Arithmetizing proofs in analysis

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1 Introduction

In this paper we continue our investigations started in [15] and [16] on the question:

What is the impact on the growth of extractable uniform bounds the use of various analytical principles Γ in a given proof of an $\forall\exists$–sentence might have?

To be more specific, we are interested in analyzing proofs of sentences having the form

$$(1)\ \forall u^1, k^0 \forall v \leq_\rho tuk \exists w^0 A_0(u,k,v,w),$$

where A_0 is a quantifier–free formula[1] (containing only u,k,v,w as free variables) in the language of a suitable subsystem $\mathcal{T}^\omega$ of arithmetic in all finite types, t is a closed term and $\leq_\rho$ is defined pointwise (ρ being an arbitrary finite type).
From a proof of (1) carried out in $\mathcal{T}^\omega$ one can extract an effective uniform bound Φuk on $\exists w$, i.e.

$$(2)\ \forall u^1, k^0 \forall v \leq_\rho tuk \exists w \leq_0 \Phi uk\, A_0(u,k,v,w),$$

where the complexity (and in particular the growth) of Φ is limited by the complexity of the system $\mathcal{T}^\omega$ (see [13],[15]).
By the predicate 'uniform' we refer to the fact that the bound Φ does not depend on $v \leq_\rho tuk$.
In [13] we have discussed in detail, how sentences (1) arise naturally in analysis and why such uniform bounds are of numerical interest (e.g. in the context of approximation theory).
Proofs in analysis can be formalized in a suitable base theory $\mathcal{T}^\omega$ plus certain (in general non–constructive) analytical principles Γ (usually not derivable in $\mathcal{T}^\omega$). In order to determine faithfully the contribution of the use of Γ to the growth of extractable bounds Φ we introduced in [15] a hierarchy of weak subsystems $\mathrm{G}_n\mathrm{A}^\omega$ of arithmetic in all finite types whose definable type–1–objects correspond to the well–known Grzegorczyk hierarchy of functions.
As the essential proof–theoretic tool, monotone functional interpretation (which was introduced in [13]) was used to extract bounds Φ (given by closed term of $\mathrm{G}_n\mathrm{A}^\omega$) from proofs

$$(3)\ \mathrm{G}_n\mathrm{A}^\omega + \Delta + \text{AC–qf} \vdash (1),$$

[1] Throughout this paper $A_0, B_0, C_0, \ldots$ always denote quantifier–free formulas.

where

$$\mathrm{AC}^{\rho,\tau}\text{–qf} \;:\; \forall x^\rho \exists y^\tau A_0(x,y) \to \exists Y^{\tau\rho} \forall x^\rho A_0(x, Yx)$$

is the schema of choice for quantifier–free formulas and Δ is a set of 'axioms' having the form

$$(4)\ \forall x^\delta \exists y \leq_\rho sx \forall z^\tau G_0(x,y,z),$$

where G_0 is a quantifier–free formula containing only x,y,z free and s is a closed term.
In particular for $n=2$ (resp. $n=3$) the extractability of a bound Φuk which is a polynomial (resp. a finitely iterated exponential function) in $u^M x := \max\limits_{i \leq x} u(i)$ and k is guaranteed (see [15] for details).
In [14] we have shown that for suitable Δ already $\mathrm{G}_2\mathrm{A}^\omega + \Delta +$ AC–qf covers a substantial part of standard analysis. In fact essentially only analytical axioms (4) having types $\delta, \rho \leq 1, \tau = 0$ are sufficient.
The proof of the verification of the extracted bound Φ also relies on these non–constructive principles Δ, in fact even on their strengthened versions

$$(5)\ \tilde{\Delta} := \{\exists Y \leq_{1(1)} s \forall x, z G_0(x, Yx, z) | \forall x^1 \exists y \leq_1 sx \forall z^0 G_0(x,y,z) \in \Delta\}$$

relatively to the intuitionistic variant $\mathrm{G}_n\mathrm{A}^\omega_i$ of $\mathrm{G}_n\mathrm{A}^\omega$.
However combining the methods from [15] with techniques from [12] one can replace the use of (5) by the use of the 'ε–weakenings' of (5) thereby achieving

$$(6)\ \mathrm{G}_n\mathrm{A}^\omega_i + \Delta_\varepsilon \vdash \forall u^1, k^0 \forall v \leq_\rho tuk \exists w \leq_0 \Phi uk A_0(u,k,v,w),$$

where

$$(7)\ \Delta_\varepsilon := \left\{ \forall x^1, z^0 \exists y \leq_1 sx \bigwedge_{i=0}^{z} G_0(x,y,i) | \forall x^1 \exists y \leq_1 sx \forall z^0 G_0(x,y,z) \in \Delta \right\}.$$

The ε–weakening Δ_ε of Δ usually is constructively provable in suitable subsystems of intuitionistic arithmetic in all finite types. This passage from $\tilde{\Delta}$ to Δ_ε – which may be viewed as an ε-arithmetization of the original proof – however is not necessary for the extraction of Φ but only for a **constructive verification** of Φ.

Whereas a number of important analytical principles can be expressed directly as axioms (4) – in particular relatively to systems like $\widehat{\mathrm{PA}}^\omega{\restriction}$ or $\mathrm{G}_n\mathrm{A}^\omega$ for $n \geq 3$ the binary König's lemma WKL can be expressed in this form (see [12] for details) – there are many theorems not having this form but which can be proved from WKL relatively to base systems like $\widehat{\mathrm{PA}}^\omega{\restriction} +$ AC–qf which essentially is a finite type extension of the second–order theory RCA_0 known from reverse mathematics. Examples of such theorems are the following principles:

- Every pointwise continuous function $f : [0,1]^d \to \mathbb{R}$ is uniformly continuous.
- The attainment of the maximum value of $f \in C([0,1]^d, \mathbb{R})$ on $[0,1]^d$.[2]
- The sequential form of the Heine–Borel covering property for $[0,1]^d$.
- Dini's theorem.

[2]This statement can be expressed as an axiom (4). However this requires a very complicated representation of the elements $f \in C([0,1]^d, \mathbb{R})$ which can be avoided using the principle of uniform boundedness discussed below.

- The existence of a uniformly continuous inverse function for every strictly increasing continuous function $f : [0,1] \to \mathbb{R}$.

The problem in treating these principles relative to weak base theories as G_2A^ω is that their usual proofs (using WKL) are not formalizable within e.g. G_2A^ω. In particular WKL can not even be expressed in its usual formulation in this system, since this involves the coding functional $f_{\langle\rangle}x := \langle f0, \ldots, f(x-1)\rangle$ which is available in G_nA^ω only for $n \geq 3$. In order to treat the principles above faithfully we introduced in [15] the axiom (having the form (4))

$\boldsymbol{F^-} :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0, z^1, n^0 \big(\bigwedge_{i<_0 n} (zi \leq_0 yki) \to \Phi k(\overline{z,n}) \leq_0 \Phi k(y_0 k)\big)$,

where, for $z^{\rho 0}$, $(\overline{z,n})(k^0) :=_\rho zk$, if $k <_0 n$ and $:= 0^\rho$, otherwise.

This axiom implies (already relatively to G_2A^ω+$AC^{1,0}$–qf) the following **principle of uniform Σ^0_1–boundedness**

$$\boldsymbol{\Sigma^0_1}\textbf{–UB}^- :\equiv \begin{cases} \forall y^{1(0)} \big(\forall k^0 \forall x \leq_1 yk \exists z^0 \ A(x,y,k,z) \to \exists \chi^1 \forall k^0, x^1, n^0 \\ \qquad \big(\bigwedge_{i<_0 n} (xi \leq_0 yki) \to \exists z \leq_0 \chi k \ A((\overline{x,n}), y, k, z)\big)\big), \end{cases}$$

where $A \equiv \exists l^0 A_0(l)$ is a purely existential formula (see [15] for a detailed discussion of this principle). In $G_2A^\omega + \Sigma^0_1$–UB^- and hence in $G_2A^\omega + F^-$+$AC^{1,0}$–qf one can give very short and perspicuous proofs of the analytical theorems listed above and since F^- has the form of an axiom Δ we can extract a polynomial bound from such a proof (see [17] for details). The verification of this so far still depends on the non–standard axiom F^- which does not hold classically, i.e. it does not hold in the full set–theoretic type structure $\mathcal{S}^\omega$ (but only in the type structure of all so–called strongly majorizable functionals $\mathcal{M}^\omega$). Nevertheless, using the ε-arithmetization technique mentioned above, one can replace the use of F^- by its ε–weakening and this ε–weakening is provable e.g. in $G_3A^\omega_i$ (see [15]). In this case ε-arithmetization still is not needed for the extraction of an uniform bound but now it is needed even for a **classical verification**.

On the other hand there are central theorems in analysis whose proofs use arithmetical comprehension, more precisely instances of

$$AC_{ar}: \ \forall x^0 \exists y^0 A(x,y) \to \exists f^1 \forall x^0 A(x, fx),$$

where $A \in \Pi^0_\infty$ (A may contain parameters of arbitrary type), and which are not covered by the results mentioned above.
Examples are the following theorems

1) The principle of convergence for bounded monotone sequences of real numbers (or equivalently: every bounded monotone sequence of reals has a Cauchy modulus (PCM)).

2) For every sequence of real numbers which is bounded from above there exists a least upper bound.

3) The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$ (for every fixed d).

4) The Arzelà–Ascoli lemma.

5) The existence of the limit superior for bounded sequences of real numbers.

Using a convenient representation of real numbers, (PCM) can be formalized as follows:

$$(\mathrm{PCM}) \ : \ \begin{cases} \forall a^{1(0)}_{(\cdot)}, c^1 \big(\forall n^0 (c \le_{\mathbb{R}} a_{n+1} \le_{\mathbb{R}} a_n) \\ \qquad \to \exists h^1 \forall k^0 \forall m, \tilde{m} \ge_0 hk(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \le_{\mathbb{R}} \frac{1}{k+1})\big). \end{cases}$$

(PCM) immediately follows from its arithmetical weakening

$$(\mathrm{PCM}^-) \ : \ \begin{cases} \forall a^{1(0)}_{(\cdot)}, c^1 \big(\forall n^0 (c \le_{\mathbb{R}} a_{n+1} \le_{\mathbb{R}} a_n) \\ \qquad \to \forall k^0 \exists n^0 \forall m, \tilde{m} \ge_0 n(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \le_{\mathbb{R}} \frac{1}{k+1})\big) \end{cases}$$

by an application of AC_{ar} to

$$A :\equiv \forall m, \tilde{m} \ge n(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \le_{\mathbb{R}} \frac{1}{k+1}) \in \Pi^0_1$$

($\le_{\mathbb{R}} \in \Pi^0_1$ follows from the fact that real numbers are given as Cauchy sequences of rationals with fixed rate of convergence in our theories).
It is well–known that a constructive functional interpretation of the negative translation of AC_{ar} requires so–called bar-recursion and cannot be caried out e.g. in Gödel's term calculus T (see [23] and [18]). AC_{ar} is (using classical logic) equivalent to $\mathrm{CA}_{ar}+\mathrm{AC}^{0,0}$–qf, where

$$\mathrm{CA}_{ar} : \ \exists g^1 \forall x^0 \big(g(x) =_0 0 \leftrightarrow A(x)\big) \text{ with } A \in \Pi^0_\infty,$$

and therefore causes an immense rate of growth (when added to e.g. $\mathrm{G}_2\mathrm{A}^\omega$). From the work in the context of 'reverse mathematics' (see e.g. [6],[22]) it is known that 1)–5) imply CA_{ar} relatively to (a second-order version of) $\widehat{\mathrm{PA}}^\omega\!\restriction + \mathrm{AC}^{0,0}$–qf (see [5] for the definition of $\widehat{\mathrm{PA}}^\omega\!\restriction$). In [14] it is shown that this holds even relatively to $\mathrm{G}_2\mathrm{A}^\omega$.
In contrast to these general facts on huge growth we prove in this paper a theorem which in particular implies that if (PCM) is applied in a proof only to sequences (a_n) which are given explicitely in the parameters of the proposition (which is proved) then this proof can be (effectively) transformed (without causing new growth) into a proof of the same conclusion which uses only (PCM^-) for these sequences. By this transformation the use of AC_{ar} is eliminated and the determination of the growth caused (potentially by (PCM)) reduces to the determination of the growth caused by (PCM^-). This reduction is achieved using the method of elimination of Skolem function for monotone formulas (developed in [16]).
In difference to (PCM) the (negative translation of the) principle (PCM^-) has a simple constructive monotone functional interpretation which is fulfilled by a functional Ψ which is primitive recursive in the sense of [9]. Because of the nice behaviour of the monotone functional interpretation with respect to the modus ponens one obtains (by applying Φ to Ψ) a monotone functional interpretation of (1) and so, using tools from [13],[15], a uniform bound ξ for $\exists w$, i.e.

$$\forall u^1, k^0 \forall v \le_\rho tuk \exists w \le_0 \xi uk A_0(u,k,v,w),$$

where ξ is **primitive recursive in the sense of Kleene [9]** (and not only in the generalized sense of Gödel's calculus T).
(This conclusion also holds for sequences of instances $\forall n^0 \mathrm{PCM}(\chi uvn)$ of $\mathrm{PCM}(a)$ instead of

PCM(χuv).)

In this case ε-arithmetization – namely the reduction of the use of instances of (PCM) to corresponding instances of its arithmetical weakening (PCM^-) – is necessary already for the **construction of the bound Φ**.
In our treatment of the Bolzano–Weierstraß theorem (as well as the Arzelà–Ascoli lemma) in section 5 below the use of the method of elimation of Skolem functions is combined with the use of the non–standard axiom F^- mentioned above: Single (sequences of) instances of the Bolzano–Weierstraß theorem can be proved (relatively to $\mathrm{G}_2\mathrm{A}^\omega + \mathrm{AC}^{1,0}$–qf) from single instances of the second–order axiom Π^0_1–CA plus F^-. Π^0_1–CA is studied in [16] where it is shown that single instances of this principle (in contrast to its full second–order universal closure, which is equivalent to full arithmetical comprehension over numbers) also contribute at most by a **primitive recursive functional in the sense of Kleene**. By the method of F^-–elimination discussed above, the resulting bound from a proof which uses single instances of the Bolzano–Weierstraß theorem then can be classically (and even constructively) verified. Here ε-arithmetization of a given proof is used twice for the construction of a bound (by elimination of Skolem functions) and for a classical verification (by elimination of the non–standard axiom F^-).
Finally we investigate the principle of the existence of the limit superior of a bounded sequence of real numbers. It turns out that the use of single instances of this principle in the proof of a theorem (1) can be reduced to an arithmetical Π^0_5–principle whose monotone functional interpretation can be fulfilled by a functional from the fragment T_1 of Gödels claculus T with the recursor constants R_ρ for $\rho \le 1$ (this fragment of T is sufficient to define the Ackermann function but no functions of essentially greater rate of growth).

In section 2 we present the theorems from [16] on which our investigations in the present paper are based in order to make this paper independent from the reading of [16]. However we assume the reader to be familiar with [15] and all undefined notions in this paper are used in the sense of [15].

2 Proof–theoretic tools

In this section we recall some of our proof–theoretic results from [16] which will be used in section 5 below.

Definition 2.1 ([16]) *Let $A \in \mathcal{L}(G_nA^\omega)$ be a formula having the form*

$$A \equiv \forall u^1 \forall v \le_\tau tu \exists y_1^0 \forall x_1^0 \ldots \exists y_k^0 \forall x_k^0 \exists w^\gamma A_0(u, v, y_1, x_1, \ldots, y_k, x_k, w),$$

where A_0 is quantifier–free and contains only $u, v, \underline{y}, \underline{x}, w$ free. Furthermore let t be $\in G_nR^\omega$ and τ, γ are arbitrary finite types.

1) A is called (arithmetically) **monotone** *if*

$$Mon(A) :\equiv \begin{cases} \forall u^1 \forall v \le_\tau tu \forall x_1, \tilde{x}_1, \ldots, x_k, \tilde{x_k}, y_1, \tilde{y}_1, \ldots y_k, \tilde{y}_k \\ \qquad \Big(\bigwedge\limits_{i=1}^{k} (\tilde{x}_i \le_0 x_i \wedge \tilde{y}_i \ge_0 y_i) \wedge \exists w^\gamma A_0(u, v, y_1, x_1, \ldots, y_k, x_k, w) \\ \qquad\qquad \to \exists w^\gamma A_0(u, v, \tilde{y}_1, \tilde{x}_1, \ldots, \tilde{y}_k, \tilde{x}_k, w) \Big). \end{cases}$$

2) The **Herbrand normal form** A^H *of* A *is defined to be*

$$A^H :\equiv \forall u^1 \forall v \leq_\tau tu \forall h_1^{\rho_1},\dots,h_k^{\rho_k} \exists y_1^0,\dots,y_k^0, w^\gamma$$
$$\underbrace{A_0(u,v,y_1,h_1y_1,\dots,y_k,h_ky_1\dots y_k,w)}_{A_0^H:\equiv},\ \text{where } \rho_i = 0\underbrace{(0)\dots(0)}_{i}.$$

Theorem 2.2 ([16]) *Let* $n \geq 1$ *and* $\Psi_1,\dots,\Psi_k \in G_nR^\omega$. *Then*

$$G_nA^\omega + Mon(A) \vdash \forall u^1 \forall v \leq_\tau tu \forall h_1,\dots,h_k \Big(\bigwedge_{i=1}^{k} (h_i \text{ monotone}) \rightarrow \exists y_1 \leq_0 \Psi_1 u\underline{h} \dots \exists y_k \leq_0 \Psi_k u \underline{h} \exists w^\gamma A_0^H \Big) \rightarrow A,$$

where $(h_i \text{ monotone}) :\equiv \forall x_1,\dots,x_i,y_1,\dots,y_i \big(\bigwedge_{j=1}^{i} (x_j \geq_0 y_j) \rightarrow h_i \underline{x} \geq_0 h_i \underline{y} \big)$.

Definition 2.3 (Bounded choice) $b\text{–}AC := \bigcup_{\delta,\rho \in \mathbf{T}} \Big\{ (b\text{–}AC^{\delta,\rho}) \Big\}$ *denotes the schema of bounded choice*

$$(b\text{–}AC^{\delta,\rho}) \ :\ \forall Z^{\rho\delta} \big(\forall x^\delta \exists y \leq_\rho Zx\ A(x,y,Z) \rightarrow \exists Y \leq_{\rho\delta} Z \forall x A(x,Yx,Z) \big).$$

Theorem 2.4 ([16]) *Let* A *be as in thm.2.2 and* Δ *be a set of sentences* $\forall x^\delta \exists y \leq_\rho sx \forall z^\eta G_0(x,y,z)$ *where* s *is a closed term of* G_nA^ω *and* G_0 *a quantifier-free formula, and let* A' *denote the negative translation*[3] *of* A. *Then the following rule holds:*

$$\begin{cases} G_nA^\omega + AC\text{–}qf + \Delta \vdash A^H \wedge Mon(A) \Rightarrow \\ G_nA^\omega + \tilde{\Delta} \ \vdash A \text{ and} \\ \text{by monotone functional interpretation one can extract a tuple } \underline{\Psi} \in \ G_nR^\omega \text{ such that} \\ G_nA_i^\omega + \tilde{\Delta} \vdash \underline{\Psi} \text{ satisfies the monotone functional interpretation of } A', \end{cases}$$

where $\tilde{\Delta} := \{ \exists Y \leq_{\rho\delta} s \forall x^\delta, z^\eta G_0(x,Yx,z) : \forall x^\delta \exists y \leq_\rho sx \forall z^\eta G_0(x,y,z) \in \Delta \}$. *(In particular the second conclusion can be proved in* $G_nA_i^\omega + \Delta +$ *b-AC).*

Remark 2.5 *In theorems 2.2,2.4 one may also have tuples '*$\exists \underline{w}$*' instead of '*$\exists w^\gamma$*' in* A.

For our applications in the next paragraph we need the following corollary of theorem 2.4:

Corollary 2.6 ([16]) *Let* $\forall x^0 \exists y^0 \forall z^0 A_0(u^1,v^\tau,x,y,z) \in \mathcal{L}(G_nA^\omega)$ *be a formula which contains only* u,v *as free variables and satisfies provably in* $G_nA^\omega + \Delta + AC\text{–}qf$ *the following monotonicity property:*

$$(*)\ \forall u,v,x,\tilde{x},y,\tilde{y}(\tilde{x} \leq_0 x \wedge \tilde{y} \geq_0 y \wedge \forall z^0 A_0(u,v,x,y,z) \rightarrow \forall z^0 A_0(u,v,\tilde{x},\tilde{y},z)),$$

(i.e. $Mon(\exists x \forall y \exists z \neg A_0)$*). Furthermore let* $B_0(u,v,w^\gamma) \in \mathcal{L}(G_nA^\omega)$ *be a (quantifier-free) formula which contains only* u,v,w *as free variables and* $\gamma \leq 2$. *Then from a proof*

$$G_nA^\omega + \Delta + AC\text{–}qf \ \vdash \forall u^1 \forall v \leq_\tau tu \big(\exists f^1 \forall x,z\, A_0(u,v,x,fx,z) \rightarrow \exists w^\gamma B_0(u,v,w) \big) \wedge (*)$$

[3] Here we can use Gödel's [8] translation or any of the various negative translations. For a systematical treatment of negative translations see [18].

one can extract a term $\chi \in G_nR^\omega$ *such that*

$$G_nA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \forall \Psi^*\big((\Psi^* \textit{ satisfies the mon.funct.interpr. of } \forall x^0, g^1 \exists y^0 A_0(u,v,x,y,gy)) \to \exists w \leq_\gamma \chi u \Psi^* B_0(u,v,w)\big)^4.$$

In the conclusion $\Delta+$ *b-AC can be replaced by* $\tilde{\Delta}$ *as defined in thm.2.4. If* $\tau \leq 1$ *and the types of existential quantifiers in the axioms* Δ *are* ≤ 1*, then* $G_nA^\omega + \Delta + AC\text{–}qf$ *may be replaced by* $E\text{–}G_nA^\omega + \Delta + AC^{\alpha,\beta}\text{–}qf$*, where* $(\alpha = 0 \wedge \beta \leq 1)$ *or* $(\alpha = 1 \wedge \beta = 0)$*.*

The mathematical significance of corollary 2.6 for the extraction of bounds from given proofs by arithmetization rests on the following fact: Direct monotone functional interpretation of

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC\text{–}qf} \vdash \forall u^1 \forall v \leq_\tau tu\big(\exists f^1 \forall x, z\, A_0(u,v,x,fx,z) \to \exists w^\gamma B_0(u,v,w)\big)$$

yields only a bound on $\exists w$ which depends on a functional which satisfies the monotone functional interpretation of (1) $\exists f \forall x, z A_0$ or if we let remain the double negation in front of $\exists$ (which comes from the negative translation) (2) $\neg\neg\exists f \forall x, z\, A_0$. However in our applications the monotone functional interpretation of (1) would require non–computable functionals (since f is not recursive) and the monotone functional interpretation of (2) can be carried out only using bar-recursive functionals (see [23]). In contrast to this the bound χ only depends on a functional which satisfies the monotone functional interpretation of the negative translation of $\forall x \exists y \forall z\, A_0(x,y,z)$: In our applications in section 5 such a functional can be constructed in $\widehat{PR}^\omega$ except for the existence of the limit superior of a bounded sequence of real numbers where the fragment T_1 of Gödel's calculus T with R_ρ for $\rho \leq 1$ is needed (note that the Ackermann function is definable in T_1).
In particular the use of the **analytical** premise $\exists f^1 \forall x, z A_0$ has been reduced to the **arithmetical** premise $\forall x^0 \exists y^0 \forall z^0 A_0$.

3 Real numbers in $\mathbf{G_2A_i^\omega}$

Suppose that a proposition $\forall x \exists y A(x,y)$ is proved in one of the theories $\mathcal{T}^\omega$ from [16], where the variables x, y may range over $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or e.g. C[0,1] etc. What sort of numerical information on '$\exists y$' relatively to the 'input' x can be extracted from a given proof depends in particular on how x is represented, i.e. on the numerical data by which x is given:
Suppose e.g. x that is a variable on $\mathbb{R}$ and real numbers are represented by **arbitrary** Cauchy sequences of rational numbers x_n, i.e.

$$(1)\ \forall k^0 \exists n^0 \forall m, \tilde{m} \geq n\big(|x_m - x_{\tilde{m}}| \leq \frac{1}{k+1}\big).$$

Let us consider the (obviously true) proposition

$$(2)\ \forall x \in \mathbb{R} \exists l \in \mathbb{N}(x \leq l).$$

Given x by a representative (x_n) in the sense of (1) it is not possible to compute an l which satisfies (2) on the basis of this representation, since this would involve the computation of a number n which

[4]'Ψ^* satisfies the mon. funct.interpr. of $\forall x, g \exists y A_0(u,v,x,y,gy)$' is meant here for fixed u, v (and not uniformly as a functional in u, v), i.e. $\exists \Psi\big(\Psi^* \text{ s–maj } \Psi \wedge \forall x, g\, A_0(u,v,x,\Psi xg, g(\Psi xg))\big)$.

fulfils a (in general undecidable) universal property like $\forall m, \tilde{m} \geq n(|x_m - x_{\tilde{m}}| \leq 1)$ to define l as $\lceil |x_n| \rceil + 1$.

If however real numbers are represented by Cauchy sequences with a **fixed Cauchy modulus**, e.g. $1/(k+1)$, i.e.

$$(3)\ \forall m, \tilde{m} \geq k\big(|x_m - x_{\tilde{m}}| \leq \frac{1}{k+1}\big),$$

then the computation of l is trivial: $l := \Phi\left((x_n)\right) := \lceil |x_0| \rceil + 1$. Φ is not a function : $\mathbb{R} \to \mathbb{N}$ since it is not extensional: Different Cauchy sequences $(x_n), (\tilde{x}_n)$ which represent the same real number, i.e. $\lim_{n\to\infty}(x_n - \tilde{x}_n) = 0$, yield in general different numbers $\Phi\left((x_n)\right) \neq \Phi\left((\tilde{x}_n)\right)$. Following E. Bishop [3] , [4] we call Φ an **operation** : $\mathbb{R} \to \mathbb{N}$. This phenomenon is a general one (and not caused by the special definition of Φ): The only computable operations $\mathbb{R} \to \mathbb{N}$, which are extensional, are operations which are constant, since the computability of Φ implies its continuity as a functional[5] : $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and therefore (if it is extensional w.r.t. $=_{\mathbb{R}}$) the continuity as a function $\mathbb{R} \to \mathbb{N}$.

The importance of the representation of complex objects as e.g. real numbers is also indicated by the fact that the logical form of properties of these objects depends essentially on the representation: If $(x_n), (\tilde{x}_n)$ are arbitrary Cauchy sequences (in the sense of (1)) then the property that both sequences represent the same real number is expressed by the Π^0_3–formula

$$(4)\ \forall k \exists n \forall m, \tilde{m} \geq n\big(|x_m - \tilde{x}_m| \leq \frac{1}{k+1}\big).$$

For Cauchy sequences with fixed Cauchy modulus as in (2) this property can be expressed by the (logically much simpler) Π^0_1–formula

$$(5)\ \forall k\big(|x_k - \tilde{x}_k| \leq \frac{3}{k+1}\big).$$

For Cauchy sequences with modulus $1/(k+1)$ (4) and (5) are equivalent (provably in $\mathrm{G_2A}^\omega_i$). But for arbitrary Cauchy sequences (4) does not imply (5) in general.

If $(x_n) \subset \mathbb{Q}$ is an arbitrary Cauchy sequence then $\mathrm{AC}^{0,0}$ applied to

$$\forall k \exists n \forall m, \tilde{m} \geq n\big(|x_m - x_{\tilde{m}}| \leq \frac{1}{k+1}\big)$$

yields the existence of a function f^1 such that $\forall k \forall m, \tilde{m} \geq fk\big(|x_m - x_{\tilde{m}}| \leq \frac{1}{k+1}\big)$.
For $m, \tilde{m} \geq k$ this implies $|x_{fm} - x_{f\tilde{m}}| \leq \frac{1}{k+1}$ (choose $k' \in \{m, \tilde{m}\}$ with $fk' \leq fm, f\tilde{m}$ and apply the Cauchy property to $m' := fm, \tilde{m}' := f\tilde{m}$), i.e. the sequence $(x_{fn})_{n\in\mathbb{N}}$ is a Cauchy sequence with modulus $1/(k+1)$ which has the same limit as $(x_n)_{n\in\mathbb{N}}$.
Thus in the presence of $\mathrm{AC}^{0,0}$ (or more precisely the restriction $\mathrm{AC}^{0,0}$–$\forall$ of $\mathrm{AC}^{0,0}$ to Π^0_1–formulas) both representations (1) and (2) equivalent. However $\mathrm{AC}^{0,0}$–$\forall$ is not provable in any of our theories and the addition of this schema to the axioms would yield an explosion of the rate of growth of the provably recursive functions. In fact every $\alpha(<\varepsilon_0)$–recursive function is provably recursive in

[5]An operation $\Phi : \mathbb{R} \to \mathbb{N}$ is given by a functional : $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ (which is extensional w.r.t. $=_1$!) since sequences of rational numbers are coded as sequences of natural numbers.

G_2A^ω+ $AC^{0,0}$–$\forall$. This follows from the fact that iterated use of $AC^{0,0}$–$\forall$ combined with classical logic yields full arithmetical comprehension

$$CA_{ar} \ : \ \exists f^1 \forall x^0 (fx =_0 0 \leftrightarrow A(x)),$$

where A is an arithmetical formula, i.e. a formula containing only quantifiers of type 0. CA_{ar} applied to QF–IA proves the induction principle for every arithmetical formula. Hence full Peano–arithmetic PA is a subsystem of G_2A^ω+ $AC^{0,0}$–$\forall$.
As a consequence of this situation we have to specify the representation of real numbers we choose:

Definition 3.1 *A real number is given by a Cauchy sequence of rational numbers with modulus* $1/(k+1)$.

The reason for this representation is two–fold:

1) As we have seen above any numerically interesting application of the extraction of a bound presupposes that the input is given as a numerically reasonable object. This is also the reason why in constructive analysis (in the sense of Bishop) as well as in complexity theory for analysis (in the sense of H. Friedman and K.–I. Ko, see [11]) real numbers are always endowed with a rate of convergence, continuous functions with a modulus of continuity and so on. Also in the work by H. Friedman, S. Simpson (see e.g. [22]) and others on the program of so–called 'reverse mathematics', real numbers are always given with a fixed rate of convergence.

2) For our representation of real numbers we can achieve that quantification over real numbers is nothing else then quantification over $\mathbb{N}^{\mathbb{N}}$, i.e. $\forall x^1, \exists y^1$. Because of this many interesting theorems in analysis have the logical form $\forall \exists F_0$ (see [13] for a discussion on that) so that our method of extracting feasible bounds applies.

1) and 2) are in fact closely related: If real numbers would be represented as arbitrary Cauchy sequences then a proposition $\forall x \in \mathbb{R} \exists y \in \mathbb{N}\ A(x,y)$ would have the logical form

$$\forall x^1 (\forall k \exists n \forall m F_0 \rightarrow \exists y^0 A),$$

where $(*)$ $\forall k \exists n \forall m F_0$ expresses the Cauchy property of the sequence of rational numbers coded by x^1. By our reasoning in [15] we know that in general we can only obtain an effective bound on y which depends on x together with a Skolem function for $(*)$. But this just means that the computation of the bound requires that x is given with a Cauchy modulus.
As concerned with provability in our theories like G_nA^ω+AC–qf the representation with fixed modulus is no real restriction: In section 5 we will show in particular that the a proof of

$$\forall (x_n) \big(\exists f^1 \forall k \forall m, \tilde{m} \geq fk (|x_m - \tilde{x}_m| \leq \frac{1}{k+1}) \rightarrow \exists y^0 A \big)$$

can be transformed into a proof of

$$\forall (x_n) \big(\forall k \exists n \forall m, \tilde{m} \geq n (|x_m - \tilde{x}_m| \leq \frac{1}{k+1}) \rightarrow \exists y^0 A \big).$$

within the same theory (i.e. without any use of $AC^{0,0}$) for a large class of formulas A.

The representation of $\mathbb{R}$ presupposes a **representation of** $\mathbb{Q}$: Rational numbers are represented as codes $j(n,m)$ of pairs (n,m) of natural numbers n,m. $j(n,m)$ represents

$$\text{the rational number } \frac{\frac{n}{2}}{m+1}, \text{ if } n \text{ is even, and the negative rational } -\frac{\frac{n+1}{2}}{m+1} \text{ if } n \text{ is odd.}$$

By the surjectivity of our pairing function j from [15] every natural number can be conceived as code of a uniquely determined rational number. On the codes of $\mathbb{Q}$, i.e. on $\mathbb{N}$, we define an equivalence relation by

$$n_1 =_{\mathbb{Q}} n_2 :\equiv \frac{\frac{j_1 n_1}{2}}{j_2 n_1 + 1} = \frac{\frac{j_1 n_2}{2}}{j_2 n_2 + 1} \text{ if } j_1 n_1, j_1 n_2 \text{ both are even}$$

and analgously in the remaining cases, where $\frac{a}{b} = \frac{c}{d}$ is defined to hold iff $ad =_0 cb$ (for $bd > 0$).
On $\mathbb{N}$ one easily defines functions $|\cdot|_{\mathbb{Q}}, +_{\mathbb{Q}}, -_{\mathbb{Q}}, \cdot_{\mathbb{Q}} :_{\mathbb{Q}}, \max_{\mathbb{Q}}, \min_{\mathbb{Q}} \in \mathrm{G}_2\mathrm{R}^\omega$ and (quantifier–free) relations) $<_{\mathbb{Q}}, \leq_{\mathbb{Q}}$ which represent the corresponding functions and relations on $\mathbb{Q}$. In the following we sometimes omit the index $\mathbb{Q}$ if this does not cause any confusion.

Notational convention: For better readability we often write e.g. $\frac{1}{k+1}$ instead of its code $j(2,k)$ in $\mathbb{N}$. So e.g. we write $x^0 \leq_{\mathbb{Q}} \frac{1}{k+1}$ for $x \leq_{\mathbb{Q}} j(2,k)$.

By the coding of rational numbers as natural numbers, **sequences of rationals** are just functions f^1 (and every function f^1 can be conceived as a sequence of rational numbers in a unique way). In particular representatives of real numbers are functions f^1 modulo this coding. We now show that **every** function can be conceived as an representative of a uniquely determined Cauchy sequence of rationals with modulus $1/(k+1)$ and therefore can be conceived as an representative of a uniquely determined real number.[6]
To achieve this we need the following functional $\widehat{f}$.

Definition 3.2 *The functional $\lambda f^1.\widehat{f} \in G_2R^\omega$ is defined such that*

$$\widehat{f}n = \begin{cases} fn, \text{ if } \forall k,m,\tilde{m} \leq_0 n\big(m,\tilde{m} \geq_0 k \to |fm -_{\mathbb{Q}} f\tilde{m}| \leq_{\mathbb{Q}} \frac{1}{k+1}\big) \\ f(n_0 - 1) \text{ for } n_0 := \min l \leq_0 n[\exists k,m,\tilde{m} \leq_0 l\big(m,\tilde{m} \geq_0 k \wedge |fm -_{\mathbb{Q}} f\tilde{m}| >_{\mathbb{Q}} \frac{1}{k+1}\big)], \\ \text{otherwise.} \end{cases}$$

One easily verifies (within $\mathrm{G}_2\mathrm{A}^\omega_i$) that

1) if f^1 represents a Cauchy sequence of rational numbers with modulus $1/(k+1)$, then $\forall n^0(fn =_0 \widehat{f}n)$,

2) for every f^1 the function $\widehat{f}$ represents a Cauchy sequence of rational numbers with modulus $1/(k+1)$.

Hence every function f gives a uniquely determined real number, namely that number which is represented by $\widehat{f}$. Quantification $\forall x \in \mathbb{R}\, A(x)$ $(\exists x \in \mathbb{R}\, A(x))$ so reduces to the quantification $\forall f^1 A(\widehat{f})$ $(\exists f^1 A(\widehat{f}))$ for properties A which are extensional w.r.t. $=_{\mathbb{R}}$ below (i.e. which are really properties of real numbers). **Operations** $\Phi : \mathbb{R} \to \mathbb{R}$ are given by functionals $\Phi^{1(1)}$ (which are

[6] A related representation of real numbers is sketched in [1] .

extensional w.r.t.$=_1$). A real function : $\mathbb{R} \to \mathbb{R}$ is given by a functional $\Phi^{1(1)}$ which (in addition) is extensional w.r.t. $=_{\mathbb{R}}$. Following the usual notation we write (x_n) instead of fn and $(\widehat{x}_n)$ instead of $\widehat{fn}$.
In the following we define various relations and operations on functions which correspond to the usual relations and operations on $\mathbb{R}$ for the real numbers represented by the respective functions:

Definition 3.3 *1)* $(x_n) =_{\mathbb{R}} (\tilde{x}_n) :\equiv \forall k^0 \big(|\widehat{x}_k -_{\mathbb{Q}} \widehat{\tilde{x}}_k| \leq_{\mathbb{Q}} \frac{3}{k+1} \big)$;

2) $(x_n) <_{\mathbb{R}} (\tilde{x}_n) :\equiv \exists k^0 \big(\widehat{\tilde{x}}_k - \widehat{x}_k >_{\mathbb{Q}} \frac{3}{k+1} \big)$;

3) $(x_n) \leq_{\mathbb{R}} (\tilde{x}_n) :\equiv \neg (\widehat{\tilde{x}}_n) <_{\mathbb{R}} (\widehat{x}_n)$;

4) $(x_n) +_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2n+1} +_{\mathbb{Q}} \widehat{\tilde{x}}_{2n+1})$;

5) $(x_n) -_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2n+1} -_{\mathbb{Q}} \widehat{\tilde{x}}_{2n+1})$;

6) $|(x_n)|_{\mathbb{R}} := (|\widehat{x}_n|_{\mathbb{Q}})$;

7) $(x_n) \cdot_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2(n+1)k} \cdot_{\mathbb{Q}} \widehat{\tilde{x}}_{2(n+1)k})$, *where* $k := \lceil \max_{\mathbb{Q}} (|x_0|_{\mathbb{Q}} + 1, |\tilde{x_0}|_{\mathbb{Q}} + 1) \rceil$;

8) For (x_n) *and* l^0 *we define*

$$(x_n)^{-1} := \begin{cases} (\max_{\mathbb{Q}} (\widehat{x}_{(n+1)(l+1)^2}, \frac{1}{l+1})^{-1}), & \textit{if } \widehat{x}_{2(l+1)} >_{\mathbb{Q}} 0 \\ (\min_{\mathbb{Q}} (\widehat{x}_{(n+1)(l+1)^2}, \frac{-1}{l+1})^{-1}), & \textit{otherwise}; \end{cases}$$

9) $\max_{\mathbb{R}} \big((x_n), (\tilde{x}_n) \big) := \big(\max_{\mathbb{Q}} (\widehat{x}_n, \widehat{\tilde{x}}_n) \big), \quad \min_{\mathbb{R}} \big((x_n), (\tilde{x}_n) \big) := \big(\min_{\mathbb{Q}} (\widehat{x}_n, \widehat{\tilde{x}}_n) \big)$.

One easily verifies the following

Lemma 3.4 *1)* $(x_n) =_{\mathbb{R}} (\tilde{x}_n)$ *resp.* $(x_n) <_{\mathbb{R}} (\tilde{x}_n)$, $(x_n) \leq_{\mathbb{R}} (\tilde{x}_n)$ *hold iff the correponding relations hold for those real numbers which are represented by* $(x_n), (\tilde{x}_n)$.

2) Provably in $G_2A^\omega_i$, $(x_n) +_{\mathbb{R}} (\tilde{x}_n)$, $(x_n) -_{\mathbb{R}} (\tilde{x}_n)$, $(x_n) \cdot_{\mathbb{R}} (\tilde{x}_n)$, $\max_{\mathbb{R}} \big((x_n), (\tilde{x}_n) \big)$, $\min_{\mathbb{R}} \big((x_n), (\tilde{x}_n) \big)$ *and* $|(x_n)|_{\mathbb{R}}$ *also represent Cauchy sequences with modulus* $1/(k+1)$ *which represent the real number obtained by addition (subtraction,...) of those real numbers which are represented by* $(x_n), (\tilde{x}_n)$. *This also holds for* $(x_n)^{-1}$ *if* $|(x_n)|_{\mathbb{R}} \geq_{\mathbb{R}} \frac{1}{l+1}$ *for the number* l *used in the definition of* $(x_n)^{-1}$. *In particular the operations* $+_{\mathbb{R}}, -_{\mathbb{R}}$ *etc. are extensional w.r.t. to* $=_{\mathbb{R}}$ *and therefore represent functions*[7].

3) The functionals $+_{\mathbb{R}}, -_{\mathbb{R}}, \cdot_{\mathbb{R}}, \max_{\mathbb{R}}, \min_{\mathbb{R}}$ *of type* $1(1)(1)$, $|\cdot|_{\mathbb{R}}$ *of type* $1(1)$ *and* $()^{-1}$ *of type* $1(1)(0)$ *are definable in* G_2R^ω.

Remark 3.5 *Since our theories* $G_nA^\omega_i$ *contain all* $\mathbb{N}, \mathbb{N}^{\mathbb{N}}$*–true purely universal sentences* $\forall \underline{x}^{0/1} A_0(\underline{x})$ *as axioms (because they do not contribute to the growth of extractable bounds at all, see [15] for details), it is easy to check that the basic properties of* $=_{\mathbb{R}}, \leq_{\mathbb{R}}, +_{\mathbb{R}}, \ldots$ *can be proved in* $G_2A^\omega_i$. *They are either directly purely universal or can be strengthened to universal statements, e.g.*

[7] The functional $()^{-1}$ is extensional for all l and $(x_n), (y_n)$ such that $|(x_n)|_{\mathbb{R}}, |(y_n)|_{\mathbb{R}} \geq \frac{1}{l+1}$.

$x =_{\mathbb{R}} y \wedge y =_{\mathbb{R}} z \to x =_{\mathbb{R}} z$ *follows from the universal axiom*

$$\forall x^1, y^1, k^0 \big(|\widehat{x}(6(k+1)) -_{\mathbb{Q}} \widehat{y}(6(k+1))| \leq_{\mathbb{Q}} \frac{3}{6(k+1)+1} \wedge |\widehat{y}(6(k+1)) -_{\mathbb{Q}} \widehat{z}(6(k+1))| \leq_{\mathbb{Q}} \frac{3}{6(k+1)+1} \to |\widehat{x}(k) -_{\mathbb{Q}} \widehat{z}(k)| \leq_{\mathbb{Q}} \frac{3}{k+1} \big).$$

Rational numbers q coded by r_q have as canonical representative in $\mathbb{R}$ (besides other representatives) the constant function $\lambda n^0.r_q$. One easily shows that $\forall k \big(|(x_n) -_{\mathbb{R}} \lambda n.\hat{x}_k| \leq_{\mathbb{R}} \frac{1}{k+1} \big)$ for every function (x_n).

Notational convention: For notational simplicity we often omit the emmbedding $\mathbb{Q} \hookrightarrow \mathbb{R}$, e.g. $x^1 \leq_{\mathbb{R}} y^0$ stands for $x \leq_{\mathbb{R}} \lambda n.y^0$. From the type of the objects it will be always clear what is meant.

If $(f_n)_{n \in \mathbb{N}}$ of type 1(0) represents a $\frac{1}{k+1}$–Cauchy sequence of **real** numbers, then $f(n) := \widehat{f}_{3(n+1)}(3(n+1))$ represents the limit of this sequence, i.e. $\forall k \big(|f_k -_{\mathbb{R}} f| \leq_{\mathbb{R}} \frac{1}{k+1} \big)$. One easily verifies this fact in $\mathrm{G}_2\mathrm{A}^\omega_i$.

Representation of $\mathbb{R}^d$ in $\mathbf{G}_2\mathbf{A}^\omega_i$:

For every fixed d we represent $\mathbb{R}^d$ as follows: Elements of $\mathbb{R}^d$ are represented by functions f^1 in the following way: Using the construction $\widehat{f}$ from above, every f^1 can be conceived as a representative of such a d–tuple of Cauchy sequences of real numbers, namely the sequence which is represented by

$$\big(\widehat{\nu^d_1(f)}, \ldots, \widehat{\nu^d_d(f)} \big), \text{ where } \nu^d_i(f) := \lambda x^0.\nu^d_i(fx),$$

(ν^d_i are the coding functions $\in \mathrm{G}_2\mathrm{R}^\omega$ from [15]).

Since the $\widehat{\nu^d_i(f)}$ represent Cauchy sequences of rationals with Cauchy modulus $\frac{1}{k+1}$, elements of $\mathbb{R}^d$ are so represented as Cauchy sequences of elements in $\mathbb{Q}^d$ which have the Cauchy modulus $\frac{1}{k+1}$ w.r.t. the maximum norm $\|f^1\|_{\max} := \max_{\mathbb{R}} \big(|\nu^d_1(f)|_{\mathbb{R}}, \ldots, |\nu^d_d(f)|_{\mathbb{R}} \big)$.

Quantification $\forall (x_1, \ldots, x_d) \in \mathbb{R}^d$ so reduces to $\forall f^1 A(\widehat{\nu^d_1(f)}, \ldots, \widehat{\nu^d_d(f)})$ for $\mathbb{R}^d$–extensional properties A (likewise for $\exists$).

The operations $+_{\mathbb{R}^d}, -_{\mathbb{R}^d}, \ldots$ are defined via the corresponding operations on the components, e.g. $x^1 +_{\mathbb{R}^d} y^1 :\equiv \nu^d(\nu^d_1 x +_{\mathbb{R}} \nu^d_1 y, \ldots, \nu^d_d x +_{\mathbb{R}} \nu^d_d y)$.

Sequences of elements in $\mathbb{R}^d$ are represented by (f_n) of type 1(0).

Representation of [0,1]$\subset \mathbb{R}$ in $\mathbf{G}_2\mathbf{A}^\omega_i$

We now show that every element of $[0,1]$ can be represented already by a bounded function $f \in \{f : f \leq_1 M\}$, where M is a fixed function from $\mathrm{G}_2\mathrm{R}^\omega$ and that every function from this set can be conceived as an (representative of an) element in [0,1]: Firstly we define a function $q \in \mathrm{G}_2\mathrm{R}^\omega$ by

$$q(n) := \begin{cases} \min l \leq_0 n[l =_{\mathbb{Q}} n], \text{ if } 0 \leq_{\mathbb{Q}} n \leq_{\mathbb{Q}} 1 \\ 0^0, \text{ otherwise.} \end{cases}$$

It is clear that every rational number $\in [0,1] \cap \mathbb{Q}$ has a unique code by a number $\in q(\mathbb{N})$ and $\forall n^0 (q(q(n)) =_0 q(n))$. Also every such number codes an element of $\in [0,1] \cap \mathbb{Q}$. We may conceive

every number n as a representative of a rational number $\in [0,1] \cap \mathbb{Q}$, namely of the rational coded by $q(n)$.
In contrast to $\mathbb{R}$ we can restrict the set of representing functions for [0,1] to the compact (in the sense of the Baire space) set $f \in \{f : f \leq_1 M\}$, where $M(n) := j(6(n+1), 3(n+1)-1)$ (here j is the Cantor pairing function):
Each fraction r having the form $\frac{i}{3(n+1)}$ (with $i \leq 3(n+1)$) is represented by a number $k \leq M(n)$, i.e. $k \leq M(n) \wedge q(k)$ codes r. Thus $\{k : k \leq M(n)\}$ contains (modulo this coding) an $\frac{1}{3(n+1)}$–net for [0,1].
We define a functional $\lambda f.\tilde{f} \in \mathrm{G_2R}^\omega$ such that

$$\tilde{f}(k) = q(i_0), \text{ where } i_0 = \mu i \leq_0 M(k)[\forall j \leq_0 M(k)(|\widehat{f}(3(k+1)) -_{\mathbb{Q}} q(j)| \geq_{\mathbb{Q}} |\widehat{f}(3(k+1)) -_{\mathbb{Q}} q(i)|)].$$

$\tilde{f}$ has (provably in $\mathrm{G_2A}^\omega_i$) the following properties:

1) $\forall f^1(\tilde{f} \leq_1 M)$.

2) $\forall f^1(\widehat{\tilde{f}} =_1 \tilde{f})$.

3) $\forall f^1(0 \leq_{\mathbb{R}} \tilde{f} \leq_{\mathbb{R}} 1)$.

4) $\forall f^1(0 \leq_{\mathbb{R}} f \leq_{\mathbb{R}} 1 \to f =_{\mathbb{R}} \tilde{f})$.

5) $\forall f^1(\tilde{\tilde{f}} =_{\mathbb{R}} \tilde{f})$.

By this construction quantification $\forall x \in [0,1]\ A(x)$ and $\exists x \in [0,1]\ A(x)$ reduces to quantification having the form $\forall f \leq_1 M\ A(\tilde{f})$ and $\exists f \leq_1 M\ A(\tilde{f})$ for properties A which are $=_{\mathbb{R}}$–extensional (for f_1, f_2 such that $0 \leq_{\mathbb{R}} f_1, f_2 \leq_{\mathbb{R}} 1$), where $M \in \mathrm{G_2R}^\omega$. Similarly one can define a representation of $[a,b]$ for variable a^1, b^1 such that $a <_{\mathbb{R}} b$ by bounded functions $\{f^1 : f \leq_1 M(a,b)\}$. However by remark 3.6 below one can easily reduce the quantification over $[a,b]$ to quantification over $[0,1]$ so that we do not need this generalization. But on some occasions it is convenient to have an explicit representation for $[-k,k]$ for all natural numbers k. This representation is analogous to the representation of $[0,1]$ except that we now define $M_k(n) := j(6k(n+1), 3(n+1)-1)$ as the bounding function. The construction corresponding to $\lambda f.\tilde{f}$ is also denoted by $\tilde{f}$ since it will be always clear from the context what interval we have in mind.

Representation of $[0,1]^d$ in $\mathbf{G_2A}^\omega_i$

Using the construction $f \mapsto \tilde{f}$ from the representation of [0,1] we also can represent $[0,1]^d$ for every fixed number d by a bounded set $\{f^1 : f \leq_1 M_d\}$ of functions, where $M_d : \nu^d(M, \ldots, M) \in \mathrm{G_2R}^\omega$ for every fixed d:
$f(\leq M_d)$ represents the vector in $[0,1]^d$ which is represented by $((\widetilde{\nu_1^d f}), \ldots, (\widetilde{\nu_d^d f}))$. If (in the other direction) $f_1, \ldots, f_d$ represent real numbers $x_1, \ldots, x_d \in [0,1]$, then $f := \nu^d(\tilde{f}_1, \ldots, \tilde{f}_d) \leq_1 \nu^d(M, \ldots, M)$ represents $(x_1, \ldots, x_d) \in [0,1]^d$ in this sense.

Remark 3.6 *For $a, b \in \mathbb{R}$ with $a \leq_{\mathbb{R}} b$, quantification $\forall x \in [a,b]\ A(x)$ ($\exists x \in [a,b]\ A(x)$) reduces to quantification over $[0,1]$ (and therefore –modulo our representation– over $\{f : f \leq_1 M\}$) by $\forall \lambda \in [0,1]\ A((1-\lambda)a + \lambda b)$ and analogously for $\exists x$. This transformation immediately generalizes to $[a_1, b_1] \times \cdots \times [a_d, b_d]$ using $\lambda_1, \ldots, \lambda_d$.*

4 Sequences and series in $G_2A_i^\omega$: Convergence with moduli involved

By our representation of real numbers by functions f^1 developed in the previous section, sequences of real numbers are given as functions $f^{1(0)}$ in $G_2A_i^\omega$. We will use the usual notation (a_n) instead of f. In this section we are concerned with the following properties of sequences of real numbers:

1) (a_n) is a Cauchy sequence, i.e. $\forall k^0 \exists n^0 \forall m, \tilde{m} \geq_0 n \big(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1} \big)$

2) (a_n) is convergent, i.e. $\exists a^1 \forall k^0 \exists n^0 \forall m \geq_0 n \big(|a_m -_{\mathbb{R}} a| \leq_{\mathbb{R}} \frac{1}{k+1} \big)$.

3) (a_n) is convergent with a modulus of convergence, i.e.

$$\exists a^1, h^1 \forall k^0 \forall m \geq_0 hk (|a_m -_{\mathbb{R}} a| \leq_{\mathbb{R}} \frac{1}{k+1}).$$

4) (a_n) is a Cauchy sequence with a Cauchy modulus, i.e.

$$\exists h^1 \forall k^0 \forall m, \tilde{m} \geq_0 hk (|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1}).$$

One easily shows within $G_2A_i^\omega$ that 4) $\leftrightarrow$ 3) $\to$ 2) $\to$ 1). Using $AC^{0,0}$-$\forall^0$ one can prove that 1) $\to$ 4) (and therefore 1) $\leftrightarrow$ 2) $\leftrightarrow$ 3) $\leftrightarrow$ 4)).
However, as we already have discussed in the previuous section, the addition of $AC^{0,0}$–$\forall^0$ to G_2A^ω would make all $\alpha(<\varepsilon_0)$–recursive functions provably recursive.
Thus since we are working in (extensions of) G_2A^ω we have to distinguish carefully between e.g. 1) and 4). In the next section we will study the relationship between 1) and 4) in detail and show in particular that the use of sequences of single instances of 4) in proofs of $\forall u^1 \forall v \leq_\rho tu \exists w^2 A_0$–sentences relatively to e.g. $G_2A^\omega + \Delta$+AC–qf (where Δ is defined as in thm.2.4) can be reduced the use of the same instances of 1).

For **monotone** sequences (a_n) the equivalence of 2) and 3) (and hence that of 2) and 4)) is already provable using only the **quantifier–free** choice $AC^{0,0}$–qf:
Let (a_n) be say increasing, i.e.

$$(i)\ \forall n^0 (a_n \leq_{\mathbb{R}} a_{n+1}),$$

and a^1 be such that

$$(ii)\ \forall k^0 \exists n^0 \forall m \geq_0 n \big(|a_m - a| \leq_{\mathbb{R}} \frac{1}{k+1} \big).$$

$AC^{0,0}$–qf applied to $\forall k^0 \exists n^0 \big(\underbrace{|a_n - a| <_{\mathbb{R}} \frac{1}{k+1}}_{\in \Sigma^0_1} \big)$ yields $\exists h^1 \forall k^0 \big(|a_{hk} - a| <_{\mathbb{R}} \frac{1}{k+1} \big)$, which gives $\exists h^1 \forall k^0 \forall m \geq_0 hk \big(|a_m - a| <_{\mathbb{R}} \frac{1}{k+1} \big)$, since –by (i),(ii)– $a_{hk} \leq a_m \leq a$ for all $m \geq_0 hk$. (Here we use the fact that $\forall n (a_n \leq_{\mathbb{R}} a_{n+1}) \to \forall m, \tilde{m} (m \geq \tilde{m} \to a_{\tilde{m}} \leq_{\mathbb{R}} a_m)$. This follows in G_2A^ω from the universal sentence
$(+)\ \forall a^{1(0)}_{(\cdot)}, n, l \big(\forall k < n (\widehat{a}_k(l) \leq_{\mathbb{Q}} \widehat{a}_{k+1}(l) + \frac{3}{l+1}) \to \forall m, \tilde{m} \leq n (m \geq \tilde{m} \to a_{\tilde{m}} \leq_{\mathbb{R}} a_m + \frac{5n}{l+1}) \big)$. (+) is true (and hence an axiom of G_2A^ω) since $\widehat{a}_k(l) \leq_{\mathbb{Q}} \widehat{a}_{k+1}(l) + \frac{3}{l+1} \to a_k \leq_{\mathbb{R}} a_{k+1} + \frac{5}{l+1}$.)

If one of the properties 1), ...,4) –say $i \in \{1,\ldots,4\}$– is fulfilled for two sequences $(a_n),(b_n)$, then $i)$ is also fulfilled (provably in $G_2A^\omega_i$) for $(a_n +_{\mathbb{R}} b_n),(a_n -_{\mathbb{R}} b_n),(a_n \cdot_{\mathbb{R}} b_n)$ and (if $b_n \neq 0$ and $b_n \to b \neq 0$) for $(\frac{a_n}{b_n})$, where in the later case the modulus in 3),4) depends on an estimate $l \in \mathbb{N}$ such that $|b| \geq \frac{1}{l+1}$ (The construction of the moduli for $(a_n +_{\mathbb{R}} b_n),(a_n -_{\mathbb{R}} b_n),(a_n \cdot_{\mathbb{R}} b_n),(\frac{a_n}{b_n})$ from the moduli for $(a_n),(b_n)$ (for i=3,4) is similar to our definition of $+_{\mathbb{R}},-_{\mathbb{R}},\cdot_{\mathbb{R}},(\cdot)^{-1}$ given in the previous section.

The most important property of bounded monotone sequences (a_n) of real numbers is their convergence. We call this fact 'principle of convergence for monotone sequences' (PCM). Because of the difference between 1) and 4) above we have in fact to consider two versions of this principle:

$$(PCM1)\ :\ \begin{cases} \forall a^{1(0)}_{(\cdot)},c^1\big(\forall n^0(c \leq_{\mathbb{R}} a_{n+1} \leq_{\mathbb{R}} a_n) \\ \qquad\qquad \to \forall k^0\exists n^0\forall m,\tilde{m} \geq_0 n(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1})\big), \end{cases}$$

$$(PCM2)\ :\ \begin{cases} \forall a^{1(0)}_{(\cdot)},c^1\big(\forall n^0(c \leq_{\mathbb{R}} a_{n+1} \leq_{\mathbb{R}} a_n) \\ \qquad\qquad \to \exists h^1\forall k^0\forall m,\tilde{m} \geq_0 hk(|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1})\big), \end{cases}$$

Both principles cannot be derived in any of the theories $G_nA^\omega + \Delta$+AC–qf. In fact $(PCM1)$ is equivalent (relatively to G_3A^ω) to the second–order axiom of Σ^0_1–induction whereas $(PCM2)$ is equivalent (relatively to G_3A^ω+AC^{0,0}–qf) even to arithmetical comprehension over numbers (see [14]; for the system RCA$_0$, known from reverse mathematics, the equivalence between $(PCM2)$ and arithmetical comprehension is due to [6]). We now determine the contribution of the use of $(PCM1)$ to the growth of extractable uniform bounds. This will be used in the next section to determine the growth which may be caused be single sequences of instances of $(PCM2)$.

Using the construction $\tilde{a}(n) := \max_{\mathbb{R}}(0,\min\limits_{i\leq n}(a(i)))$ we can express $(PCM1)$ in the following logically more simple form[8]

$$(1)\ \forall a^{1(0)}\forall k^0\exists n^0\forall m >_0 n\big(\tilde{a}(n) -_{\mathbb{R}} \tilde{a}(m) \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

(If $a^{1(0)}$ fulfils $\forall n(0 \leq_{\mathbb{R}} a(n+1) \leq_{\mathbb{R}} a(n))$, then $\forall n(\tilde{a}(n) =_{\mathbb{R}} a(n))$. Furthermore $\forall n(0 \leq_{\mathbb{R}} \tilde{a}(n+1) \leq_{\mathbb{R}} \tilde{a}(n))$ for all $a^{1(0)}$. Thus by the transformation $a \mapsto \tilde{a}$, quantification over all decreasing sequences $\subset \mathbb{R}_+$ reduces to quantification over all $a^{1(0)}$).

By AC^{0,0}–qf (1) is equivalent to

$$(2)\ \forall a^{1(0)},k^0,g^1\exists n^0\big(gn >_0 n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

We now construct a functional Ψ which provides a bound for $\exists n$, i.e.

$$(3)\ \forall a^{1(0)},k^0,g^1\exists n \leq_0 \Psi akg\big(gn >_0 n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

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[8]Here we use that $\forall n^0\big(a(n+1) \leq_{\mathbb{R}} an\big) \to \forall n^0\big(\Phi_{\min_{\mathbb{R}}}(a,n) =_{\mathbb{R}} an\big)$, where $\Phi_{\min_{\mathbb{R}}}$ is a functional from G_2R^ω which computes the minimum of the real numbers $a(0),\ldots,a(n)$ (such a functional can be defined similarly to $\min_{\mathbb{R}}$ in section 3 noting that $\Phi_{\min_{\mathbb{Q}}}(f^1,n^0) = \min_{\mathbb{Q}}(f0,\ldots,fn)$ is definable in G_2R^ω). This follows in G_2A^ω from the purely universal sentence
$(+)\ \forall a^{1(0)},n,k\Big(\forall l < n\Big(\big(\widehat{a(l+1)}\big)(k) \leq_{\mathbb{Q}} (\widehat{al})(k) + \frac{3}{k+1}\Big) \to |\Phi_{\min_{\mathbb{R}}}(a,n) -_{\mathbb{R}} an| \leq_{\mathbb{R}} \frac{5n}{k+1}\Big)$. (+) is true (and hence an axiom of G_2A^ω) since $\big(\widehat{a(l+1)}\big)(k) \leq_{\mathbb{Q}} (\widehat{al})(k) + \frac{3}{k+1} \to a(l+1) \leq_{\mathbb{R}} al + \frac{5}{k+1}$.

Let $C(a) \in \mathbb{N}$ ($C(a) \geq 1$) be an upper bound for the real number represented by $\tilde{a}(0)$, e.g. $C(a) := (a(0))(0) + 1$. We show that

$\Psi akg := \max_{i<C(a)k'} \big(\Phi_{it} i0g\big)(= \max_{i<C(a)k'} \big(g^i(0)\big)$ satisfies (3) (provably in PRA^ω):

Claim: $\exists i < C(a)k'\big(g(g^i 0) > g^i 0 \to \tilde{a}(g^i 0) -_{\mathbb{R}} \tilde{a}(g(g^i 0)) \leq_{\mathbb{R}} \frac{1}{k+1}\big)$.

Case 1: $\exists i < C(a)k'(g(g^i 0) \leq g^i 0)$: Obvious!

Case 2: $\forall i < C(a)k'(g(g^i 0) > g^i 0)$: Assume $\forall i < C(a)k'\big(\tilde{a}(g^i 0) -_{\mathbb{R}} \tilde{a}(g(g^i 0)) >_{\mathbb{R}} \frac{1}{k+1}\big)$.

Then $\tilde{a}(0) -_{\mathbb{R}} \tilde{a}(g^{C(a)k'} 0) > C(a)$, contradicting $\tilde{a}(n) \in [0, C(a)]$ for all n.

In contrast to (2) the bounded proposition (3) has the form of an axiom Δ in the theorems from [15] and section 2. Hence the monotone functional interpretation of (3) requires just a majorant for Ψ. In particular we may use $\Psi \in \widehat{PR}^\omega$ itself since Ψ s–maj Ψ.

Thus from a proof of e.g. a sentence $\forall x^0 \forall y \leq_\rho sx \exists z^0 A_0(x, y, z)$ in $\mathrm{G}_n\mathrm{A}^\omega + \Delta + (PCM1)$+AC–qf we can (in general) extract only a bound t for z (i.e. $\forall x \forall y \leq sx \exists z \leq tx\, A_0(x, y, z)$) which is defined in $\widehat{PR}^\omega$ since the definition of Ψ uses the functional Φ_{it} which is not definable in $\mathrm{G}_\infty\mathrm{R}^\omega$ (see [15]). If however the proof uses (3) above only for functions g which can be bounded by terms in $\mathrm{G}_k\mathrm{R}^\omega$, then we can extract a $t \in \mathrm{G}_{\max(k+1,n)}\mathrm{R}^\omega$ since the iteration of a function $\in \mathrm{G}_k\mathrm{R}^\omega$ is definable in $\mathrm{G}_{k+1}\mathrm{R}^\omega$ (for $k \geq 2$).

The monotone functional interpretation of the negative translation of (1) requires (taking the quantifier hidden in $\leq_{\mathbb{R}}$ into account) a majorant for a functional Φ which bounds '$\exists n$' in

$$(3)' \ \forall a^{1(0)}, k^0, g^1, h^1 \exists n \big(gn > n \to \widehat{a(n)}(hn) -_{\mathbb{Q}} \widehat{\tilde{a}(gn)}(hn) \leq_{\mathbb{Q}} \frac{1}{k+1} + \frac{3}{h(n)+1}\big).$$

However every Φ which provides a bound for (2) a fortiori yields a bound for (3)$'$ (which does not depend on h). Hence Ψ satisfies (provably in PRA^ω_i) the monotone functional interpretation of the negative translation of (1), i.e. $(PCM1)$.

5 The rate of growth caused by sequences of instances of analytical principles whose proofs rely on arithmetical comprehension

In this section we apply the results presented in section 2 in order to determine the impact on the rate of growth of uniform bounds for provably $\forall u^1 \forall v \leq_\tau tu \exists w^\gamma A_0$–sentences which may result from the use of sequences (which however may depend on the parameters of the proposition to be proved) of instances of:

1) $(PCM2)$ and the convergence of bounded monotone sequences of real numbers.
2) The existence of a greatest lower bound for every sequence of real numbers which is bounded from below.
3) Π^0_1–CA and Π^0_1–AC.
4) The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$ (for every fixed d).
5) The Arzelà–Ascoli lemma.
6) The existence of lim sup and lim inf for bounded sequences in $\mathbb{R}$.

5.1 $(PCM2)$ and the convergence of bounded monotone sequences of real numbers

Let $a^{1(0)}$ be such that $\forall n^0(0 \leq_{\mathbb{R}} a(n+1) \leq_{\mathbb{R}} an)$[9]
$(PCM2)$ implies

$$\exists h^1 \forall k^0, m^0 (m \geq_0 hk \to a(hk) -_{\mathbb{R}} a(m) \leq_{\mathbb{R}} \frac{1}{k+1}).$$

$\big(a(hk)\big)_k$ is a Cauchy sequence with modulus $\frac{1}{k+1}$ whose limit equals the limit of $(a(m))_{n\in\mathbb{N}}$. The existence of a limit a_0 of $(a(m))_m$ now follows from the remarks below lemma 3.4 : $a_0k := (a(h(\widehat{3(k+1)})))(3(k+1))$. Thus we only have to consider $(PCM2)$. In order to simplify the logical form of $(PCM2)$ we use the construction $\tilde{a}(n) := \max_{\mathbb{R}}(0, \min_{i\leq n}(a(i))$ from the previous section (recall that this construction ensures that $\tilde{a}$ is monotone decreasing and bounded from below by 0. If a already fulfils these properties nothing is changed by the passage from a to $\tilde{a}$).

$$(PCM2)(a^{1(0)}) :\equiv \exists h^1 \forall k^0, m^0 \big(m \geq_0 hk \to \tilde{a}(hk) -_{\mathbb{R}} \tilde{a}(m) \leq_{\mathbb{R}} \frac{1}{k+1}).$$

We now show that the contribution of single instances $(PCM2)(a)$ of $(PCM2)$ to the growth of uniform bounds is (at most) given by the functional $\Psi akg := \max\limits_{i<C(a)k'} \big(\Phi_{it} i0g\big)$ (where $\mathbb{N} \ni C(a) \geq \tilde{a}(0))$ as above:

Proposition 5.1.1 *Let $n \geq 2$ and $B_0(u^1, v^\tau, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula which contains only u^1, v^τ, w^γ free, where $\gamma \leq 2$. Furthermore let $\xi, t \in G_nR^\omega$ and Δ be as in thm.2.4. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \forall u^1 \forall v \leq_\tau tu\big((PCM2)(\xi uv) \to \exists w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.)\chi, \tilde{\chi} \in G_nR^\omega \text{ such that} \\ G_nA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \forall \tilde{\Psi}^* \big((\tilde{\Psi}^* \text{ satifies the mon.funct.interpr. of} \\ \qquad \forall k^0, g^1 \exists n^0 (gn > n \to (\widetilde{\xi uv})(n) -_{\mathbb{R}} (\widetilde{\xi uv})(gn) \leq_{\mathbb{R}} \frac{1}{k+1})\big) \to \exists w \leq_\gamma \tilde{\chi} u \tilde{\Psi}^* B_0(u,v,w)\big) \\ \text{and} \\ G_nA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \forall \Psi^* \big((\Psi^* \text{ satifies the mon. funct.interpr. of} \\ \qquad \forall a^{1(0)}, k^0, g^1 \exists n^0 (gn > n \to \tilde{a}(n) -_{\mathbb{R}} \tilde{a}(gn) \leq_{\mathbb{R}} \frac{1}{k+1})) \to \exists w \leq_\gamma \chi u \Psi^* \, B_0(u,v,w)\big) \\ \text{and therefore} \\ PRA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \chi u \Psi \, B_0(u,v,w), \end{cases}$$

where $\Psi := \lambda a, k, g. \max\limits_{i<C(a)k'} \big(\Phi_{it} i0g\big) = \max\limits_{i<C(a)k'} \big(g^{(i)}(0)\big)$ *and* $C(a) := (a(0))(0) + 1$.

In the conclusion, $\Delta +$ b-AC can be replaced by $\tilde{\Delta}$, where $\tilde{\Delta}$ is defined as in theorem 2.4. If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\exists$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta + AC$–$qf$ may be replaced by E–$G_nA^\omega + \Delta + AC^{\alpha,\beta}$–$qf$, where α, β are as in cor.2.6.

[9]The restriction to the lower bound 0 is (convenient but) not essential: If $\forall n^0(c \leq_{\mathbb{R}} a(n+1) \leq_{\mathbb{R}} an)$ we may define $a'(n) := a(n) -_{\mathbb{R}} c$. $(PCM2)$ applied to a' implies $(PCM2)$ for a. Everything holds analogously for increasing sequences which are bounded from above.

Proof: The existence of $\tilde{\chi}$ follows from cor.2.6 since

$$\mathrm{G}_2\mathrm{A}^\omega \vdash \forall a^{1(0)}\forall k,\tilde{k},n,\tilde{n}\big(\tilde{k}\le_0 k\wedge \tilde{n}\ge_0 n\wedge \forall m\ge_0 n(\tilde{a}(n)-_{\mathbb{R}}\tilde{a}(m)\le_{\mathbb{R}}\tfrac{1}{k+1}) \\ \rightarrow \forall m\ge_0 \tilde{n}(\tilde{a}(\tilde{n})-_{\mathbb{R}}\tilde{a}(m)\le_{\mathbb{R}}\tfrac{1}{\tilde{k}+1})\big).$$

Ψ fulfils the monotone functional interpretation of
$\forall a^{1(0)},k^0,g^1\exists n^0(gn>n\rightarrow \tilde{a}(n)-_{\mathbb{R}}\tilde{a}(gn)\le_{\mathbb{R}}\frac{1}{k+1})$ (see the end of section 4) and hence (using lemma 2.2.11 from [15]) $\Psi(\xi^*(u^M,t^*u^M))$ satisfies the monotone functional interpretation of

$$\forall k^0,g^1\exists n^0(gn>n\rightarrow (\widetilde{\xi uv})(n)-_{\mathbb{R}}(\widetilde{\xi uv})(gn)\le_{\mathbb{R}}\frac{1}{k+1}),\ \text{where } \xi^* \text{ s–maj } \xi\wedge t^* \text{ s–maj } t.$$

χ is defined by $\chi:=\lambda u,\Psi^*.\tilde{\chi}u\big(\Psi^*(\xi^*(u^M,t^*u^M))\big)$.

Remark 5.1.2 *1) The computation of the bound $\tilde{\chi}$ in the proposition above needs only a functional $\tilde{\Psi}^*$ which satifies the monotone functional interpretation of*

$$(+)\ \forall k^0,g^1\exists n^0(gn>n\rightarrow (\widetilde{\xi uv})(n)-_{\mathbb{R}}(\widetilde{\xi uv})(gn)\le_{\mathbb{R}}\frac{1}{k+1}).$$

For special ξ such a functional may be constructable without the use of Φ_{it}. Furthermore for fixed u the number of iterations of g only depends on the k–instances of $(+)$ which are used in the proof.

2) If the given proof of the assumption of this proposition applies Ψ only to functions g of low growth, then also the bound $\chi u\Psi$ is of low growth: e.g. if only $g:=S$ is used and type/w $=0$, then $\chi u\Psi$ is a polynomial in u^M (in the sense of [15]).

Corollary to the proof of prop.5.1.1:
The rule

$$\begin{cases} \mathrm{G}_n\mathrm{A}^\omega+\Delta+\text{AC–qf}\ \vdash \forall u^1\forall v\le_\tau tu\big(\exists f^0\forall k\forall m,\tilde{m}>fk(|(\xi uv)(\tilde{m})-_{\mathbb{R}}(\xi uv)(m)|\le\tfrac{1}{k+1}) \\ \qquad\qquad\qquad\qquad \rightarrow \exists w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \\ \mathrm{G}_n\mathrm{A}^\omega+\tilde{\Delta}\vdash \forall u^1\forall v\le_\tau tu\big(\forall k\exists n\forall m,\tilde{m}>n(|(\xi uv)(\tilde{m})-_{\mathbb{R}}(\xi uv)(m)|\le\tfrac{1}{k+1}) \\ \qquad\qquad\qquad\qquad \rightarrow \exists w^\gamma B_0(u,v,w)\big)\end{cases}$$

holds for arbitrary sequences $(\xi uv)^{1(0)}$ of real numbers (this also extends to more general monotone formulas $\forall u^1\forall v\le_\tau tuB(u,v)$ in the sense of thm.2.4). The restriction to bounded monotone sequences $\widetilde{\xi uv}$ is used only to ensure the existence of a functional Ψ which satisfies the monotone functional interpretation of $(+)$ above.

We now consider a generalization $(PCM2^*)(a_{(\cdot)}^{1(0)(0)})$ of $(PCM2)(a^{1(0)})$ which asserts the existence of a sequence of Cauchy moduli for a sequence $\widetilde{a_l}$ of bounded monotone sequences:

$$(PCM2^*)(a_{(\cdot)}^{1(0)(0)}):\equiv \exists h^{1(0)}\forall l^0,k^0\forall m\ge_0 hkl\big((\widetilde{a_l})(hkl)-_{\mathbb{R}}(\widetilde{a_l})(m)\le_{\mathbb{R}}\frac{1}{k+1}\big).$$

Proposition 5.1.3 *Let $n, B_0(u,v,w), t, \Delta$ be as in prop.5.1.1. $t, \xi \in G_nR^\omega$. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \forall u^1 \forall v \leq_\tau tu\big((PCM2^*)(\xi uv) \rightarrow \exists w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.)\chi \in G_nR^\omega \text{ such that} \\ G_nA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \forall \Psi^*\big((\Psi^* \text{ satifies the mon. funct.interpr. of} \\ \forall a^{1(0)(0)}, k^0, g^1 \exists n^0 (gn > n \rightarrow \forall l \leq k(\widetilde{(a_l)}(n) -_{\mathrm{IR}} \widetilde{(a_l)}(gn) \leq_{\mathrm{IR}} \frac{1}{k+1}))) \rightarrow \exists w \leq_\gamma \chi u \Psi^* B_0(u,v,w)\big) \\ \text{and in particular} \\ PRA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \chi u \Psi' B_0(u,v,w), \end{cases}$$

where $\Psi' := \lambda a, k, g. \max\limits_{i < C(a,k)(k+1)^2} \big(\Phi_{it} i0g\big)$ *and* $\mathrm{IN} \ni C(a,k) \geq \max_{\mathrm{IR}}(\widetilde{(a_0)}(0), \ldots, \widetilde{(a_k)}(0))$.

In the conclusion, $\Delta +$ b-AC can be replaced by $\tilde{\Delta}$, where $\tilde{\Delta}$ is defined as in theorem 2.4. If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\exists$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta + AC$–$qf$ may be replaced by E–$G_nA^\omega + \Delta + AC^{\alpha,\beta}$–$qf$, where α, β are as in cor.2.6.
As in prop.5.1.1 we also have a term $\tilde{\chi}$ which needs only a $\tilde{\Psi}^*$ for the instance $a := \xi uv$.

Proof: The first part of the proposition follows from cor.2.6 since $(PCM2^*)(a)$ is implied by

$$\exists h^1 \forall k^0 \forall m \geq_0 hk \forall l \leq_0 k\big(\widetilde{(a_l)}(hk) -_{\mathrm{IR}} \widetilde{(a_l)}(m) \leq_{\mathrm{IR}} \frac{1}{k+1}\big)$$

and

$$\mathrm{G_2A}^\omega \vdash \forall a_{(\cdot)}^{1(0)(0)} \forall k, \tilde{k}, n, \tilde{n}\big(\tilde{k} \leq_0 k \wedge \tilde{n} \geq_0 n \wedge \forall m \geq_0 n \forall l \leq_0 k(\widetilde{(a_l)}(n) -_{\mathrm{IR}} \widetilde{(a_l)}(m) \leq_{\mathrm{IR}} \tfrac{1}{k+1}) \\ \rightarrow \forall m \geq_0 \tilde{n} \forall l \leq_0 \tilde{k}(\widetilde{(a_l)}(\tilde{n}) -_{\mathrm{IR}} \widetilde{(a_l)}(m) \leq_{\mathrm{IR}} \tfrac{1}{\tilde{k}+1})\big).$$

It remains to show that Ψ' satisfies the monotone functional interpretation of
$\forall a^{1(0)(0)}, k^0, g^1 \exists n^0\big(gn > n \rightarrow \forall l \leq k\big(\widetilde{(a_l)}(n) - \widetilde{(a_l)}(gn) \leq \frac{1}{k+1}\big)\big)$:
Assume

$$\forall i < C(a,k)(k+1)^2\big(g(g^i0) > g^i0 \wedge \exists l \leq k\big(\widetilde{(a_l)}(g^i0) - \widetilde{(a_l)}(g(g^i0)) > \frac{1}{k+1}\big)\big).$$

Then

$$\forall i < C(a,k)(k+1)^2\big(g(g^i0) > g^i0\big) \text{ and} \\ \exists l \leq k \exists j\big(\forall i < C(a,k)(k+1) \dot{-} 1\big((j)_i < (j)_{i+1} < C(a,k)(k+1)^2\big) \wedge \\ \forall i < C(a,k)(k+1)\big(\widetilde{(a_l)}(g^{(j)_i}0) - \widetilde{(a_l)}(g(g^{(j)_i}0)) > \tfrac{1}{k+1}\big)\big)$$

and therefore

$$\exists l \leq k \exists j\Big(\forall i < C(a,k)(k+1) \dot{-} 1\big(g^{(j)_{i+1}}0 > g^{(j)_i}0 \wedge \widetilde{(a_l)}(g^{(j)_i}0) - \widetilde{(a_l)}(g^{(j)_{i+1}}0) > \tfrac{1}{k+1}\big) \\ \wedge g(g^{(j)_{C(a,k)(k+1) \dot{-} 1}}(0)) > g^{(j)_{C(a,k)(k+1) \dot{-} 1}}(0) \\ \wedge \widetilde{(a_l)}(g^{(j)_{C(a,k)(k+1) \dot{-} 1}}(0)) - \widetilde{(a_l)}(g(g^{(j)_{C(a,k)(k+1) \dot{-} 1}}(0))) > \tfrac{1}{k+1}\Big).$$

Hence

$$\exists l \le k \exists j \forall i < C(a,k)(k+1)\big(g^{(j)_{i+1}}0 > g^{(j)_i}0 \wedge \widetilde{(a_l)}(g^{(j)_i}0) - \widetilde{(a_l)}(g^{(j)_{i+1}}0) > \frac{1}{k+1}\big),$$

which contradicts $\widetilde{(a_l)} \subset [0, C(a,k)]$.

5.2 The principle (GLB) 'every sequence of real numbers in $\mathbb{R}_+$ has a greatest lower bound'

This principle can be easily reduced to $(PCM2)$ (provably in $\mathrm{G}_2\mathrm{A}^\omega$):
Let $a^{1(0)}$ be such that $\forall n^0(0 \le_{\mathbb{R}} an)$. Then $(PCM2)(a)$ implies that the decreasing sequence $(\tilde{a}(n))_n \subset \mathbb{R}_+$ has a limit $\tilde{a}_0^1$. It is clear that $\tilde{a}_0$ is the greatest lower bound of $(a(n))_n \subset \mathbb{R}_+$. Thus we have shown

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \forall a^{1(0)}\big((PCM2)(a) \to (GLB)(a)\big).$$

By this reduction we may replace $(PCM2)(\xi uv)$ by $(GLB)(\xi uv)$ in the assumption of prop.5.1.1. There is nothing lost (w.r.t to the rate of growth) in this reduction since in the other direction we have

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{0,0}\text{–qf} \vdash \forall a^{1(0)}\big((GLB)(a) \to (PCM2)(a)\big):$$

Let $a^{1(0)}$ be as above and a_0 its greatest lower bound. Then $a_0 = \lim\limits_{n\to\infty} \tilde{a}_n$. Using $\mathrm{AC}^{0,0}$–qf one obtains (see section 4) a modulus of convergence and so a Cauchy modulus for $(\tilde{a}(n))_n$.

5.3 Π^0_1–CA and Π^0_1–AC

Definition 5.3.1 *1)* $\Pi^0_1\text{–}CA(f^{1(0)}) :\equiv \exists g^1 \forall x^0\big(gx =_0 0 \leftrightarrow \forall y^0(fxy =_0 0)\big)$.

2) Define $A_0^C(f^{1(0)}, x^0, y^0, z^0) :\equiv \forall \tilde{x} \le_0 x \exists \tilde{y} \le_0 y \forall \tilde{z} \le_0 z\big(f\tilde{x}\tilde{y} \ne_0 0 \vee f\tilde{x}\tilde{z} =_0 0\big)$.

A_0^C *can be expressed as a quantifier–free formula in* G_nA^ω *(see [15]).*

(Note that iteration of $\forall f^{1(0)}(\Pi^0_1\text{–CA}(f))$ yields CA_{ar}).
In [16] we proved (using cor.2.6)

Proposition 5.3.2 *Let* $n \ge 1$ *and* $B_0(u^1, v^\tau, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ *be a quantifier-free formula which contains only* u^1, v^τ, w^γ *free, where* $\gamma \le 2$. *Furthermore let* $\xi, t \in G_nR^\omega$ *and* Δ *be as in thm.2.4. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{-}qf \vdash \forall u^1 \forall v \le_\tau tu\big(\Pi^0_1\text{-}CA(\xi uv) \to \exists w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.)\chi \in G_nR^\omega \text{ such that} \\ G_nA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \le_\tau tu \forall \Psi^*\big((\Psi^* \text{ satifies the mon. funct.interpr.} \\ \qquad \text{of } \forall x^0, h^1 \exists y^0 A_0^C(\xi uv, x, y, hy)\big) \to \exists w \le_\gamma \chi u \Psi^*\, B_0(u,v,w)\big) \\ \text{and in particular} \\ PRA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \le_\tau tu \exists w \le_\gamma \chi u \Psi\, B_0(u,v,w), \end{cases}$$

where $\Psi := \lambda x^0, h^1. \max\limits_{i<x+1}\big(\Phi_{it} i0h\big)\big(= \lambda x^0, h^1. \max\limits_{i<x+1}(h^i 0)\big)$.

In the conclusion, Δ+b-AC can be replaced by $\tilde{\Delta}$, where $\tilde{\Delta}$ is defined as in thm.2.4. If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\exists$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta$+AC–qf may be replaced by E-$G_nA^\omega + \Delta$+$AC^{\alpha,\beta}$–qf, where α, β are as in cor.2.6.

A similar result holds for Π^0_1–AC(ξuv), where

$$\Pi^0_1\text{–AC}(f^{1(0)(0)(0)}) :\equiv \forall l^0 \big(\forall x^0 \exists y^0 \forall z^0 (flxyz =_0 0) \to \exists g^1 \forall x^0, z^0 (flx(gx)z =_0 0)\big).$$

5.4 The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$ (for every fixed d)

We now consider the Bolzano–Weierstraß principle for sequences in $[-1,1]^d \subset \mathbb{R}^d$. The restriction to the special bound 1 is convenient but not essential: If $(x_n) \subset \mathbb{R}^d$ is bounded by $C > 0$, we define $x'_n := \frac{1}{C} \cdot x_n$ and apply the Bolzano–Weierstraß principle to this sequence. For simplicity we formulate the Bolzano– Weierstraß principle w.r.t. the maximum norm $\| \cdot \|_{\max}$. This of course implies the principle for the Euclidean norm $\| \cdot \|_E$ since $\| \cdot \|_E \leq \sqrt{d} \cdot \| \cdot \|_{\max}$.
We start with the investigation of the following formulation of the Bolzano–Weierstraß principle:

$$BW \ : \ \forall (x_n) \subset [-1,1]^d \exists x \in [-1,1]^d \forall k^0, m^0 \exists n >_0 m \big(\|x - x_n\|_{\max} \leq \frac{1}{k+1}\big),$$

i.e. (x_n) possesses a limit point x.
Later on we discuss a second formulation which (relatively to $\mathrm{G}_n\mathrm{A}^\omega$) is slightly stronger than BW:

$$BW^+ \ : \ \begin{cases} \forall (x_n) \subset [-1,1]^d \exists x \in [-1,1]^d \exists f^1 \big(\forall n^0 (fn <_0 f(n+1)) \\ \qquad\qquad\qquad\qquad\qquad \wedge \forall k^0 \big(\|x - x_{fk}\|_{\max} \leq \frac{1}{k+1}\big)\big), \end{cases}$$

i.e. (x_n) has a subsequence (x_{fn}) which converges (to x) with the modulus $\frac{1}{k+1}$.
Using our representation of $[-1,1]$ from section 3, the principle BW has the following form

$$\forall x_1^{1(0)}, \ldots, x_d^{1(0)} \underbrace{\exists a_1, \ldots, a_d \leq_1 M \forall k^0, m^0 \exists n >_0 m \bigwedge_{i=1}^{d} \big(|\tilde{a}_i -_{\mathbb{R}} \widetilde{x_i n}| \leq_{\mathbb{R}} \frac{1}{k+1} \big)}_{BW(\underline{x}^{1(0)}) :\equiv},$$

where M and $y^1 \mapsto \tilde{y}$ are the constructions from our representation of $[-1,1]$ in section 3. We now prove

$$(*) \ \mathrm{G}_2\mathrm{A}^\omega + \mathrm{AC}^{1,0}\text{–qf} \ \vdash F^- \to \forall x_1^{1(0)}, \ldots, x_d^{1(0)} \big(\Pi^0_1\text{–CA}(\chi \underline{x}) \to BW(\underline{x})\big),$$

for a suitable $\chi \in \mathrm{G}_2\mathrm{R}^\omega$:
$BW(\underline{x})$ is equivalent to

$$(1) \ \exists a_1, \ldots, a_d \leq_1 M \forall k^0 \exists n >_0 k \bigwedge_{i=1}^{d} \big(|\tilde{a}_i -_{\mathbb{R}} \widetilde{x_i n}| \leq_{\mathbb{R}} \frac{1}{k+1} \big)$$

which in turn is equivalent to

$$(2) \ \exists a_1, \ldots, a_d \leq_1 M \forall k^0 \exists n >_0 k \bigwedge_{i=1}^{d} \big(|\tilde{a}_i k -_{\mathbb{Q}} (\widetilde{x_i n})(k)| \leq_{\mathbb{Q}} \frac{3}{k+1} \big).$$

Assume $\neg(2)$, i.e.

$$(3)\ \forall a_1,\ldots,a_d \leq_1 M \exists k^0 \forall n >_0 k \bigvee_{i=1}^{d} \big(|\tilde{a}_i k -_{\mathbb{Q}} (\widetilde{x_i n})(k)| >_{\mathbb{Q}} \frac{3}{k+1}\big).$$

Let $\chi \in G_2R^\omega$ be such that

$$G_2A^\omega \vdash \forall x_1^{1(0)},\ldots,x_d^{1(0)} \forall l^0, n^0 \big(\chi \underline{x} l n =_0 0 \leftrightarrow \\ \big[n >_0 \nu_{d+1}^{d+1}(l) \rightarrow \bigvee_{i=1}^{d} |\nu_i^{d+1}(l) -_{\mathbb{Q}} (\widetilde{x_i n})(\nu_{d+1}^{d+1}(l))| >_{\mathbb{Q}} \tfrac{3}{\nu_{d+1}^{d+1}(l)+1}\big]\big).$$

Π^0_1–CA$(\chi\underline{x})$ yields the existence of a function h such that

$$(4)\ \forall l_1^0,\ldots,l_d^0,k^0 \big(h l_1 \ldots l_d k =_0 0 \leftrightarrow \forall n >_0 k \bigvee_{i=1}^{d} \big(|l_i -_{\mathbb{Q}} (\widetilde{x_i n})(k)| >_{\mathbb{Q}} \frac{3}{k+1}\big).$$

Using h, (3) has the form

$$(5)\ \forall a_1,\ldots,a_d \leq_1 M \exists k^0 \big(h(\tilde{a}_1 k,\ldots,\tilde{a}_d k, k) =_0 0\big).$$

By Σ^0_1–UB$^-$ (which follows from AC^{1,0}–qf and F^- by [15] (prop. 4.20)) we obtain

$$(6)\ \exists k_0 \forall a_1,\ldots,a_d \leq_1 M \forall m^0 \exists k \leq_0 k_0 \forall n >_0 k \bigvee_{i=1}^{d} \big(|(\widetilde{\overline{a_i,m}})(k) -_{\mathbb{Q}} (\widetilde{x_i n})(k)| >_{\mathbb{Q}} \frac{3}{k+1}\big)$$

and therefore

$$(7)\ \exists k_0 \forall a_1,\ldots,a_d \leq_1 M \forall m^0 \forall n >_0 k_0 \bigvee_{i=1}^{d} \big(|(\widetilde{\overline{a_i,m}}) -_{\mathbb{R}} \widetilde{x_i n}| >_{\mathbb{R}} \frac{1}{k_0+1}\big).$$

Since $|\widetilde{\overline{a_i,3(m+1)}} -_{\mathbb{R}} \tilde{a}_i| <_{\mathbb{R}} \frac{2}{m+1}$ (see the definition of $y \mapsto \tilde{y}$ from section 3) it follows

$$(8)\ \exists k_0 \forall a_1,\ldots,a_d \leq_1 M \forall n >_0 k_0 \bigvee_{i=1}^{d} \big(|\tilde{a}_i -_{\mathbb{R}} \widetilde{x_i n}| >_{\mathbb{R}} \frac{1}{2(k_0+1)}\big), \text{ i.e.}$$

$$(9)\ \exists k_0 \forall (a_1,\ldots,a_d) \in [-1,1]^d \forall n >_0 k_0 \big(\|\underline{a} - \underline{x} n\|_{\max} > \frac{1}{2(k_0+1)}\big).$$

By applying this to $\underline{a} := \underline{x}(k_0+1)$ yields the contradiction $\|\underline{x}(k_0+1) - \underline{x}(k_0+1)\|_{\max} > \frac{1}{2(k_0+1)}$, which concludes the proof of $(*)$.

Remark 5.4.1 In the proof of $(*)$ we used a combination of Π^0_1–CA(ξg) and Σ^0_1–UB$^-$ to obtain a restricted form Π^0_1–UB$^-\restriction$ of the extension of Σ^0_1–UB$^-$ to Π^0_1–formulas:

$$\Pi^0_1\text{–UB}^-\restriction\ :\ \begin{cases} \forall f \leq_1 s \exists n^0 \forall k^0 A_0(t^0[f],n,k) \rightarrow \\ \exists n_0 \forall f \leq_1 s \forall m^0 \exists n \leq_0 n_0 \forall k^0 A_0(t[\overline{f,m}],n,k), \end{cases}$$

where k does not occur in $t[f]$ and f does not occur in $A_0(0,0,0)$ and g^1 is the only free variable in $A_0(0,0,0)$.
Π^0_1–UB$^-\restriction$ follows by applying Π^0_1–CA to $\lambda n,k.t_{A_0}(a^0,n^0,k^0)$, where t_{A_0} is such that $t_{A_0}(a^0,n^0,k^0)=_0 0 \leftrightarrow A_0(a^0,n^0,k^0)$, and subsequent application of Σ^0_1–UB$^-$.
Π^0_1–CA and Σ^0_1–UB$^-$ do not imply the unrestricted form Π^0_1–UB$^-$ of Π^0_1–UB$^-\restriction$:

$$\Pi^0_1\text{–UB}^- \quad \begin{cases} \forall f \leq_1 s\exists n^0\forall k^0 A_0(f,n,k) \rightarrow \\ \exists n_0\forall f \leq_1 s\forall m^0\exists n \leq_0 n_0\forall k^0 A_0((\overline{f,m}),n,k) \end{cases}$$

since a reduction of Π^0_1–UB$^-$ to Σ^0_1–UB$^-$ would require a comprehension functional in f:

$$(+)\ \exists\Phi\forall f^1,n^0(\Phi fn =_0 0 \leftrightarrow \forall k^0 A_0(f,n,k)).$$

In fact Π^0_1–UB$^-$ can easily be refuted by applying it to $\forall f \leq_1 \lambda x.1\exists n^0\forall k^0(fk=0\rightarrow fn=0)$, which leads to a contradiction. This reflects the fact that we had to use F^- to derive Σ^0_1–UB$^-$, which is incompatible with (+) since Φ+AC^{1,0}–qf produces (see above) a non–majorizable functional, namely

$$\Psi f^1 := \begin{cases} \min n[fn=0], \text{ if existent} \\ 0^0, \text{ otherwise,} \end{cases}$$

whereas F^- is true only in the model $\mathcal{M}^\omega$ of all strongly majorizable functionals introduced in [2] (see [15] for details).

Next we prove

$$(**)\ \mathrm{G}_2\mathrm{A}^\omega + \mathrm{AC}^{0,0}\text{–qf} \vdash \forall x_1^{1(0)},\ldots,x_d^{1(0)}\big(\Sigma^0_1\text{–IA}(\chi\underline{x}) \wedge BW(\underline{x}) \rightarrow BW^+(\underline{x})\big)$$

for a suitable term $\chi \in \mathrm{G}_2\mathrm{R}^\omega$, where

$$\Sigma^0_1\text{–IA}(f) :\equiv \begin{cases} \forall l^0\big(\exists y^0(fl0y =_0 0) \wedge \forall x^0(\exists y(flxy=0) \rightarrow \exists y(flx'y=0)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \rightarrow \forall x\exists y(flxy=0)\big). \end{cases}$$

$BW(\underline{x})$ implies the existence of $a_1,\ldots,a_d \leq_1 M$ such that

$$(10)\ \forall k,m\exists n>m\bigwedge_{i=1}^{d}\big(|\tilde{a}_i(2(k+1)(k+2)) -_{\mathbb{Q}} (\widetilde{x_in})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \frac{1}{k+1}\big).$$

Define (for $x_1^{1(0)},\ldots,x_d^{1(0)},l_1^0,\ldots,l_d^0$)

$$F(\underline{x},\underline{l},k,m,n) :\equiv$$
$$\big(\underline{x}n \text{ is the } m\text{–th element in } (\underline{x}(l))_l \text{ such that } \bigwedge_{i=1}^{d}\big(|l_i -_{\mathbb{Q}} (\widetilde{x_in})(2(k+1)(k+2))| \leq_{\mathbb{Q}} \tfrac{1}{k+1}\big)\big).$$

One easily verifies that $F(\underline{x},\underline{l},k,m,n)$ can be expressed in the form $\exists a^0 F_0(\underline{x},\underline{l},k,m,n,a)$, where F_0 is a quantifier–free formula in $\mathcal{L}(\mathrm{G}_2\mathrm{A}^\omega)$, which contains only $\underline{x},\underline{l},k,m,n,a$ as free variables. Let $\tilde{\chi} \in \mathrm{G}_2\mathrm{R}^\omega$ such that

$$\tilde{\chi}(\underline{x},\underline{l},k,m,n,a) =_0 0 \leftrightarrow F_0(\underline{x},\underline{l},k,m,n,a)$$

and define $\chi(\underline{x},q,m,p) := \tilde{\chi}(\underline{x},\nu_1^{d+1}(q),\ldots,\nu_{d+1}^{d+1}(q),m,j_1(p),j_2(p))$. Σ_1^0–IA$(\chi\underline{x})$ yields

$$(11)\ \begin{cases} \forall l_1,\ldots,l_d,k\big(\exists n\, F(\underline{x},\underline{l},k,0,n)\wedge\forall m\big(\exists nF(\underline{x},\underline{l},k,m,n)\to\exists nF(\underline{x},\underline{l},k,m',n)\big)\\ \qquad\qquad\qquad\qquad\qquad\to\forall m\exists nF(\underline{x},\underline{l},k,m,n)\big).\end{cases}$$

(10) and (11) imply

$$(12)\ \begin{cases} \forall k,m\exists n\big(\underline{x}n \text{ is the } m\text{–th element of } (\underline{x}(l))_l \text{ such that}\\ \qquad\bigwedge_{i=1}^{d}\big(|\tilde{a}_i(2(k+1)(k+2))-_{\mathbb{Q}}(\widetilde{x_in})(2(k+1)(k+2))|\leq_{\mathbb{Q}}\frac{1}{k+1}\big)\big).\end{cases}$$

and therefore

$$(13)\ \begin{cases} \forall k\exists n\big(\underline{x}n \text{ is the } k\text{–th element of } (\underline{x}(l))_l \text{ such that}\\ \qquad\bigwedge_{i=1}^{d}\big(|\tilde{a}_i(2(k+1)(k+2))-_{\mathbb{Q}}(\widetilde{x_in})(2(k+1)(k+2))|\leq_{\mathbb{Q}}\frac{1}{k+1}\big)\big).\end{cases}$$

By AC^{0,0}–qf we obtain a function g^1 such that

$$(14)\ \begin{cases} \forall k\big(\underline{x}(gk) \text{ is the } k\text{–th element of } (\underline{x}(l))_l \text{ such that}\\ \qquad\bigwedge_{i=1}^{d}\big(|\tilde{a}_i(2(k+1)(k+2))-_{\mathbb{Q}}(\widetilde{x_i(gk)})(2(k+1)(k+2))|\leq_{\mathbb{Q}}\frac{1}{k+1}\big)\big).\end{cases}$$

We show (15) $\forall k(gk<g(k+1))$:

Define $A_0(\underline{x}l,k):\equiv\bigwedge_{i=1}^{d}\big(|\tilde{a}_i(2(k+1)(k+2))-_{\mathbb{Q}}(\widetilde{x_il})(2(k+1)(k+2))|\leq_{\mathbb{Q}}\frac{1}{k+1}\big)$. Let l be such that $A_0(\underline{x}l,k+1)$. Because of

$$\begin{aligned}&|\tilde{a}_i(2(k+1)(k+2))-_{\mathbb{Q}}(\widetilde{x_il})(2(k+1)(k+2))|\leq\\ &|\tilde{a}_i(2(k+2)(k+3))-_{\mathbb{Q}}(\widetilde{x_il})(2(k+2)(k+3))|+\frac{2}{2(k+1)(k+2)}\overset{A_0(\underline{x}l,k+1)}{\leq}\\ &\frac{1}{k+2}+\frac{2}{2(k+1)(k+2)}=\frac{1}{k+1},\end{aligned}$$

this yields $A_0(\underline{x}l,k)$. Thus the $(k+1)$–th element $\underline{x}l$ such that $A_0(\underline{x}l,k+1)$ is at least the $(k+1)$–th element such that $A_0(\underline{x}l,k)$ and therefore occurs later in the sequence than the k-th element such that $A_0(\underline{x}l,k)$, i.e. $gk<g(k+1)$.

It remains to show

$$(16)\ \forall k\bigwedge_{i=1}^{d}\big(|\tilde{a}_i-_{\mathbb{R}}x_i(\widetilde{fk})|\leq_{\mathbb{R}}\frac{1}{k+1}\big),\text{ where } fk:=g(2(k+1)):$$

This follows since $\bigwedge_{i=1}^{d}\big(|\tilde{a}_i(2(k+1)(k+2))-_{\mathbb{Q}}(x_i\widetilde{(gk)})(2(k+1)(k+2))|\leq_{\mathbb{Q}}\frac{1}{k+1}\big)$ implies $\bigwedge_{i=1}^{d}\big(|\tilde{a}_i-_{\mathbb{R}}x_i\widetilde{(gk)}|\leq_{\mathbb{R}}\frac{1}{k+1}+\frac{2}{2(k+1)(k+2)+1}\leq\frac{2}{k+1}\big)$.

(15) and (16) imply $BW^+(\underline{x})$ which concludes the proof of $(**)$.

Remark 5.4.2 One might ask why we did not use the following obvious proof of $BW^+(\underline{x})$ from $BW(\underline{x})$: Let $\underline{a}$ be such that $\forall k \exists n > k \bigwedge_{i=1}^{d} \big(|\tilde{a}_i -_{\mathbb{R}} \widetilde{x_i n}| <_{\mathbb{R}} \frac{1}{k+1}\big)$. $\mathrm{AC}^{0,0}$–qf yields the existence of a function g such that $\forall k(gk > k \wedge \bigwedge_{i=1}^{d} (|\tilde{a}_i -_{\mathbb{R}} \widetilde{x_i(gk)}| <_{\mathbb{R}} \frac{1}{k+1}))$. Now define $fk := g^{(k+1)}(0)$. It is clear that f fulfils $BW^+(\underline{x})$.
The problem with this proof is that we cannot use our results from section 2 in the presence of the iteration functional Φ_{it} (see [16] for more information in this point) which is needed to define f as a functional in g. To introduce the graph of Φ_{it} by Σ^0_1–IA and AC–qf does not help since this would require an application of Σ^0_1–IA which involves (besides $\underline{x}$) also g as a genuine function parameter. In contrast to this situation, our proof of $BW(\underline{x}) \to BW^+(\underline{x})$ uses Σ^0_1–IA only for a formula with (besides $\underline{x}$) only $k, \underline{a}k$ as parameters. Since k (as a parameter) remains fixed throughout the induction, $\underline{a}$ only occurs as the **number parameter $\underline{a}k$** but **not as genuine function parameter**. This is the reason why we are able to construct a term χ such that $\Sigma^0_1\text{–IA}(\chi\underline{x}) \wedge BW(\underline{x}) \to BW^+(\underline{x})$.

Using $(*)$ and $(**)$ we are now able to prove

Proposition 5.4.3 *Let $n \geq 2$ and $B_0(u^1, v^\tau, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula which contains only u^1, v^τ, w^γ free, where $\gamma \leq 2$. Furthermore let $\underline{\xi}, t \in G_nR^\omega$ and Δ be as in thm.2.4. Then for a suitable $\xi' \in G_nR^\omega$ the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \forall u^1 \forall v \leq_\tau tu\big(BW^+(\underline{\xi}uv) \to \exists w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.)\chi \in G_nR^\omega \text{ such that} \\ G_{\max(n,3)}A^\omega_i + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \forall \Psi^* \big((\Psi^* \text{ satifies the mon. funct.interpr. of} \\ \qquad \forall x^0, h^1 \exists y^0 A^C_0(\xi' uv, x, y, hy)) \to \exists w \leq_\gamma \chi u \Psi^*\, B_0(u,v,w)\big) \\ \text{and in particular} \\ PRA^\omega_i + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \chi u \Psi\, B_0(u,v,w), \end{cases}$$

where $\Psi := \lambda x^0, h^1. \max_{i<x+1} \big(\Phi_{it} i 0 h\big) \big(= \lambda x^0, h^1. \max_{i<x+1} (h^i 0)\big)$.
In the conclusion, Δ+b-AC can be replaced by $\tilde{\Delta}$, where $\tilde{\Delta}$ is defined as in thm.2.4. If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\exists$–quantifiers in Δ are ≤ 1, then G_nA^ω + Δ+AC–qf may be replaced by E–G_nA^ω + Δ+$AC^{\alpha,\beta}$–qf, where α, β are as in cor.2.6.
This results also holds (for a suitable ξ'' instead of ξ') if instead of the single instance $BW^+(\underline{\xi}uv)$, a sequence $\forall l^0 BW^+(\underline{\xi}uvl)$ of instances is used in the proof.

Proof: By $(*)$,$(**)$ and the proof of prop.3.11 from [16] there are functionals $\varphi_1, \varphi_2 \in \mathrm{G_2R}^\omega$ such that

$$\mathrm{G_2A}^\omega + \mathrm{AC}^{1,0}\text{–qf} \vdash F^- \to \forall \underline{x}\big(\Pi^0_1\text{–CA}(\varphi_1\underline{x}) \wedge \Pi^0_1\text{–CA}(\varphi_2\underline{x}) \to BW^+(\underline{x})\big).$$

Furthermore $\mathrm{G_2A}^\omega \vdash \Pi^0_1\text{–CA}(\psi f_1 f_2) \to \Pi^0_1\text{–CA}(f_1) \wedge \Pi^0_1\text{–CA}(f_2)$, where

$$\psi f_1 f_2 x^0 y^0 =_0 \begin{cases} f_1(j_2 x, y), \text{ if } j_1 x = 0 \\ f_2(j_2 x, y), \text{ otherwise.} \end{cases}$$

Hence $G_2A^\omega + AC^{1,0}\text{–qf} \vdash F^- \to \forall \underline{x}\big(\Pi^0_1\text{–CA}(\varphi_3\underline{x}) \to BW^+(\underline{x})\big)$, for a suitable $\varphi_3 \in G_2R^\omega$ and thus

$$G_nA^\omega + \Delta + AC\text{–qf} \vdash F^- \to \forall u^1 \forall v \leq_\tau tu\big(\Pi^0_1\text{–CA}(\varphi_3(\underline{\xi}uv)) \to \exists w\, B_0\big).$$

By the proof of theorem 4.21 from [15] we obtain

$$G_nA^\omega + \widetilde{\Delta} + (*) + \ AC\text{–qf} \vdash \forall u^1 \forall v \leq_\tau tu\big(\Pi^0_1\text{–CA}(\varphi_3(\underline{\xi}uv)) \to \exists w\, B_0\big),$$

where $\widetilde{\Delta} := \{\exists Y \leq_{\rho\delta} s \forall x^\delta, z^\eta A_0(x, Yx, z) : \forall x \exists y \leq sx \forall z^\eta A_0 \in \Delta\}$,

$$(*) :\equiv \forall n_0 \exists Y \leq \lambda \Phi^{2(0)}, y^{1(0)}.y \forall \Phi, \tilde{y}^{1(0)}, k^0, \tilde{z}^1 \forall n \leq_0 n_0\big(\bigwedge_{i<n}(\tilde{z}i \leq \tilde{y}ki) \to \Phi k(\overline{\tilde{z}, n}) \leq \Phi k(Y\Phi\tilde{y}k)\big).$$

Prop.5.3.2 (with $\Delta' := \widetilde{\Delta} \cup \{(*)\}$) yields the conclusion of our proposition in $G_nA^\omega_i + \Delta + (*) +$ b–AC and so (since, again by the proof of theorem 4.21 from [15], $G_3A^\omega_i \vdash (*)$ and even $G_3A^\omega_i \vdash (\tilde{*})$) in $G_{\max(3,n)}A^\omega_i + \Delta$+b–AC.
This proof also extends to sequences $\forall l^0 BW^+(\underline{\xi}uvl)$ of instances of BW^+ since by the reasoning above such a sequence reduces to a suitable sequence $\forall l^0 \Pi^0_1\text{–CA}(\varphi uvl)$ of instances of Π^0_1–CA which can be reduced in turn to a single instance using coding (see [16] for this).

5.5 The Arzelà–Ascoli lemma

Under the name 'Arzelà–Ascoli lemma' we understand (as in the literature on 'reverse mathematics') the following proposition:
Let $(f_l) \subset C[0,1]$ be a sequence of functions[10] which are equicontinuous and have a common bound, i.e. there exists a common modulus of uniform continuity ω for all f_l and a bound $C \in \mathbb{N}$ such that $\|f_l\|_\infty \leq C$. Then

(i) (f_l) possesses a limit point w.r.t. $\|\cdot\|_\infty$ which also has the modulus ω, i.e.

$$\exists f \in C[0,1]\big(\forall k^0 \forall m \exists n >_0 m\big(\|f - f_n\|_\infty \leq \frac{1}{k+1}\big) \wedge f \text{ has modulus } \omega\big);$$

(ii) there is a subsequence (f_{gl}) of (f_l) which converges with modulus $\frac{1}{k+1}$.

As in the case of the Bolzano–Weierstraß principle we deal first with (i). The sligthly stronger assertion (ii) can then be obtained from (i) using Σ^0_1–IA(f) and $AC^{0,0}$–qf analogously to our proof of $BW^+(\underline{x})$ from $BW(\underline{x})$. For notational simplicity we may assume that $C = 1$. When formalized in G_nA^ω, the version (i) of the Arzelà–Ascoli lemma has the form[11]

$$\text{A–A}(f^{1(0)(0)}_{(\cdot)}, \omega^1) :\equiv \Big(f_{(\cdot)} \leq_{1(0)(0)} \lambda l^0, n^0.M \wedge$$
$$\forall l^0, m^0, u^0, v^0\big(\overbrace{|qu -_{\mathbb{Q}} qv| \leq_{\mathbb{Q}} \frac{1}{\omega(m)+1} \to |\widetilde{f_l u} -_{\mathbb{R}} \widetilde{f_l v}| \leq_{\mathbb{R}} \frac{1}{m+1}}^{\Pi^0_1 \ni F(f_l,m,u,v) :\equiv \forall a^0 F_0(f_l,m,u,v,a) :\equiv}\big)$$
$$\to \exists g \leq_{1(0)} \lambda n.M\big(\forall m, u, v F(g, m, u, v) \wedge \forall k \exists n >_0 k(\|\lambda x^1.g(x)_{\mathbb{R}} - \lambda x^1.f_n(x)_{\mathbb{R}}\|_\infty \leq \tfrac{1}{k+1})\big)\Big).$$

[10] The restriction to the unit interval $[0,1]$ is convenient for the following proofs but not essential.
[11] $g(x)_{\mathbb{R}}$ denotes the continuation of $g : [0,1] \cap \mathbb{Q} \to \mathbb{R}$ to $[0,1]$ which is definable in g and its modulus ω.

Here M, q and $y^1 \mapsto \tilde{y}$ are the constructions from our representation of $[0,1],[-1,1]$ in section 3. For notational simplicity we omit in the following $\tilde{()}$.
A–A(f,ω) is equivalent to[12]

$$f_{(\cdot)} \leq l^0, n^0.M \wedge \forall l^0, m^0, u^0, v^0 F(f_l, m, u, v) \to \exists g \leq_{1(0)} \lambda n.M \big(\forall m, u, v F(g, m, u, v) \wedge$$
$$\forall k \exists n >_0 k \bigwedge_{i=0}^{\omega(k)+1} \big(|g(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k)| \leq_{\mathbb{Q}} \tfrac{5}{k+1}\big)\big).$$

Assume $\neg$A–A(f,ω), i.e. $f_{(\cdot)} \leq \lambda l^0, n^0 M \wedge \forall l, m, u, v F(f_l, m, u, v)$ and

$$(1) \begin{cases} \forall g \leq_{1(0)} \lambda n.M \Big(\forall m, u, v\, F(g, m, u, v) \to \\ \exists k \forall n \big(n >_0 k \to \bigvee_{i=0}^{\omega(k)+1} \big(|g(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \tfrac{5}{k+1}\big)\big)\Big). \end{cases}$$

Let α be such that

$$\forall l, k, n \big(\alpha(l^0, k^0, n^0) =_0 0 \leftrightarrow \big[n > k \to \bigvee_{i=0}^{\omega(k)+1} \big(|(l)_i -_{\mathbb{Q}} f_n(\frac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \frac{5}{k+1}\big)\big]\big).$$

Π^0_1–CA(α') (where $\alpha' in := \alpha(j_1 i, j_2 i, n)$) yields the existence of a function h such that

$$\forall l, k \big(hlk =_0 0 \leftrightarrow \forall n \big(\alpha(l, k, n) = 0\big)\big).$$

Hence

$$(2) \begin{cases} \forall g, k \Big(h(\overline{\lambda i.g(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k)}(\omega(k)+2), k) =_0 0 \leftrightarrow \\ \forall n >_0 k \bigvee_{i=0}^{\omega(k)+1} \big(|g(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \tfrac{5}{k+1}\big)\Big). \end{cases}$$

(1),(2) and Σ^0_1–UB$^-$ yield (using the fact that g can be coded into a type–1–object by $g'x^0 := g(j_1 x, j_2 x)$)

$$(3) \begin{cases} \exists k_0 \forall g' \leq_1 \lambda x.M(j_2 x) \forall l^0 \Big(\forall m, u, v, a \leq k_0 F_0(\lambda x, y.(\overline{g', l})(j(x,y)), m, u, v, a) \to \\ \exists k \leq k_0 \forall n > k_0 \bigvee_{i=0}^{\omega(k)+1} \big(|(\lambda x, y.(\overline{g', l})(j(x,y)))(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \tfrac{5}{k+1}\big)\Big), \end{cases}$$

and therefore using

$$g_l mn := \begin{cases} gmn, \text{ if } m, n \leq l \\ 0^0, \text{ otherwise}, \quad \text{and } g_l =_{1(0)} \lambda x, y.(\overline{(g_l)', r})(j(x,y)) \text{ for } r > j(l,l) \end{cases}$$

$$(4) \begin{cases} \exists k_0 \forall g \leq_{1(0)} \lambda n.M \forall l^0 \Big(\forall m, u, v, a \leq k_0 F_0(g_l, m, u, v, a) \to \\ \exists k \leq k_0 \forall n > k_0 \bigvee_{i=0}^{\omega(k)+1} \big(|g_l(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k) -_{\mathbb{Q}} f_n(\tfrac{i}{\omega(k)+1})_{\mathbb{R}}(k)| >_{\mathbb{Q}} \tfrac{5}{k+1}\big)\Big), \end{cases}$$

[12]For better readability we write $\frac{i}{\omega(k)+1}$ instead of its code.

By putting $g := f_{k_0+1}$ and $l^0 := 3(c+1)$, where c is the maximum of k_0+1 and the codes of all $\frac{i}{\omega(k)+1}$ for $i \le \omega(k)+1$ and $k \le k_0$, (4) yields the contradiction

$$\exists k \le k_0 \bigvee_{i=0}^{\omega(k)+1} \big(|f_{k_0+1}(\frac{i}{\omega(k)+1})(k) -_{\mathbb{Q}} f_{k_0+1}(\frac{i}{\omega(k)+1})(k)| >_{\mathbb{Q}} \frac{5}{k+1}\big).$$

α' can be defined as a functional ξ in $f_{(\cdot)}, \omega$, where $\xi \in \mathrm{G}_2\mathrm{R}^\omega$. Since the proof above can be carried out in $\mathrm{G}_3\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$–qf[13] (under the assumption of F^- and Π^0_1–CA$(\xi(f,\omega))$ using prop. 4.20 from [15]) we have shown that

$$\mathrm{G}_3\mathrm{A}^\omega + \mathrm{AC}^{1,0}\text{–qf} \vdash F^- \to \forall f^{1(0)(0)}, \omega^1 \big(\Pi^0_1\text{–CA}(\xi(f,\omega)) \to \text{A–A}(f,\omega)\big).$$

Analogously to BW^+ one defines a formalization A–A$^+(f,\omega)$ of the version (ii) of the Arzelà–Ascoli lemma. Similarly to the proof of $BW(\underline{x}) \to BW^+(\underline{x})$ one shows (using Σ^0_1–IA$(\chi(f,\omega))$ for a suitable $\chi \in \mathrm{G}_2\mathrm{R}^\omega$ and $\mathrm{AC}^{0,0}$–qf) that A–A$(f,\omega) \to$ A–A$^+(f,\omega)$. Aanalogously to prop.5.4.3 one so obtains

Proposition 5.5.1 *For $n \ge 3$ proposition 5.4.3 holds with $BW^+(\xi uv)$ (resp. $\forall l^0 BW^+(\xi uvl)$) replaced by A–A$(\xi uv)$ or A–A$^+(\xi uv)$ (resp. $\forall l^0$A–A(ξuvl) or $\forall l^0$A–A$^+(\xi uvl)$).*

5.6 The existence of lim sup and lim inf for bounded sequences in $\mathbb{R}$

Definition 5.6.1 *$a \in \mathbb{R}$ is the* lim sup *of $(x_n) \subset \mathbb{R}$ iff*

$$(*)\ \forall k^0 \big(\forall m \exists n >_0 m(|a - x_n| \le \frac{1}{k+1}) \wedge \exists l \forall j >_0 l(x_j \le a + \frac{1}{k+1})\big).$$

Remark 5.6.2 *This definition of* lim sup *is equivalent to the following one:*
$(**)$ a is the greatest limit point of (x_n).
The implication $(*) \to (**)$ is trivial and can be proved e.g. in G_2A^ω. The implication $(**) \to (*)$ uses the Bolzano–Weierstraß principle.
In the following we determine the rate of growth caused by the assertion of the existence of lim sup *(for bounded sequences) in the sense of $(*)$ and thus a fortiori in the sense of $(**)$.*

We may restrict ourselves to sequences of rational numbers: Let $x^{1(0)}$ represent a sequence of real numbers with $\forall n(|x_n| \le_{\mathbb{R}} C)$. Then $y_n := \widehat{x_n}(n)$ represents a sequence of rational numbers which is bounded by $C+1$. Let a^1 be the $\limsup$ of (y_n), then a also is the $\limsup$ of x. Hence the existence of $\limsup x_n$ follows from the existence of $\limsup y_n$. Furthermore we may assume that $C = 1$.

The existence of $\limsup$ for a sequence of rational numbers $\in [-1,1]$ is formalized in $\mathrm{G}_n\mathrm{A}^\omega$ (for $n \ge 2$) as follows:

$$\exists \limsup(x^1) :\equiv \exists a^1 \forall k^0 \big(\forall m \exists n >_0 m(|a -_{\mathbb{R}} \breve{x}(n)| \le_{\mathbb{R}} \frac{1}{k+1}) \wedge \exists l \forall j >_0 l(\breve{x}(j) \le_{\mathbb{R}} a + \frac{1}{k+1})\big),$$

where $\breve{x}(n) := \max_{\mathbb{Q}}(-1, \min_{\mathbb{Q}}(xn, 1))$. In the following we use the usual notation $\breve{x}_n$ instead of $\breve{x}(n)$.

[13] We have to work in $\mathrm{G}_3\mathrm{A}^\omega$ instead of $\mathrm{G}_2\mathrm{A}^\omega$ since we have used the functional $\Phi_{\langle\rangle} f x = \overline{f} x$.

We now show that $\exists \lim\sup(x^1)$ can be reduced to a purely **arithmetical** assertion $L(x^1)$ on x^1 in proofs of $\forall u^1 \forall v \leq_\tau tu\exists w^\gamma A_0$–sentences:

$$L(x^1) :\equiv \forall k \exists l >_0 k \forall K \geq_0 l \exists j \forall q,r \geq_0 j \underbrace{\forall m,n(K \geq_0 m,n \geq_0 l \to |x_q^m -_{\mathbb{Q}} x_r^n| \leq_{\mathbb{Q}} \frac{1}{k+1})}_{L_0(x,k,l,K,q,r):\equiv},$$

where $x_q^m := \max_{\mathbb{Q}}(\breve{x}_m, \ldots, \breve{x}_{m+q})$ (Note that L_0 can be expressed as a quantifier–free formula in $\mathrm{G}_n\mathrm{A}^\omega$).

Lemma 5.6.3 *1)* $G_2A^\omega \vdash Mon(\exists k \forall l \exists K \forall j \exists q,r(l > k \to K \geq l \wedge q,r \geq j \wedge \neg L_0)$.

2) $G_2A^\omega \vdash \forall x^1 \big(\exists \lim\sup(x) \to L(x)\big)$.

3) $G_2A^\omega \vdash \forall x^1 \big((L(x)^s \to \exists \lim\sup(x)\big)$.
(The facts 1)–3) combined with the results of section 2 imply that $\exists \lim\sup(\xi uv)$ *can be reduced to* $L(\xi uv)$ *in proofs of sentences* $\forall u^1 \forall v \leq_\tau tu\exists w^\gamma A_0$*, see prop. 5.6.4 below).*

4) $G_3A^\omega + \Sigma_2^0\text{–}IA \vdash \forall x^1 L(x)$.

Proof: **1)** is obvious.
2) By $\exists \lim\sup(x^1)$ there exists an a^1 such that (1) $\forall k^0 \forall m \exists n >_0 m(|a -_{\mathbb{R}} \breve{x}_n| \leq_{\mathbb{R}} \frac{1}{k+1})$ and (2) $\forall k^0 \exists l \forall j >_0 l(\breve{x}_j \leq_{\mathbb{R}} a + \frac{1}{k+1})$. Assume $\neg L(x)$, i.e. there exists a k_0 such that

$$(3)\ \forall l > k_0 \exists K \geq l \forall j \exists q,r \geq j \exists m,n\big(K \geq m,n \geq l \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big).$$

Applying (2) to $2k_0+1$ yields an u_0 such that (4) $\forall j \geq u_0(\breve{x}_j \leq_{\mathbb{R}} a + \frac{1}{2(k_0+1)})$. (3) applied to $l := \max_0(k_0, u_0)+1$ provides a K_0 with

$$(5)\ K_0 \geq u_0 \wedge \forall j \exists q,r \geq j \exists m,n\big(K_0 \geq m,n \geq u_0 \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big).$$

(1) applied to $k := 2k_0+1$ and $m := K_0$ yields a d_0 such that

$$(6)\ d_0 > K_0 \wedge \big(|a - \breve{x}_{d_0}| \leq \frac{1}{2(k_0+1)}\big).$$

By (5) applied to $j := d_0$ we obtain

$$(7) \begin{cases} K_0 \geq u_0 \wedge d_0 > K_0 \wedge \big(|a -_{\mathbb{R}} \breve{x}_{d_0}| \leq \frac{1}{2(k_0+1)}\big)\wedge \\ \exists q,r \geq d_0 \exists m,n\big(K_0 \geq m,n \geq u_0 \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big). \end{cases}$$

Let q,r,m,n be such that

$$(8)\ q,r \geq d_0 \wedge K_0 \geq m,n \geq u_0 \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}.$$

Then $x_q^m \geq \breve{x}_{d_0} \overset{(6)}{\geq} a - \frac{1}{2(k_0+1)}$ since $m \leq K_0 \leq d_0 \leq m+q$. Analogously: $x_r^n \geq a - \frac{1}{2(k_0+1)}$. On the other hand, (4) implies $x_q^m, x_r^n \leq a + \frac{1}{2(k_0+1)}$. Thus $|x_q^m -_{\mathbb{Q}} x_r^n| \leq \frac{1}{k_0+1}$ which contradicts (8).

3) Let f, g be such that L^s is fulfilled, i.e.

$$(*) \begin{cases} \forall k \big(fk > k \wedge \forall K \geq fk \forall q, r \geq gkK \\ \qquad\qquad \forall m, n (K \geq m, n \geq fk \to |x_q^m -_{\mathbb{Q}} x_r^n| \leq_{\mathbb{Q}} \frac{1}{k+1}) \big). \end{cases}$$

We may assume that f, g are monotone for otherwise we could define $f^M k := \max_0(f0, \ldots, fk), g^M kK := \max_0 \{gxy : x \leq_0 k \wedge y \leq_0 K\}$ (f^M, g^M can be defined in $\mathrm{G_1R}^\omega$ using Φ_1 and λ–abstraction). If f, g satisfy $(*)$, then f^M, g^M also satisfy $(*)$.
Define

$$h(k) :=_0 \begin{cases} \min i [f(k) \leq_0 i \leq_0 f(k) + gk(fk) \wedge \breve{x}_i =_{\mathbb{Q}} x_{gk(fk)}^{fk}], \text{ if existent} \\ 0^0, \text{ otherwise.} \end{cases}$$

h can be defined in $\mathrm{G_2A}^\omega$ as a functional in f, g. The case 'otherwise' does not occur since

$$\forall m, q \exists i (m \leq_0 i \leq_0 m + q \wedge \breve{x}_i =_{\mathbb{Q}} \max_{\mathbb{Q}}(\breve{x}_m, \ldots, \breve{x}_{m+q})).$$

By the definition of h we have $(+)$ $\breve{x}_{hk} =_{\mathbb{Q}} x_{gk(fk)}^{fk}$ for all k. Assume that $m \geq k$. By the monotonicity of f, g we obtain $fm \geq_0 fk \wedge gm(fm) \geq_0 gk(fm) \geq_0 gk(fk)$. Hence $(*)$ implies

$$(1)\ |x_{gk(fm)}^{fk} -_{\mathbb{Q}} x_{gm(fm)}^{fm}| \leq \frac{1}{k+1} \text{ and } (2)\ |x_{gk(fk)}^{fk} -_{\mathbb{Q}} x_{gk(fm)}^{fk}| \leq \frac{1}{k+1}$$

and therefore (3) $|x_{gk(fk)}^{fk} -_{\mathbb{Q}} x_{gm(fm)}^{fm}| \leq \frac{2}{k+1}$. Thus for $m, \tilde{m} \geq k$ we obtain

$$(4)\ |x_{gm(fm)}^{fm} -_{\mathbb{Q}} x_{g\tilde{m}(f\tilde{m})}^{f\tilde{m}}| \leq \frac{4}{k+1}.$$

For $\tilde{h}(k) := h(4(k+1))$ this yields (5) $\forall k \forall m, \tilde{m} \geq k \big(\breve{x}_{\tilde{h}m} -_{\mathbb{Q}} \breve{x}_{\tilde{h}\tilde{m}}| \leq \frac{1}{k+1}\big)$. Hence for $a :=_1 \lambda m^0.\breve{x}_{\tilde{h}m}$ we have $\widehat{a} =_1 a$, i.e. a represents the limit of the Cauchy sequence $(\breve{x}_{\tilde{h}m})$.
Since $\tilde{h}(k) = h(4(k+1)) \geq f(4(k+1)) \overset{(*)}{\geq} 4(k+1) > k$, we obtain

$$(6)\ \forall k \big(\tilde{h}(k) > k \wedge |\breve{x}_{\tilde{h}k} -_{\mathbb{R}} a| \leq_{\mathbb{R}} \frac{1}{k+1}\big),$$

i.e. a is a limit point of x. It remains to show that (7) $\forall k \exists l \forall j >_0 l \big(\breve{x}_j \leq_{\mathbb{R}} a + \frac{1}{k+1}\big)$:
Define $c(k) := g(4(k+1), f(4(k+1)))$. Then by $(*)$

$$\forall q, r \geq c(k) \big(|x_q^{f(4(k+1))} -_{\mathbb{Q}} x_r^{f(4(k+1))}| \leq \frac{1}{4(k+1)}\big)$$

and by $(+)$ $a(k) =_{\mathbb{Q}} x_{g(4(k+1), f(4(k+1)))}^{f(4(k+1))}$ and therefore

$$\forall j \geq c(k) \big(|x_j^{f(4(k+1))} -_{\mathbb{Q}} a(k)| \leq \frac{1}{4(k+1)}\big).$$

Hence $\forall j \geq c(k)\big(\breve{x}_{f(4(k+1))+j} \leq_{\mathbb{Q}} a(k) + \frac{1}{4(k+1)}\big)$ which implies

$$\forall j \geq c(k) + f(4(k+1))\big(\breve{x}_j \leq_{\mathbb{R}} a + \frac{1}{4(k+1)} + \frac{1}{k+1}\big).$$

Thus (7) is satisfied by $l := c(2(k+1)) + f(4(2k+1)+1)$.

4) Assume $\neg L(x)$, i.e. there exists a k_0 such that

$$(+)\ \forall \tilde{l} > k_0 \exists K \geq \tilde{l} \forall j \exists q, r \geq j \exists m, n\big(K \geq m, n \geq \tilde{l} \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big).$$

We show (using Σ_1^0–IA on l^0): $(++) :\equiv$

$$\forall l \geq_0 1 \exists i^0 \underbrace{\big(lth(i) = l \wedge \forall j < l \dot{-} 1\big((i)_j < (i)_{j+1}\big) \wedge \forall j, j' \leq l \dot{-} 1(j \neq j' \to |\breve{x}_{(i)_j} -_{\mathbb{Q}} \breve{x}_{(i)_{j'}}| > \frac{1}{k_0+1}\big)}_{A_0(i,l):\equiv}.$$

$l = 1$: Obvious. $l \mapsto l+1$: By the induction hypothesis their exists an i which satisfies $A_0(i,l)$.
Case 1: $\forall j \leq l \dot{-} 1 \exists a \forall b > a\big(|\breve{x}_b -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1}\big)$.
Then by the collection principle for Π_1^0-formulas Π_1^0–CP there exists an a_0 such that

$$\forall j \leq l \dot{-} 1 \forall b > a_0\big(|\breve{x}_b -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1}\big).$$

Hence $i' := i * \langle \max_0(a_0, (i)_{l \dot{-} 1}) + 1\rangle$ satisfies $A_0(i', l+1)$.
Case 2: $\neg$ Case 1. Let us assume that $\breve{x}_{(i)_0} < \ldots < \breve{x}_{(i)_{l \dot{-} 1}}$ (If not we use a permutation of $(i)_0, \ldots, (i)_{l \dot{-} 1}$). Let $j_0 \leq_0 l \dot{-} 1$ be maximal such that

$$(1)\ \forall \tilde{m} \exists n \geq_0 \tilde{m}\big(|\breve{x}_n -_{\mathbb{Q}} \breve{x}_{(i)_{j_0}}| \leq \frac{1}{k_0+1}\big).$$

(The existence of j_0 follows from the least number principle for Π_2^0–formulas Π_2^0–LNP: Let j_1 be the least number such that $(l \dot{-} 1) \dot{-} j_1$ satisfies (1). Then $j_0 = (l \dot{-} 1) \dot{-} j_1$).
The definition of j_0 implies $\forall j \leq l \dot{-} 1\big(j > j_0 \to \exists a \forall b > a(|\breve{x}_b -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1})\big)$. Hence (again by Π_1^0–CP)

$$(2)\ \exists a_1 > j_0 \forall j \leq l \dot{-} 1\big(j > j_0 \to \forall b > a_1(|\breve{x}_b -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1})\big).$$

Let $c \in \mathbb{N}$ be arbitrary. By (+) (applied to $\tilde{l} := \max_0(k_0, c) + 1$) there exists a K_1 such that

$$(3)\ \forall j \exists q, r \geq j \exists m, n\big(K_1 \geq m, n \geq c, k_0 \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1}\big).$$

By (1) applied to $\tilde{m} := K_1$ there exists a $u \geq K_1$ such that $|\breve{x}_u -_{\mathbb{Q}} \breve{x}_{(i)_{j_0}}| \leq \frac{1}{k_0+1}$.
(3) applied to $j := u$ yields q, r, m, n such that

$$(5)\ q, r \geq u \wedge K_1 \geq m, n \geq c, k_0 \wedge |x_q^m -_{\mathbb{Q}} x_r^n| > \frac{1}{k_0+1} \wedge x_q^m, x_r^n \geq_{\mathbb{Q}} \breve{x}_{(i)_{j_0}} - \frac{1}{k_0+1}$$

(since $m,n \le u \le m+q, n+r$).
Because of $m,n \ge c, k_0$ this implies the existence of an $\alpha \ge c, k_0$ such that $\breve{x}_\alpha > \breve{x}_{(i)_{j_0}}$. Thus we have shown

$$(6)\ \forall c \exists \alpha \ge_0 c, k_0 (\breve{x}_\alpha > \breve{x}_{(i)_{j_0}}).$$

For $c := \max_0(a_1, (i)_{l \dot{-} 1}) + 1$ this yields the existence of an $\alpha_1 > a_1, (i)_{l \dot{-} 1}, k_0$ such that $\breve{x}_{\alpha_1} > \breve{x}_{(i)_{j_0}}$.
Let K_{α_1} be (by (+)) such that

$$(7)\ \forall j \exists q, r \ge j \exists m, n \big(K_{\alpha_1} \ge m, n \ge \alpha_1 (\ge a_1, k_0) \wedge |x^m_q -_{\mathbb{Q}} x^n_r| > \frac{1}{k_0+1}\big).$$

(6) applies to $c := K_{\alpha_1}$ provides an $\alpha_2 \ge K_{\alpha_1}$ such that $\breve{x}_{\alpha_2} > \breve{x}_{(i)_{j_0}}$. Hence (7) applied to $j := \alpha_2$ yields q, r, m, n with

$$(8)\ q, r \ge \alpha_2 \wedge K_{\alpha_1} \ge m, n \ge \alpha_1 \wedge |x^m_q -_{\mathbb{Q}} x^n_r| > \frac{1}{k_0+1} \wedge x^m_q, x^n_r \ge_{\mathbb{Q}} \breve{x}_{\alpha_2}.$$

Since $m, n \ge \alpha_1 > a_1, (i)_{l \dot{-} 1}$, (8) implies the existence of an $\alpha_3 > (i)_{l \dot{-} 1}, a_1$ such that

$$(9)\ \breve{x}_{\alpha_3} >_{\mathbb{Q}} \breve{x}_{(i)_{j_0}} + \frac{1}{k_0+1}.$$

Since $\breve{x}_{(i)_j} \le \breve{x}_{(i)_{j_0}}$ for $j \le j_0$, this implies (10) $\forall j \le j_0 \big(\breve{x}_{\alpha_3} >_{\mathbb{Q}} \breve{x}_{(i)_j} + \frac{1}{k_0+1}\big)$.
Let $j \le l \dot{-} 1$ be $> j_0$. Then by (2) and $\alpha_3 > a_1$: $|\breve{x}_{\alpha_3} -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1}$. Put together we have shown

$$(11)\ \alpha_3 > (i)_{l \dot{-} 1} \wedge \forall j \le l \dot{-} 1 \big(|\breve{x}_{\alpha_3} -_{\mathbb{Q}} \breve{x}_{(i)_j}| > \frac{1}{k_0+1}\big).$$

Define $i' := i * \langle \alpha_3 \rangle$. Then $A_0(i, l)$ implies $A_0(i', l+1)$, which concludes the proof of (++).
(++) applied to $l := 2(k_0+1)+1$ yields the existence of indices $i_0 < \ldots < i_{2(k_0+1)}$ such that $|\breve{x}_{(i)_j} -_{\mathbb{Q}} \breve{x}_{(i)_{j'}}| > \frac{1}{k_0+1}$ for $j, j' \le 2(k_0+1) \wedge j \ne j'$, which contradicts $\forall j^0(-1 \le_{\mathbb{Q}} \breve{x}_j \le_{\mathbb{Q}} 1)$.
Hence we have proved $L(x)$. This proof has used Σ^0_1–IA, Π^0_1–CP and Π^0_2–LNP. Since Π^0_2–LNP is equivalent to Σ^0_2–IA (see [20]), and Π^0_1–CP follows from Σ^0_2–IA by [19] (where CP is denoted by M), the proof above can be carried out in $\mathrm{G_3A}^\omega + \Sigma^0_2$–IA (these results from [19],[20] are proved there in a purely first–order context but immediately generalize to the case where function parameters are present).

Proposition 5.6.4 *Let $n \ge 2$ and $B_0(u^1, v^\tau, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ be a quantifier–free formula which contains only u^1, v^τ, w^γ free, where $\gamma \le 2$. Furthermore let $\xi, t \in G_nR^\omega$ and Δ be as in thm.2.4. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \forall u^1 \forall v \le_\tau tu\big(\exists \limsup(\xi uv) \to \exists w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists (eff.) \chi \in G_nR^\omega \textit{ such that} \\ G_nA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \le_\tau tu \forall \underline{\Psi}^* \big((\underline{\Psi}^* \textit{ satifies the mon. funct.interpr. of} \\ \qquad \textit{the negative translation } L(\xi uv)' \textit{ of } L(\xi uv)) \to \exists w \le_\gamma \chi u \underline{\Psi}^*\ B_0(u,v,w)\big) \\ \textit{and in particular } \exists \Psi \in \mathrm{T}_1 \textit{ such that} \\ PA_i^\omega + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \le_\tau tu \exists w \le_\gamma \Psi u\, B_0(u,v,w). \end{cases}$$

where T_1 is the restriction of Gödel's T which contains only the recursor R_ρ for $\rho \leq 1$. The Ackermann function (but no functions having an essentially greater order of growth) can be defined in T_1.
In the conclusion, Δ+b-AC can be replaced by $\tilde{\Delta}$, where $\tilde{\Delta}$ is defined as in thm.2.4. If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\exists$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta$+AC–qf may be replaced by E–$G_nA^\omega + \Delta$+$AC^{\alpha,\beta}$–qf, where α, β are as in cor.2.6.

Proof: Prenexation of $\forall u^1 \forall v \leq_\tau tu\big(L(\xi uv) \to \exists w^\gamma B_0(u,v,w)\big)$ yields

$$G :\equiv \forall u^1 \forall v \leq_\tau tu \exists k \forall l \exists K \forall j \exists q, r, w\big[(l > k \wedge (K \geq l \wedge q, r \geq j \to L_0)) \to B_0(u,v,w)\big].$$

Lemma 5.6.3.1) implies

$$(1)\ \mathrm{G}_2\mathrm{A}^\omega \vdash Mon(G).$$

The assumption of the proposition combined with lemma 5.6.3.3) implies

$$(2)\ \mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC–qf}\ \vdash \forall u^1 \forall v \leq_\tau tu\big(L(\xi uv)^S \to \exists w^\gamma B_0(u,v,w)\big)$$

and therefore

$$(3)\ \mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC–qf}\ \vdash G^H.$$

Theorem 2.4 applied to (1) and (3) provides the extractability of a tuple $\underline{\varphi} \in \mathrm{G}_n\mathrm{R}^\omega$ such that

$$(4)\ \mathrm{G}_n\mathrm{A}_i^\omega + \Delta + \text{b-AC}\ \vdash \big(\underline{\varphi} \text{ satisfies the monotone functional interpretation of } G'\big).$$

G' intuitionistically implies

$$(5)\ \forall u^1 \forall v \leq_\tau tu\big(L(\xi uv)' \to \neg\neg\exists w^\gamma B_0(u,v,w)\big).$$

Hence from $\underline{\varphi}$ one obtains a term $\tilde{\varphi} \in \mathrm{G}_n\mathrm{R}^\omega$ such that (provably in $\mathrm{G}_n\mathrm{A}_i^\omega + \Delta$ + b-AC)

$$(6)\ \exists \psi\big(\tilde{\varphi} \text{ s–maj } \psi \wedge \forall u^1 \forall v \leq_\tau tu \forall \underline{a}\big(\forall \underline{b}(L(\xi uv)')_D \to B_0(u,v,\psi uv\underline{a})\big)\big),$$

where $\exists \underline{a} \forall \underline{b}\big(L(\xi uv)'\big)_D$ is the usual functional interpretation of $L(\xi uv)'$.
Let $\underline{\Psi}^*$ satisfy the monotone functional interpretation of $L(\xi uv)'$ then

$$(7)\ \exists \underline{a}\big(\underline{\Psi}^* \text{ s–maj } \underline{a} \wedge \forall \underline{b}\big(L(\xi uv)'\big)_D\big).$$

Hence for such a tuple $\underline{a}$ we have

$$(8)\ \lambda u^1.\tilde{\varphi}u(t^*u)\underline{\Psi}^* \text{ s–maj } \psi uv\underline{a} \text{ for } v \leq tu$$

(Use lemma 2.2.11 from [15]. t^* in $\mathrm{G}_n\mathrm{R}^\omega$ is a majorant for t).
Since $\gamma \leq 2$ this yields a $\geq_2$ bound $\chi u \underline{\Psi}^*$ for $\psi uv\underline{a}$ (lemma 2.2.11 from [15]).
The second part of the proposition follows from lemma 5.6.3.4) and the fact that $\mathrm{G}_n\mathrm{A}^\omega + \Sigma_2^0$–IA has a monotone functional interpretation in PA_i^ω by terms $\in \mathrm{T}_1$ (By [20] Σ_2^0–IA has a functional interpretation in T_1. Since every term in T_1 has a majorant in T_1, also the monotone functional interpretation can be satisfied in T_1).

Remark 5.6.5 *By the theorem above the use of the analytical axiom $\exists \limsup(\xi uv)$ in a given proof of $\forall u^1 \forall v \leq_\tau tu \exists w^\gamma B_0$ can be reduced to the use of the arithmetical principle $L(\xi uv)$. By lemma 5.6.3.2) this reduction is optimal (relatively to G_2A^ω).*

References

[1] Beeson, M.J., Foundations of Constructive Mathematics. Springer Ergebnisse der Mathematik und ihrer Grenzgebiete 3.Folge, Bd.6., Berlin Heidelberg New York Tokyo 1985.

[2] Bezem, M.A., Strongly majorizable functional of finite type: a model for bar recursion containing discontinuous functionals. J. Symb. Logic **50**, pp. 652–660 (1985).

[3] Bishop, E., Foundations of constructive analysis. McGraw–Hill, New–York (1967).

[4] Bishop, E.– Bridges, D. , Constructive analysis. Springer Grundlehren der mathematischen Wissenschaften vol.279, Berlin 1985.

[5] Feferman, S., Theories of finite type related to mathematical practice. In: Barwise, J. (ed.), Handbook of Mathematical Logic, North–Holland, Amsterdam, pp. 913–972 (1977).

[6] Friedman, H., Systems of second order arithmetic with restricted induction (abstract), J. Symbolic Logic **41**, pp. 558-559 (1976).

[7] Gödel, K., Zur intuitionistischen Arithmetik und Zahlentheorie. Ergebnisse eines Mathematischen Kolloquiums, vol. 4 pp. 34–38 (1933).

[8] Gödel, K., Über eine bisher noch nicht benutzte Erweiterung des finiten Standpunktes. Dialectica **12**, pp. 280–287 (1958).

[9] Kleene, S.C.,Introduction to Metamathematics. North– Holland (Amsterdam), Noordhoff (Groningen), New–York (Van Nostrand) (1952).

[10] Kleene, S.C., Recursive functionals and quantifiers of finite types, I. Trans. A.M.S. **91**, pp.1–52 (1959).

[11] Ko, K.–I., Complexity theory of real functions. Birkhäuser; Boston, Basel, Berlin (1991).

[12] Kohlenbach, U., Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. J. Symbolic Logic **57**, pp. 1239–1273 (1992).

[13] Kohlenbach, U., Analysing proofs in analysis. In: W. Hodges, M. Hyland, C. Steinhorn, J. Truss, editors, *Logic: from Foundations to Applications. European Logic Colloquium* (Keele, 1993), pp. 225–260, Oxford University Press (1996).

[14] Kohlenbach, U., Real growth in standard parts of analysis. Habilitationsschrift, xv+166 p., Frankfurt (1995).

[15] Kohlenbach, U., Mathematically strong subsystems of analysis with low rate of provably recursive functionals. Arch. Math. Logic **36**, pp. 31–71 (1996).

[16] Kohlenbach, U., Elimination of Skolem functions for monotone formulas. To appear in: Archive for Mathematical Logic.

[17] Kohlenbach, U., The use of a logical principle of uniform boundedness in analysis. To appear in: Proc. 'Logic in Florence 1995'.

[18] Luckhardt, H., Extensional Gödel functional interpretation. A consistency proof of classical analysis. Springer Lecture Notes in Mathematics **306** (1973).

[19] Parsons, C., On a number theoretic choice schema and its relation to induction. In: Intuitionism and proof theory, pp. 459-473. North-Holland, Amsterdam (1970).

[20] Parsons, C., On n-quantifier induction. J. Symbolic Logic **37**, pp. 466–482 (1972).

[21] Sieg, W., Fragments of arithmetic. Ann. Pure Appl. Logic **28**, pp. 33-71 (1985).

[22] Simpson, S.G., Reverse Mathematics. Proc. Symposia Pure Math. **42**, pp. 461–471, AMS, Providence (1985).

[23] Spector, C., Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics. In: Recursive function theory, Proceedings of Symposia in Pure Mathematics, vol. 5 (J.C.E. Dekker (ed.)), AMS, Providence, R.I., pp. 1–27 (1962).