) := \left\lceil \frac{b^2\cdot\lambda}{\pi(1-\lambda)\varepsilon^2} \right\rceil$.
For the last statement no completeness of X or closedness of C is needed.

Proof: For $x \in C_b$ it follows from a deep result due to Baillon and Bruck [1] (and using that $\lambda\|x_n - T(x_n)\| = \|T_\lambda^{n+1}x - T_\lambda^n x\|$) that α is a rate of asymptotic regularity for T_λ (this result even holds in arbitrary normed spaces).[1] With T also T_λ satisfies (W) since

$$\begin{aligned}\|T_\lambda x + T_\lambda y\| &= \|(1-\lambda)x + \lambda Tx + (1-\lambda)y + \lambda Ty\| \\ &\le (1-\lambda)\|x+y\| + \lambda\|Tx+Ty\| \\ &\le (1-\lambda)\|x+y\| + \lambda\|x+y\| = \|x+y\|.\end{aligned}$$

Hence the corollary follows from theorem 2.2 applied to T_λ (note that the proof for the metastability statement did not use the completeness of X nor the closedness of C). $\square$

Remark 2.6. *For nonexpansive T the condition (W), in particular, holds when $C = -C$ and T is odd, i.e. $T(-x) = -T(x)$.*

The proof of theorem 2.2 (and theorem 2.5) immediately yields an effective rate of convergence of $(T^n x)_{n\in\mathbb{N}}$ (instead of a rate of metastability only) provided one has a rate $\Psi_{x,T}$ of convergence for $(\|T^n x\|)_{n\in\mathbb{N}}$ given, i.e. for $d := \lim\limits_{n\to\infty} \|T^n x\|$

$$\forall\varepsilon > 0\, \forall n \ge \Psi_{x,T}(\varepsilon)\ \left(\|T^n x\| - d < \varepsilon\right).$$

Then $\Psi_{x,T}\left(\frac{\delta_b(\varepsilon)}{2}\right)$ is a rate of convergence of $(T^n x)_{n\in\mathbb{N}}$. This leads to the following (using the notion of computability for Banach spaces and mappings between Banach spaces from [9] and [13]).

Corollary 2.7. *Let X be a computable uniformly convex Banach space with a computable modulus of uniform convexity η and C be a closed and convex subset. Let $T : C \to C$ be a computable nonexpansive mapping satisfying condition (W) and $x \in C$ be a computable point. Finally, let $\lambda \in (0,1)$ be computable. Then $(T_\lambda^n x)_{n\in\mathbb{N}}$ converges effectively (i.e. with a computable rate of convergence) to its limit $p := \lim\limits_{n\to\infty} T_\lambda^n x$ if and only if $\|p\|$ is computable.*

Proof: The assumptions yields that (x_n) with $x_n := T_\lambda^n x$ is a computable sequence in X. If (x_n) converges effectively, then also p and hence $\|p\|$ is computable. Conversely, suppose that $\|p\|$ is computable. Then there is a computable function $\rho : \mathbb{Q}_+^* \to \mathbb{N}$ such that

$$\forall q \in \mathbb{Q}_+^*\ \left(\|T_\lambda^{\rho(q)} x\| - \|p\| < q\right)$$

since '$\|T_\lambda^n x\| - \|p\| < q$' is computably enumerable in n, q. Since $(\|T^n x\|)_{n\in\mathbb{N}}$ is nonincreasing, ρ in fact is a rate of convergence. The comments preceding this corollary now yield a computable rate of convergence for $(T_\lambda^n x)_{n\in\mathbb{N}}$. $\square$

[1] The bound in [1] is stated for sequences in $B_1(0)$ but can easily be adapted to $B_b(0)$ by switching to the norm $\|x\|_b := \frac{1}{b}\cdot\|x\|$. The α in our theorem results from this adaptation.

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