A note on Goodman's theorem

Ulrich Kohlenbach

Department of Mathematics

University of Michigan

Ann Arbor

MI 48109, USA

April 1997

Abstract

Goodman's theorem states that intuitionistic arithmetic in all finite types plus full choice, $\mathbf{HA}^\omega +$ $\mathbf{AC}$, is conservative over first-order intuitionistic arithmetic $\mathbf{HA}$. We show that this result does not extend to various subsystems of $\mathbf{HA}^\omega$, $\mathbf{HA}$ with restricted induction.

1 Introduction

Let $\mathbf{E}\text{-}\mathbf{HA}^\omega$ denote the system of extensional intuitionistic arithmetic and $\mathbf{HA}^\omega$ its 'neutral' variant as defined in [10]).[1] $\mathbf{E}\text{-}\mathbf{PA}^\omega$ and $\mathbf{PA}^\omega$ are the corresponding theories with classical logic. $\mathbf{PA}$ (resp. $\mathbf{HA}$) is first-order Peano arithmetic (resp. its intuitionistic version) with all primitive recursive functions. $\mathbf{T}$ denotes the set of finite types.

The schema $\mathbf{AC}$ of full choice is given by $\bigcup\limits_{\rho,\tau \in \mathbf{T}} \mathbf{AC}^{\rho,\tau}$, where

$$\mathbf{AC}^{\rho,\tau} \ : \ \forall x^\rho \exists y^\tau A(x,y) \to \exists Y^{\tau(\rho)} \forall x^\rho A(x, Yx)$$

and A is an arbitrary formula of $\mathcal{L}(\mathbf{HA}^\omega)$.

We also consider a restricted (arithmetical) form of $\mathbf{AC}$:

$$\mathbf{AC}_{ar}^{0,0} \ : \ \forall x^0 \exists y^0 A(x,y) \to \exists f^{0(0)} \forall x^0 A(x, fx),$$

where A contains only quantifiers of type 0 (but maybe parameters of arbitrary type).

By the rule of choice we mean

$$\mathbf{ACR} \ : \ \frac{\forall x^\rho \exists y^\tau A(x,y)}{\exists Y^{\tau(\rho)} \forall x^\rho A(x, Yx)}$$

---

[1] Note that both theories contain only equality of type 0 as a primitive relation symbol.

for all ρ, τ and all formulas A of $\mathcal{L}(\mathbf{HA}^\omega)$.

Classically we have the following well-known situation:

$$(\mathbf{E\text{-}})\mathbf{PA}^\omega + \mathbf{AC} \ = (\mathbf{E\text{-}})\mathbf{PA}^\omega + \mathbf{ACR}$$

has the strength of full simple type theory.[2]

Furthermore, even $\mathbf{PA}^\omega + \mathbf{AC}_{ar}^{0,0}$ is not Π_1^0-conservative over $\mathbf{PA}^\omega$: $\mathbf{PA}^\omega + \mathbf{AC}_{ar}^{0,0}$ proves the consistency of Peano arithmetic[3] whereas $\mathbf{PA}^\omega$ is conservative over $\mathbf{PA}$.

These facts are in sharp contrast to the intuitionistic case where – in particular – the following is known:

1) $(\mathbf{E\text{-}})\mathbf{HA}^\omega$ is closed under $\mathbf{ACR}$.

2) $(\mathbf{E\text{-}})\mathbf{HA}^\omega + \mathbf{AC}$ is conservative over $(\mathbf{E\text{-}})\mathbf{HA}^\omega$ for all formulas A which are implied (relative to $(\mathbf{E\text{-}})\mathbf{HA}^\omega$) by their modified-realizability interpretation.

3) $\mathbf{E\text{-}HA}^\omega + \mathbf{AC}$ is conservative over $\mathbf{HA}$.[4]

1) and 2) are proved e.g. in [10],[11] using the 'modified-realizability-with-truth' resp. the 'modified-realizability' interpretation.

3) for $\mathbf{HA}^\omega$ instead of $\mathbf{E\text{-}HA}^\omega$ is due to [4],[5] (a different proof is given in [7]). The generalization to $\mathbf{E\text{-}HA}^\omega$ was established by Beeson in [1],[2] (using [6] and solving a problem stated by H. Friedman) who refers to 3) for $\mathbf{HA}^\omega$ as 'Goodman's theorem' (see also [9] for an even stronger result).

Let us switch now to the subsystems, $(\mathbf{E\text{-}})\mathbf{PA}_-^\omega\!\restriction$, $(\mathbf{E\text{-}})\mathbf{HA}_-^\omega\!\restriction$ of $(\mathbf{E\text{-}})\mathbf{PA}^\omega$, $(\mathbf{E\text{-}})\mathbf{HA}^\omega$ where the full schema of induction is replaced by the schema of quantifier–free induction

$$\mathbf{QF\text{-}IA} : \ A_0(0) \wedge \forall x^0 \big(A_0(x) \to A_0(x')\big) \to \forall x^0 A_0(x),$$

where A_0 is a quantifier-free formula, and instead of the constants of $(\mathbf{E\text{-}})\mathbf{HA}^\omega$ we only have 0^0 and symbols for every primitive recursive function plus their defining equations (but no higher type functional constants). If we add to $(\mathbf{E\text{-}})\mathbf{PA}_-^\omega\!\restriction$, $(\mathbf{E\text{-}})\mathbf{HA}_-^\omega\!\restriction$ the combinators $\Pi_{\rho,\tau}$ and $\Sigma_{\delta,\rho,\tau}$ (from $\mathbf{HA}^\omega$) as well as the predicative Kleene recursors $\widehat{R}_\rho$ (and – in the case of $\mathbf{HA}_-^\omega\!\restriction$ – also a functional allowing definition by cases which is redundant in $\mathbf{E\text{-}HA}_-^\omega\!\restriction$) we obtain the systems, $(\mathbf{E\text{-}})\widehat{\mathbf{PA}}^\omega\!\restriction$, $(\mathbf{E\text{-}})\widehat{\mathbf{HA}}^\omega\!\restriction$ due to Feferman (see [3]).

$\mathbf{PA}\!\restriction$, $\mathbf{HA}\!\restriction$ are the restrictions of $\mathbf{PA}$, $\mathbf{HA}$ with quantifier-free induction only.

Since both the modified-realizability interpretation with and without truth hold for $(\mathbf{E\text{-}})\widehat{\mathbf{HA}}^\omega\!\restriction$ one easily obtains the results 1),2) above also for this restricted system. In this note we observe that 3) fails for $\widehat{\mathbf{HA}}^\omega\!\restriction$ (and even for $\mathbf{HA}_-^\omega\!\restriction$) and $\mathbf{HA}\!\restriction$ instead of $(\mathbf{E\text{-}})\mathbf{HA}^\omega$ and $\mathbf{HA}$.

---

[2]Every instance of $\mathbf{AC}$ is derivable in $\mathbf{PA}^\omega + \mathbf{ACR}$ and $\mathbf{ACR}$ is a derivable rule in $(\mathbf{E\text{-}})\mathbf{PA}^\omega + \mathbf{AC}$.

[3]$\mathbf{PA}^\omega + \mathbf{AC}_{ar}^{0,0}$ proves the schema of arithmetical comprehension (with arbitrary parameters) and therefore is a finite type extension of (a function variable version of) the second-order system $(\Pi_\infty^0\text{-}\mathbf{CA})$ which is known to prove the consistency of $\mathbf{PA}$ (see e.g. [3](5.5.2)).

[4]$\mathbf{HA}$ can be considered as a subsystem of $\mathbf{E\text{-}HA}^\omega$ either by switching to a definitorial extension of $\mathbf{E\text{-}HA}^\omega$ by adding all primitive recursive functions or modulo a suitable bi-unique mapping Δ which translates $\mathbf{HA}$ into a subsystem of $\mathbf{E\text{-}HA}^\omega$, see [10] (1.6.9) for the latter.

2 Results

Proposition 1 *Let $A \in \mathcal{L}(\mathbf{PA})$ be such that $\mathbf{PA} \vdash A$. Then one can construct a sentence $\tilde{A} \in \mathcal{L}(\mathbf{PA})$ such that $A \leftrightarrow \tilde{A}$ holds by* **classical** *logic and* $\mathbf{HA}^{\omega}_{-} \!\upharpoonright + \mathbf{AC}^{0,0}_{ar} \vdash \tilde{A}$.

Proof:
Let A be such that $\mathbf{PA} \vdash A$. By negative translation $(F \rightsquigarrow F')$ it follows that $\mathbf{HA} \vdash A'$. So there exists a finite set $G_1, \ldots, G_k$ of instances of the schema of induction **IA** in $\mathbf{HA}$ (more precisely G_i is the universal closure of the corresponding instance) such that

$$\mathbf{HA}\!\upharpoonright \vdash \bigwedge_{i=1}^{k} G_i \to A'.$$

Let $\widehat{G}_i(x, \underline{a})$ be the induction formula belonging to G_i (x being the induction variable and $\underline{a}$ the (number) parameters of the induction formula), i.e.

$$G_i \equiv \forall \underline{a}\big(\widehat{G}_i(0, \underline{a}) \wedge \forall x\big(\widehat{G}_i(x, \underline{a}) \to \widehat{G}_i(x', \underline{a})\big) \to \forall x \widehat{G}_i(x, \underline{a})\big).$$

Define

$$\check{G}_i :\equiv \forall x, \underline{a} \exists y \big(y =_0 0 \leftrightarrow \widehat{G}_i(x, \underline{a})\big)$$

and consider the formula

$$\tilde{A} :\equiv \bigwedge_{i=1}^{k} \check{G}_i \to A'.$$

With **classical** logic (and $0 \neq S0$) we have $\tilde{A} \leftrightarrow A' \leftrightarrow A$.
Using $\mathbf{AC}^{0,0}_{ar}$ applied to $\check{G}_i$ one shows

$$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \ \vdash \check{G}_i \to \exists f \forall x, \underline{a}\big(fx\underline{a} =_0 0 \leftrightarrow \widehat{G}_i(x, \underline{a})\big).$$

QF-IA applied to $A_0(x) :\equiv (fx\underline{a} =_0 0)$ now yields G_i. Hence

$$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \ \vdash \bigwedge_{i=1}^{k} \check{G}_i \to \bigwedge_{i=1}^{k} G_i$$

and therefore

$$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \ \vdash \bigwedge_{i=1}^{k} \check{G}_i \to A'.$$

Hence $\tilde{A}$ satisfies the claim of the proposition.

Corollary 2 *For every $n \in \mathbb{N}$ one can construct a sentence $A \in \mathcal{L}(\mathbf{HA}\!\upharpoonright)$ (involving only $0, S, +, \cdot$ and logic) such that*

$$\mathbf{HA}^{\omega}_{-}\!\upharpoonright + \ \mathbf{AC}^{0,0}_{ar} \vdash A, \ \textit{but} \ \mathbf{PA}\!\upharpoonright + \Sigma^0_n\text{-IA} \not\vdash A.$$

Proof:

It is well-known (see e.g. [8]) that for every $n \in \mathbb{N}$ there exists an arithmetical sentence $\tilde{A}$ (involving only $0, S, +, \cdot$ and logic) such that $\mathbf{PA} \vdash \tilde{A}$ (and even $\mathbf{PA}|\!\backslash + \Sigma_{n+1}^0\text{-IA} \vdash \tilde{A}$), but $\mathbf{PA}|\!\backslash + \Sigma_n^0\text{-IA} \not\vdash \tilde{A}$. The corollary now follows from the proposition above.

Corollary 3 $(\text{E-})\mathbf{HA}_{-}^{\omega}|\!\backslash + \mathbf{AC}_{ar}^{0,0}$ *is not conservative over* $\mathbf{HA}|\!\backslash$. *In particular, Goodman's theorem fails for* $\mathbf{HA}_{-}^{\omega}|\!\backslash$, $\mathbf{HA}|\!\backslash$ *and therefore also for* $\widehat{\mathbf{HA}}^{\omega}|\!\backslash$, $\mathbf{HA}|\!\backslash$.

For $\widehat{\mathbf{HA}}^{\omega}|\!\backslash$ instead of $\mathbf{HA}_{-}^{\omega}|\!\backslash$ and without the restriction to the language $\{0, S, +, \cdot\}$ but with $A \in \mathcal{L}(\mathbf{HA}|\!\backslash)$, the proof of proposition 1 is even more easy:

It is well-known (see [8]) that for every $n \in \mathbb{N}$ there is an instance

$$\tilde{A} :\equiv \big(\forall x < a \exists y A(x, y) \to \exists y_0 \forall x < a \exists y < y_0 A(x, y)\big),$$

where $A \in \Pi_n^0$, of the so-called 'collection principle for Π_n^0-formulas' Π_n^0-CP (A containing only number parameters) such that

$$\mathbf{PA}|\!\backslash + \Sigma_n^0\text{-IA} \not\vdash \tilde{A}.$$

However $\widehat{\mathbf{HA}}^{\omega}|\!\backslash + \mathbf{AC}_{ar}^{0,0} \vdash \tilde{A}$:

$\mathbf{AC}_{ar}^{0,0}$ applied to $\forall x \exists y (x < a \to A(x, y))$ yields a function f such that $\forall x < a A(x, fx)$. Applying the functional $\Phi_{\max} f x = \max(f0, \ldots, fx)$ (definable in $\widehat{\mathbf{HA}}^{\omega}|\!\backslash$) to f and $a \dot{-} 1$ yields a y_0 such that $\forall x < a \exists y < y_0 A(x, y)$: Put $y_0 := \Phi_{\max}(f, a \dot{-} 1) + 1$.

So in particular for $n = 0$ we have a universally closed instance

$$\tilde{A} :\equiv \forall a, \underline{b}\big(\big(\forall x < a \exists y A_0(x, y, a, \underline{b}) \to \exists y_0 \forall x < a \exists y < y_0 A_0(x, y, a, \underline{b})\big),$$

of Π_0^0-CP (where A_0 is quantifier-free and $a^0, \underline{b}^0, x^0, y^0$ are all the free variables of $A_0(x, y, a, \underline{b})$) such that $\mathbf{PA}|\!\backslash \not\vdash \tilde{A}$. Using coding of pairs and bounded search (both available in $\mathbf{PA}|\!\backslash$), $\tilde{A}$ can be expressed as a sentence having the form

$$\forall a^0 \big(\forall x^0 \exists y^0 A_0(x, y, a) \to \exists y^0 B_0(y, a)\big),$$

where A_0 and B_0 both are quantifier-free. Furthermore the instance of $\mathbf{AC}_{ar}^{0,0}$ needed to prove $\tilde{A}$ in $\widehat{\mathbf{HA}}^{\omega}|\!\backslash + \mathbf{AC}_{ar}^{0,0}$ is

$$\forall a^0 \big(\forall x^0 \exists y^0 A_0(x, y, a) \to \exists f \forall x A_0(x, fx, a)\big).$$

So we have the following

Proposition 4 *One can construct a sentence* $A :\equiv \forall a^0 \big(\forall x^0 \exists y^0 A_0(x, y, a) \to \exists y^0 B_0(y, a)\big)$ *(A_0, B_0 being quantifer-free formulas of* $\mathbf{HA}|\!\backslash$*) such that*

$$\widehat{\mathbf{HA}}^{\omega}|\!\backslash + \mathbf{AC}_{-}^{0,0}\text{-}qf \vdash A, \text{ but } \mathbf{PA}|\!\backslash \not\vdash A,$$

where $\mathbf{AC}_{-}^{0,0}$*-qf is the restriction of* $\mathbf{AC}_{ar}^{0,0}$ *to* **quantifier-free** *formulas* **without function parameters**.

Remark 5 *Note that modified-realizability interpretation yields that* $\widehat{\mathbf{HA}}^{\omega}\upharpoonright + \mathbf{AC}$ *is conservative over* $\mathbf{HA}\upharpoonright$ *w.r.t. sentences having e.g. the form* $A :\equiv \forall a^0 \big(\exists x^0 \forall y^0 A_0(x,y,a) \to \exists y^0 B_0(y,a) \big).$

References

[1] Beeson, M., Goodman's theorem and beyond. Pacific J. Math. **84**, pp. 1-16 (1979).

[2] Beeson, M., Foundations of Constructive Mathematics. Springer Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band 6. Springer Berlin Heidelberg New York Tokyo 1985.

[3] Feferman, S., Theories of finite type related to mathematical practice. In: Barwise, J. (ed.), Handbook of Mathematical Logic, North–Holland, Amsterdam, pp. 913–972 (1977).

[4] Goodman, N., The faithfulness of the interpretation of arithmetic in the theory of constructions. J. Symbolic Logic **38**, pp.453-459 (1973).

[5] Goodman, N., The theory of the Gödel functionals. J. Symbolic Logic **41**, pp. 574-582 (1976).

[6] Goodman, N., Relativized realizability in intuitionistic arithmetic of all finite types. J. Symbolic Logic **43**, pp. 23-44 (1978).

[7] Mints, G.E., Finite investigations of transfinite derivations. J. Soviet Math. **10**, pp. 548-596 (1978).

[8] Parsons, C., On a number theoretic choice schema and its relation to induction. In: Intuitionism and proof theory, pp. 459-473. North-Holland, Amsterdam (1970).

[9] Renardel de Lavalette, G.R., Extended bar induction in applicative theories. Ann. Pure Applied Logic **50**, 139-189 (1990).

[10] Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes in Mathematics **344** (1973).

[11] Troelstra, A.S., Realizability. ILLC Prepublication Series for Mathematical Logic and Foundations ML–92–09, 60 pp., Amsterdam (1992).