A note on Goodman's theorem

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Abstract

Goodman's theorem states that intuitionistic arithmetic in all finite types plus full choice, $\mathbf{HA}^\omega + \mathbf{AC}$, is conservative over first-order intuitionistic arithmetic **HA**. We show that this result does not extend to various subsystems of $\mathbf{HA}^\omega$, **HA** with restricted induction.

1 Introduction

Let $\mathbf{E\text{-}HA}^\omega$ denote the system of extensional intuitionistic arithmetic and $\mathbf{HA}^\omega$ its 'neutral' variant as defined in [10]).[1] $\mathbf{E\text{-}PA}^\omega$ and $\mathbf{PA}^\omega$ are the corresponding theories with classical logic. **PA** (resp. **HA**) is first-order Peano arithmetic (resp. its intuitionistic version) with all primitive recursive functions. **T** denotes the set of finite types.

The schema **AC** of full choice is given by $\bigcup\limits_{\rho,\tau\in\mathbf{T}} \mathbf{AC}^{\rho,\tau}$, where

$$\mathbf{AC}^{\rho,\tau}: \ \forall x^\rho \exists y^\tau A(x,y) \to \exists Y^{\tau(\rho)} \forall x^\rho A(x, Yx)$$

and A is an arbitrary formula of $\mathcal{L}(\mathbf{HA}^\omega)$.

We also consider a restricted (arithmetical) form of **AC**:

$$\mathbf{AC}^{0,0}_{ar}: \ \forall x^0 \exists y^0 A(x,y) \to \exists f^{0(0)} \forall x^0 A(x, fx),$$

where A contains only quantifiers of type 0 (but maybe parameters of arbitrary type).

By the rule of choice we mean

$$\mathbf{ACR}: \ \frac{\forall x^\rho \exists y^\tau A(x,y)}{\exists Y^{\tau(\rho)} \forall x^\rho A(x, Yx)}$$

[1] Note that both theories contain only equality of type 0 as a primitive relation symbol.

for all ρ, τ and all formulas A of $\mathcal{L}(\mathbf{HA}^\omega)$.

Classically we have the following well-known situation:

$$\mathbf{(E\text{-})PA}^\omega + \mathbf{AC} \ = \mathbf{(E\text{-})PA}^\omega + \mathbf{ACR}$$

has the strength of full simple type theory.[2]
Furthermore, even $\mathbf{PA}^\omega + \mathbf{AC}^{0,0}_{ar}$ is not Π^0_1-conservative over $\mathbf{PA}^\omega$: $\mathbf{PA}^\omega + \mathbf{AC}^{0,0}_{ar}$ proves the consistency of Peano arithmetic[3] whereas $\mathbf{PA}^\omega$ is conservative over $\mathbf{PA}$.

These facts are in sharp contrast to the intuitionistic case where – in particular – the following is known:

1) $\mathbf{(E\text{-})HA}^\omega$ is closed under $\mathbf{ACR}$.
2) $\mathbf{(E\text{-})HA}^\omega + \mathbf{AC}$ is conservative over $\mathbf{(E\text{-})HA}^\omega$ for all formulas A which are implied (relative to $\mathbf{(E\text{-})HA}^\omega$) by their modified-realizability interpretation.
3) $\mathbf{E\text{-}HA}^\omega + \mathbf{AC}$ is conservative over $\mathbf{HA}$.[4]

1) and 2) are proved e.g. in [10],[11] using the 'modified-realizability-with-truth' resp. the 'modified-realizability' interpretation.
3) for $\mathbf{HA}^\omega$ instead of $\mathbf{E\text{-}HA}^\omega$ is due to [4],[5] (a different proof is given in [7]). The generalization to $\mathbf{E\text{-}HA}^\omega$ was established by Beeson in [1],[2] (using [6] and solving a problem stated by H. Friedman) who refers to 3) for $\mathbf{HA}^\omega$ as 'Goodman's theorem' (see also [9] for an even stronger result).

Let us switch now to the subsystems, $\mathbf{(E\text{-})PA}^\omega\!\restriction$, $\mathbf{(E\text{-})HA}^\omega\!\restriction$ of $\mathbf{(E\text{-})PA}^\omega$, $\mathbf{(E\text{-})HA}^\omega$ where the full schema of induction is replaced by the schema of quantifier–free induction

$$\mathbf{QF\text{-}IA}: \ A_0(0) \wedge \forall x^0 \big(A_0(x) \rightarrow A_0(x')\big) \rightarrow \forall x^0 A_0(x),$$

where A_0 is a quantifier-free formula, and instead of the constants of $\mathbf{(E\text{-})HA}^\omega$ we only have 0^0 and symbols for every primitive recursive function plus their defining equations (but no higher type functional constants). If we add to $\mathbf{(E\text{-})PA}^\omega\!\restriction$, $\mathbf{(E\text{-})HA}^\omega\!\restriction$ the combinators $\Pi_{\rho,\tau}$ and $\Sigma_{\delta,\rho,\tau}$ (from $\mathbf{HA}^\omega$) as well as the predicative Kleene recursors $\widehat{R}_\rho$ (and – in the case of $\mathbf{HA}^\omega\!\restriction$ – also a functional allowing definition by cases which is redundant in $\mathbf{E\text{-}HA}^\omega\!\restriction$) we obtain the systems, $\mathbf{(E\text{-})}\widehat{\mathbf{PA}}^\omega\!\restriction$, $\mathbf{(E\text{-})}\widehat{\mathbf{HA}}^\omega\!\restriction$ due to Feferman (see [3]).
$\mathbf{PA}\!\restriction$, $\mathbf{HA}\!\restriction$ are the restrictions of $\mathbf{PA}$, $\mathbf{HA}$ with quantifier-free induction only.

Since both the modified-realizability interpretation with and without truth hold for $\mathbf{(E\text{-})}\widehat{\mathbf{HA}}^\omega\!\restriction$ one easily obtains the results 1),2) above also for this restricted system. In this note we observe that 3) fails for $\widehat{\mathbf{HA}}^\omega\!\restriction$ (and even for $\mathbf{HA}^\omega\!\restriction$) and $\mathbf{HA}\!\restriction$ instead of $\mathbf{(E\text{-})HA}^\omega$ and $\mathbf{HA}$.

[2]Every instance of $\mathbf{AC}$ is derivable in $\mathbf{PA}^\omega + \mathbf{ACR}$ and $\mathbf{ACR}$ is a derivable rule in $\mathbf{(E\text{-})PA}^\omega + \mathbf{AC}$.

[3]$\mathbf{PA}^\omega + \mathbf{AC}^{0,0}_{ar}$ proves the schema of arithmetical comprehension (with arbitrary parameters) and therefore is a finite type extension of (a function variable version of) the second-order system $(\Pi^0_\infty\text{-}\mathbf{CA})$ which is known to prove the consistency of $\mathbf{PA}$ (see e.g. [3](5.5.2)).

[4]$\mathbf{HA}$ can be considered as a subsystem of $\mathbf{E\text{-}HA}^\omega$ either by switching to a definitorial extension of $\mathbf{E\text{-}HA}^\omega$ by adding all primitive recursive functions or modulo a suitable bi-unique mapping Δ which translates $\mathbf{HA}$ into a subsystem of $\mathbf{E\text{-}HA}^\omega$, see [10] (1.6.9) for the latter.

2 Results

Proposition 1 *Let $A \in \mathcal{L}(\mathbf{PA})$ be such that $\mathbf{PA} \vdash A$. Then one can construct a sentence $\tilde{A} \in \mathcal{L}(\mathbf{PA})$ such that $A \leftrightarrow \tilde{A}$ holds by* **classical** *logic and $\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \vdash \tilde{A}$.*

Proof:
Let A be such that $\mathbf{PA} \vdash A$. By negative translation ($F \rightsquigarrow F'$) it follows that $\mathbf{HA} \vdash A'$. So there exists a finite set $G_1, \ldots, G_k$ of instances of the schema of induction **IA** in **HA** (more precisely G_i is the universal closure of the corresponding instance) such that

$$\mathbf{HA}\!\upharpoonright \vdash \bigwedge_{i=1}^{k} G_i \to A'.$$

Let $\widehat{G}_i(x, \underline{a})$ be the induction formula belonging to G_i (x being the induction variable and $\underline{a}$ the (number) parameters of the induction formula), i.e.

$$G_i \equiv \forall \underline{a}\big(\widehat{G}_i(0, \underline{a}) \wedge \forall x\big(\widehat{G}_i(x, \underline{a}) \to \widehat{G}_i(x', \underline{a})\big) \to \forall x \widehat{G}_i(x, \underline{a})\big).$$

Define

$$\check{G}_i :\equiv \forall x, \underline{a} \exists y\big(y =_0 0 \leftrightarrow \widehat{G}_i(x, \underline{a})\big)$$

and consider the formula

$$\tilde{A} :\equiv \bigwedge_{i=1}^{k} \check{G}_i \to A'.$$

With **classical** logic (and $0 \neq S0$) we have $\tilde{A} \leftrightarrow A' \leftrightarrow A$.
Using $\mathbf{AC}^{0,0}_{ar}$ applied to $\check{G}_i$ one shows

$$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \ \vdash \check{G}_i \to \exists f \forall x, \underline{a}\big(fx\underline{a} =_0 0 \leftrightarrow \widehat{G}_i(x, \underline{a})\big).$$

QF-IA applied to $A_0(x) :\equiv (fx\underline{a} =_0 0)$ now yields G_i. Hence

$$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \ \vdash \bigwedge_{i=1}^{k} \check{G}_i \to \bigwedge_{i=1}^{k} G_i$$

and therefore

$$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \ \vdash \bigwedge_{i=1}^{k} \check{G}_i \to A'.$$

Hence $\tilde{A}$ satisfies the claim of the proposition.

Corollary 2 *For every $n \in \mathbb{N}$ one can construct a sentence $A \in \mathcal{L}(\mathbf{HA}\!\upharpoonright)$ (involving only $0, S, +, \cdot$ and logic) such that*

$$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \vdash A, \ \textit{but } \mathbf{PA}\!\upharpoonright + \Sigma^0_n\text{-IA} \not\vdash A.$$

Proof:
It is well-known (see e.g. [8]) that for every $n \in \mathbb{N}$ there exists an arithmetical sentence $\tilde{A}$ (involving only $0, S, +, \cdot$ and logic) such that $\mathbf{PA} \vdash \tilde{A}$ (and even $\mathbf{PA}\!\upharpoonright + \Sigma^0_{n+1}$-IA $\vdash \tilde{A}$), but $\mathbf{PA}\!\upharpoonright + \Sigma^0_n$-IA $\nvdash \tilde{A}$. The corollary now follows from the proposition above.

Corollary 3 **(E-)**$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \mathbf{AC}^{0,0}_{ar}$ *is not conservative over* $\mathbf{HA}\!\upharpoonright$*. In particular, Goodman's theorem fails for* $\mathbf{HA}^{\omega}_{-}\!\upharpoonright$*,* $\mathbf{HA}\!\upharpoonright$ *and therefore also for* $\widehat{\mathbf{HA}}^{\omega}\!\upharpoonright$*,* $\mathbf{HA}\!\upharpoonright$*.*

For $\widehat{\mathbf{HA}}^{\omega}\!\upharpoonright$ instead of $\mathbf{HA}^{\omega}_{-}\!\upharpoonright$ and without the restriction to the language $\{0, S, +, \cdot\}$ but with $A \in \mathcal{L}(\mathbf{HA}\!\upharpoonright)$, the proof of proposition 1 is even more easy:
It is well-known (see [8]) that for every $n \in \mathbb{N}$ there is an instance

$$\tilde{A} :\equiv \big(\forall x < a \exists y A(x,y) \to \exists y_0 \forall x < a \exists y < y_0 A(x,y)\big),$$

where $A \in \Pi^0_n$, of the so-called 'collection principle for Π^0_n-formulas' Π^0_n-CP (A containing only number parameters) such that

$$\mathbf{PA}\!\upharpoonright + \Sigma^0_n\text{-IA} \nvdash \tilde{A}.$$

However $\widehat{\mathbf{HA}}^{\omega}\!\upharpoonright + \mathbf{AC}^{0,0}_{ar} \vdash \tilde{A}$:
$\mathbf{AC}^{0,0}_{ar}$ applied to $\forall x \exists y (x < a \to A(x,y))$ yields a function f such that $\forall x < a A(x, fx)$. Applying the functional $\Phi_{\max} fx = \max(f0, \ldots, fx)$ (definable in $\widehat{\mathbf{HA}}^{\omega}\!\upharpoonright$) to f and $a \dot{-} 1$ yields a y_0 such that $\forall x < a \exists y < y_0 A(x,y)$: Put $y_0 := \Phi_{\max}(f, a \dot{-} 1) + 1$.
So in particular for $n = 0$ we have a universally closed instance

$$\tilde{A} :\equiv \forall a, \underline{b}\big(\big(\forall x < a \exists y A_0(x,y,a,\underline{b}) \to \exists y_0 \forall x < a \exists y < y_0 A_0(x,y,a,\underline{b})\big)\big),$$

of Π^0_0-CP (where A_0 is quantifier-free and $a^0, \underline{b}^0, x^0, y^0$ are all the free variables of $A_0(x,y,a,\underline{b})$) such that $\mathbf{PA}\!\upharpoonright \nvdash \tilde{A}$. Using coding of pairs and bounded search (both available in $\mathbf{PA}\!\upharpoonright$), $\tilde{A}$ can be expressed as a sentence having the form

$$\forall a^0 \big(\forall x^0 \exists y^0 A_0(x,y,a) \to \exists y^0 B_0(y,a)\big),$$

where A_0 and B_0 both are quantifier-free. Furthermore the instance of $\mathbf{AC}^{0,0}_{ar}$ needed to prove $\tilde{A}$ in $\widehat{\mathbf{HA}}^{\omega}\!\upharpoonright + \mathbf{AC}^{0,0}_{ar}$ is

$$\forall a^0 \big(\forall x^0 \exists y^0 A_0(x,y,a) \to \exists f \forall x A_0(x, fx, a)\big).$$

So we have the following

Proposition 4 *One can construct a sentence* $A :\equiv \forall a^0 \big(\forall x^0 \exists y^0 A_0(x,y,a) \to \exists y^0 B_0(y,a)\big)$ *(*A_0, B_0 *being quantifer-free formulas of* $\mathbf{HA}\!\upharpoonright$*) such that*

$$\widehat{\mathbf{HA}}^{\omega}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{-}\text{-}qf \vdash A, \ \textit{but} \ \ \mathbf{PA}\!\upharpoonright \nvdash A,$$

where $\mathbf{AC}^{0,0}_{-}$*-qf is the restriction of* $\mathbf{AC}^{0,0}_{ar}$ *to* **quantifier-free** *formulas* **without function parameters**.

Remark 5 *Note that modified-realizability interpretation yields that* $\widehat{\mathbf{HA}}^{\omega}\upharpoonright + \mathbf{AC}$ *is conservative over* $\mathbf{HA}\upharpoonright$ *w.r.t. sentences having e.g. the form* $A :\equiv \forall a^0\big(\exists x^0 \forall y^0 A_0(x,y,a) \to \exists y^0 B_0(y,a)\big)$.

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