On the arithmetical content of restricted forms of comprehension, choice and general uniform boundedness*

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Abstract

In this paper the numerical strength of fragments of arithmetical comprehension, choice and general uniform boundedness is studied systematically. These principles are investigated relative to base systems $\mathcal{T}_n^\omega$ in all finite types which are suited to formalize substantial parts of analysis but nevertheless have provably recursive function(al)s of low growth. We reduce the use of instances of these principles in $\mathcal{T}_n^\omega$-proofs of a large class of formulas to the use of instances of certain arithmetical principles thereby determining faithfully the arithmetical content of the former. This is achieved using the method of elimination of Skolem functions for monotone formulas which was introduced by the author in a previous paper.

As corollaries we obtain new conservation results for fragments of analysis over fragments of arithmetic which strengthen known purely first-order conservation results.

We also characterize the provably recursive function(al)s of type ≤ 2 of the extensions of $\mathcal{T}_n^\omega$ based on these fragments of arithmetical comprehension, choice and uniform boundedness.

1 Introduction

This paper studies the numerical strength of fragments Γ of arithmetical comprehension, choice and uniform boundedness relative to weak base systems, formulated in the language of all finite types, which are suited to formalize substantial parts of analysis.

In a previous paper ([12]) we have introduced a hierarchy $\mathrm{G}_n\mathrm{A}^\omega$ of systems where the definable functions correspond to the well-known Grzegorczyk hierarchy. These systems extended by the schema of full quantifier-free choice

$$AC^{\rho,\tau}\text{-qf} \; : \; \forall x^\rho \exists y^\tau A_0(x,y) \rightarrow \exists Y^{\tau(\rho)} \forall x^\rho A_0(x,Yx), \quad AC\text{-qf} \; := \bigcup_{\rho,\tau\in\mathbf{T}} \{AC^{\rho,\tau}\text{-qf}\},$$

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where A_0 is a quantifier-free formula,[1] and various non-constructive analytical axioms Δ, having the form

$$\forall x^\delta \exists y \leq_\rho sx \forall z^\eta A_0(x,y,z),$$

including a generalized version of the binary König's lemma WKL, allow to carry out a great deal of classical analysis even for $n = 2,3$. The axioms Δ and AC-qf do not contribute to the growth of extractable uniform bounds which in the particular case of $\mathrm{G}_2\mathrm{A}^\omega$ are polynomials (see [12],[14] and in particular [10] for more information).

In contrast to this, fragments of arithmetical comprehension and choice as well as generalizations of our principle of uniform Σ^0_1-boundedness (from [12]) to more complex formulas do contribute significantly to the arithmetical strength of the base systems. In [13] we developed a general method to calibrate faithfully this contribution and applied it to instances of Π^0_1-comprehension and Π^0_1-choice. These results were then used in [15] to determine the arithmetical strength of single sequences of instances of the Bolzano-Weierstraß theorem for bounded sequences in $\mathbb{R}^d$, the Ascoli-lemma and others.

In this paper we give a systematic treatment of the whole arithmetical hierarchy for comprehension, choice and uniform boundedness and determine precisely their arithmetical strength as well as their provably recursive function(al)s of type ≤ 2. We also consider much more complex formulas to be proved in these systems than we did in our previous papers.

In the following let us discuss now some of the difficulties one has to deal with in order to achieve this goal and which indicate already the type of results one can expect. For simplicity we restrict ourselves for the moment to the second-order system $\mathrm{EA}^2+\ \mathrm{AC}^{0,0}$-qf instead of $\mathrm{G}_n\mathrm{A}^\omega+$ AC-qf $+\Delta$ (which we actually are going to consider below).
EA^2 is an extension of Kalmar-elementary arithmetic (with number quantifiers) EA obtained by adding n-ary function quantifiers (for every $n \geq 1$)[2] and the schema of explicit definition of functions

$$ED: \ \exists f \forall \underline{x}\big(f(\underline{x}) = t[\underline{x}]\big),$$

where t is a number term of EA^2 and $\underline{x}$ is a tuple of number variables. Furthermore EA^2 contains the schema of quantifier-free induction for all quantifier-free formulas of EA^2 which may contain function parameters. Finally EA^2 contains constants and their defining equations for all elementary recursive functionals of type ≤ 2.
In EA^2 the schema of quantifier-free induction can be expressed equivalently as a single axiom

$$\text{QF-IA} \ : \ \forall f\big(f(0) = 0 \wedge \forall x\big(f(x) = 0 \to f(x') = 0\big) \to \forall x(f(x) = 0)\big).$$

Analogously Σ^0_k-IA is the induction axiom for $\exists y^0_1 \forall y^0_2 \ldots \forall^{(d)} y^0_k f(x,\underline{y}) = 0$ instead of $fx = 0$. In first-order contexts this is replaced by a schema with $\exists y^0_1 \forall y^0_2 \ldots \forall^{(d)} y^0_k A_0(x,\underline{y})$ as induction formulas. Let us consider furthermore the restriction of arithmetical choice to Π^0_1- (or equivalently to Σ^0_2-) formulas of $\mathcal{L}(\mathrm{EA}^2)$ which like QF-IA can be expressed as a single second-order axiom $\forall f\, \Pi^0_1\text{-AC}(f)$,

[1] Throughout this paper $A_0, B_0, C_0, \ldots$ denote quantifier-free formulas. We allow bounded number quantifiers $\forall x \leq_0 t$, $\exists x \leq_0 t$ to occur in $A_0, B_0, C_0, \ldots$ since they can be expressed in a quantifier-free way using the bounded search-functional μ_b from $\mathrm{G}_n\mathrm{A}^\omega$. $\mathbf{T}$ denotes the set of all finite types.

[2] Since coding of finite tuples of numbers is available in EA one can in fact restrict oneself to unary function variables.

where[3]

$$\Pi^0_1\text{-AC}(f) :\equiv \forall a^0 \big(\forall x^0 \exists y^0 \forall z^0 (f(a,x,y,z)=0) \to \exists g \forall x,z (f(a,x,gx,z)=0)\big).$$

Now by iteration one easily verifies that $\mathrm{EA}^2 + \forall f\, \Pi^0_1\text{-AC}(f)$ proves already full arithmetical choice. So in order to prevent the arithmetical hierarchy of choice principles from collapsing we restrict ourselves to single instances of $\forall f \Pi^0_1\text{-AC}(f)$ which later on are allowed however to depend on the parameters of the theorem to be proved. For the moment we forbid completely the occurrence of function parameters in Π^0_1-AC, i.e. we consider the schema

$$\Pi^0_1\text{-AC}^- \,:\, \forall x^0 \exists y^0 A(x,y) \to \exists g \forall x\, A(x,gx),$$

where $A(x,y)$ is a Π^0_1-formula without **function parameters**.
As a starting point for the introduction into our general program let us consider now the following question:

What arithmetical statements are provable in EA^2+ $\mathrm{AC}^{0,0}$-qf $+\Pi^0_1\text{-AC}^-$?

A first observation is that $\Pi^0_1\text{-AC}^-$ proves $\Pi^0_1\text{-CA}^-$, i.e.

$$\exists f \forall x \big(f(x)=0 \leftrightarrow A(x)\big),$$

where $A(x)$ is a Π^0_1-formula without function parameters. Combined with the axiom QF-IA this yields every function parameter-free instance of Σ^0_1-IA. Hence the first-order system EA $+\Sigma^0_1$-IA is a subsystem of EA^2+ $\mathrm{AC}^{0,0}$-qf $+\Pi^0_1\text{-AC}^-$.

What is the precise relationship between EA^2+ $\mathrm{AC}^{0,0}$-qf $+\Pi^0_1\text{-AC}^-$ and EA $+\Sigma^0_1$-IA?
It will turn out that the former theory is conservative over the latter for **some** formulas, including Π^0_3-sentences, but not for all formulas.
That EA^2+ $\mathrm{AC}^{0,0}$-qf cannot be conservative over EA $+\Sigma^0_1$-IA without some restriction imposed on the formulas follows from the following observation:

By applying the functional $\Phi_{\max} fx := \max\limits_{i\le x}(f(i))$ to the function g in $\Pi^0_1\text{-AC}^-$ one obtains the corresponding instance of the so-called (bounded) collection principle for Π^0_1-formulas

$$\Pi^0_1\text{-CP} \;:\; \forall x \le a \exists y\, A(x,y) \to \exists z \forall x \le a \exists y \le z\, A(x,y),$$

where $A \in \Pi^0_1$.
So EA^2+ $\mathrm{AC}^{0,0}$-qf $+\Pi^0_1\text{-AC}^-$ proves every function parameter-free instance of Π^0_1-CP, i.e. EA $+\Pi^0_1$-CP is a subsystem of EA^2+ $\mathrm{AC}^{0,0}$-qf $+\Pi^0_1\text{-AC}^-$.
It is well-known (see [19]) that there exists an instance A of Π^0_1-CP which is not provable in EA $+\Sigma^0_1$-IA. On the other hand EA $+\Pi^0_1$-CP is Π^0_3-conservative over EA $+\Sigma^0_1$-IA by a result due to H. Friedman and (implicitly) J.Paris/L.Kirby [18] (see e.g. [7] for details). The universal closure of the instance A of Π^0_1-CP can be shown to be equivalent to a Π^0_4-sentence in EA $+\Sigma^0_1$-IA. Hence EA^2+ $\mathrm{AC}^{0,0}$-qf $+\Pi^0_1\text{-AC}^-$ is not Π^0_4-conservative over EA $+\Sigma^0_1$-IA.

[3]The universal closure with respect to number parameters a^0 is superfluous for $\forall f\, \Pi^0_1\text{-AC}(f)$ since it can be captured by the universal closure $\forall f$. However below we consider single instances $\Pi^0_1\text{-AC}(\xi)$ of $\forall f\, \Pi^0_1\text{-AC}(f)$ where it does make a difference. Because of the closure w.r.t. arithmetical parameters a^0 a single instance $\Pi^0_1\text{-AC}(\xi)$ contains a whole sequence of instances of Π^0_1-AC.

Here is another arithmetical use of Π^0_1-AC$^-$ we can make relative to EA^2+ AC^{0,0}-qf:

As mentioned above, Π^0_1-CA$^-$ is a trivial consequence of Π^0_1-AC$^-$ (in the presence of classical logic). Now combining Π^0_1-CA$^-$ with AC^{0,0}-qf one can easily prove Δ^0_2-CA$^-$ and therefore every function parameter-free instance of Δ^0_2-IA. Hence EA $+\Delta^0_2$-IA is a subsystem of EA^2+ AC^{0,0}-qf $+\Pi^0_1$-AC$^-$ as well even if the functional $\Phi_{\max}$ would not be included in EA^2.

So the arithmetical strength of Π^0_1-AC$^-$ depends heavily on the second-order axioms, like QF-IA, AC^{0,0}-qf and the characterizing axioms for functionals as $\Phi_{\max}$, which are available in the context in which Π^0_1-AC$^-$ is considered.[4]

As a special corollary of the results of this paper it follows that EA^2+ AC^{0,0}-qf $+\Pi^0_k$-AC$^-$ is Π^0_{k+2}-conservative over EA $+\Sigma^0_k$-IA, which implies the result of H. Friedman, J.Paris/L.Kirby. Furthermore we show that EA^2+ AC^{0,0}-qf $+\Pi^0_k$-AC$^-$ is conservative over EA $+\Sigma^0_k$-IA w.r.t. monotone formulas of arbitrary complexity. These results are sensitive to small changes of the base system EA^2: E.g. if we add the primitive recursive functional Φ_{it} defined by

$$\Phi_{it} fg0 := g(0) \quad \Phi_{it} fgx' := f(x, \Phi_{it} fgx)$$

to EA^2, then the Ackermann-function becomes provably total in EA$^2+\Phi_{it}+$ AC^{0,0}-qf $+\Pi^0_1$-AC$^-$ and the resulting system proves the consistency of EA $+\Sigma^0_1$-IA: EA$^2+\Phi_{it}+$ AC^{0,0}-qf proves the second-order axiom of Σ^0_1-induction. Combined with Π^0_1-CA$^-$ one obtains every function parameter-free instance of Σ^0_2-IA. Hence EA $+\Sigma^0_2$-IA (which is known to prove the totality of the Ackermann-function as well as the consistency of EA $+\Sigma^0_1$-IA) is a subsystem of EA$^2+\Phi_{it}+$ AC^{0,0}-qf $+\Pi^0_1$-AC$^-$.
Using a more involved argument one can show that already EA$^2+\Phi_{it}+\Pi^0_1$-AC$^-$ proves the totality of the Ackermann function (see chapter 12 of [10] for details on this).

So any proof of conservation of systems based on Π^0_k-AC$^-$ over Σ^0_k-IA has to take into account carefully the structure of the functionals of type 2 which are definable in the given system.

Things become of course even more complicated for the systems G$_n$A$^\omega$+ AC-qf $+\Delta$ instead of EA^2+ AC^{0,0}-qf which we are treating in this paper.
Among other things we show that relative to base systems $\mathcal{T}^\omega_n :=$ G$_n$A$^\omega$+ AC-qf $(+\Delta)$ the use of Δ^0_{k+1}-CA$(\xi_1 f)$ and Π^0_k-AC$(\xi_2 f)$ in a proof of a formula $B_{ar}(f) \in \Pi^0_{k+2}$ can be reduced to the use of Σ^0_k-IA.

This is true also for $B_{ar}(f)$ of arbitrary complexity in the arithmetical hierarchy if $B_{ar}(f)$ is monotone in the sense of definition 2.3 below.

We also show that the provably recursive function(al)s of type ≤ 2 of G$_n$A$^\omega$+ AC-qf + WKL $+\Delta^0_{k+1}$-CA$^-$ + Π^0_k-AC$^-$ are just the functionals of these types definable in T_{k-1} $(k \geq 1)$, where T_k is the fragment of Gödel's T with recursion up to the type k only.

[4] Both aspects are not taken into account appropriately in [22] where Π^0_k-CA$^-$ and Π^0_k-AC$^-$ are studied systematically for the first time. As a consequence of this, theorems 5.8,5.13 and corollaries 5.9,5.14 in [22] are not correct as stated (see [11] and in particular chapter 12 of [10] for a thorough investigation of this matter).

These results are used to prove new conservation results for EA $+\Pi^0_k$-CP over EA $+\Sigma^0_k$-IA which strengthen the Friedman-Paris-Kirby result.[5]

Finally we consider generalizations Π^0_k-UB$^-\restriction$ of the principle of uniform Σ^0_1-boundedness Σ^0_1-UB$^-$ which was studied in [12].[6] In [14] we showed that Σ^0_1-UB$^-$ proves already relative to G$_2$A$^\omega$+ AC-qf many important analytical theorems (like Dini's theorem, the attainment of the maximum for $f \in C([0,1]^d, \mathbb{R})$, the sequential Heine-Borel property for $[0,1]^d$, the existence of an inverse function for every strictly monotone function $f \in C[0,1]$ and others) but does not contribute to the growth of extractable bounds, thereby guaranteeing the extractability of polynomial bounds when applied in the context of G$_2$A$^\omega$+ AC-qf.
Whereas the straightforward generalization of Σ^0_1-UB$^-$ to Π^0_k-formulas is inconsistent with G$_2$A$^\omega$ already for $k=1$, our restricted version Π^0_k-UB$^-\restriction$ (introduced in the present paper) is consistent. In [15] we implicitly used (a special case of) Π^0_1-UB$^-\restriction$ to prove the Bolzano-Weierstraß principle and the Ascoli-lemma and it were these proofs which were used to calibrate faithfully the arithmetical strength of these principles.

We show that our results on fragments of arithmetical comprehension and choice mentioned above remain valid if in addition to Δ^0_{k+1}-CA$(\xi_1 f) \wedge \Pi^0_k$-AC$(\xi_2 f)$ also Π^0_k-UB$^-\restriction(\xi_3 f)$ is used in the proof of $B_{ar}(f)$.

2 Monotone formulas and their Skolem normal forms

In this section we review some of the proof-theoretic tools from [13] on which the present paper is based and also recall some of the basic concepts and definitions from [12].
The set $\mathbf{T}$ of all finite types is defined inductively by

$$(i)\ 0 \in \mathbf{T} \text{ and } (ii)\ \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type ρ to objects of type τ.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

$$(i)\ 0 \in \mathbf{P} \text{ and } (ii)\ \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}.$$

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write $0(00)$ instead of $0(0(0))$. Furthermore we write for short $\tau\rho_k \ldots \rho_1$ instead of $\tau(\rho_k)\ldots(\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n+1$. The types $0, 00, 0(00), 0(0(00))\ldots$ are so represented by $0,1,2,3\ldots$. For arbitrary types $\rho \in \mathbf{T}$ the degree of ρ (for short deg(ρ)) is defined by deg$(0) := 0$ and deg$(\tau(\rho)) :=$ max(deg(τ),deg$(\rho)+1$). For pure types the degree is just the number which represents this type.

Description of the theories (E)–G$_n$A$^\omega$

[5] A proof-theoretic treatment of the Friedman-Paris-Kirby result was first given in [22]. However the proof in [22] contains gap. See [1] for a correction of Sieg's proof. Another proof-theoretic treatment can be found in [3].

[6] Whereas we generally use the superscript '$-$' to denote the restriction S^- of a schema S to function parameter-free instances of S, this superscript has a different meaning in the context of principles of uniform boundedness. Although this might be troublesome we wish to stick to the notation for these principles from [12] where they were introduced.

Our theories $\mathcal{T}^\omega$ used in this paper are based on many–sorted classical logic formulated in the language of all finite types plus the combinators $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$ which allow the definition of λ–abstraction. $\mathcal{T}^\omega_i$ denotes the intuitionistic variant of $\mathcal{T}^\omega$.
The systems $\mathrm{G}_n\mathrm{A}^\omega$ (for all $n \geq 1$) are introduced in [12] to which we refer for details. $\mathrm{G}_n\mathrm{A}^\omega$ has as primitive relations $=_0, \leq_0$ for objects of type 0, the constant 0^0, functions $\min_0, \max_0, S^{00}$ (successor), $A_0, \ldots, A_n$, where A_i is the i–th branch of the Ackermann function (i.e. $A_0(x,y) = y', A_1(x,y) = x+y, A_2(x,y) = x \cdot y, A_3(x,y) = x^y, \ldots$), functionals of degree 2: $\Phi_1, \ldots, \Phi_n$, where $\Phi_1 fx = \max_0(f0, \ldots, fx)$ and Φ_i is the iteration of A_{i-1} on the f–values for $i \geq 2$, i.e. $\Phi_2 fx = \sum\limits_{i=0}^{x} fi, \Phi_3 fx = \prod\limits_{i=0}^{x} fi, \ldots$. We also have a bounded search functional μ_b and bounded predicative recursion provided by recursor constants $\tilde{R}_\rho$ (where 'predicative' means that recursion is possible only at the type 0 as in the case of the (unbounded) Kleene-Feferman recursors $\widehat{R}_\rho$). Moreover $\mathrm{G}_n\mathrm{A}^\omega$ contains a quantifier-free rule of extensionality QF–ER.
In addition to the defining axioms for the constants of our theories all true sentences having the form $\forall x^\rho A_0(x)$, where A_0 is quantifier–free and $deg(\rho) \leq 2$, are added as axioms. By 'true' we refer to the full set–theoretic model $\mathcal{S}^\omega$. In given proofs however only very special universal axioms will be used which can be proved in suitable extensions of our theories. Nevertheless we include them all as axioms in order to emphasize that (proofs of) universal sentences do not contribute to the growth of extractable bounds. In particular this covers all instances of the schema of quantifier-free induction (The main results in this paper are also valid for the variant of $\mathrm{G}_n\mathrm{A}^\omega_i$ where the universal axioms are replaced by the schema of quantifier–free induction). The restriction $deg(\rho) \leq 2$ has a technical reason discussed in [12].

$\mathrm{G}_\infty\mathrm{A}^\omega := \bigcup\limits_{n \in \mathbb{N}} \mathrm{G}_n\mathrm{A}^\omega$.

PA^ω, PA^ω_i are the extensions of $\mathrm{G}_n\mathrm{A}^\omega$, $\mathrm{G}_n\mathrm{A}^\omega_i$ obtained by the addition of the schema of full induction and all (impredicative) primitive recursive functionals in the sense of [5].
E–$\mathcal{T}^\omega_{(i)}$ denotes the theory which results from $\mathcal{T}^\omega_{(i)}$ when the quantifier–free rule of extensionality is replaced by the axioms of extensionality (E)

$$\forall x^\rho, y^\rho, z^{\tau\rho}(x =_\rho y \to zx =_\tau zy)$$

for all finite types ($x =_\rho y$ is defined as $\forall z_1^{\rho_1}, \ldots, z_k^{\rho_k}(xz_1 \ldots z_k =_0 yz_1 \ldots z_k)$ where $\rho = 0\rho_k \ldots \rho_1$).
$\mathrm{G}_n\mathrm{R}^\omega$ and T denote the sets of all closed terms of (E)–$\mathrm{G}_n\mathrm{A}^\omega_{(i)}$ and (E)–$\mathrm{PA}^\omega_{(i)}$. T_k is the subset of all closed terms of T which contain the Gödel-recursors R_ρ for ρ of degree $\leq k$ only.

Definition 2.1 *Between functionals of type ρ we define relations $\leq_\rho$ ('less or equal') and s–maj$_\rho$ ('strongly majorizes') by induction on the type:*

$$\begin{cases} x_1 \leq_0 x_2 :\equiv (x_1 \leq_0 x_2), \\ x_1 \leq_{\tau\rho} x_2 :\equiv \forall y^\rho (x_1 y \leq_\tau x_2 y); \end{cases}$$

$$\begin{cases} x^* \ s\text{–}maj_0 \ x :\equiv x^* \geq_0 x, \\ x^* \ s\text{–}maj_{\tau\rho} \ x :\equiv \forall y^{*\rho}, y^\rho (y^* \ s\text{–}maj_\rho \ y \to x^* y^* \ s\text{–}maj_\tau \ x^* y, \ xy). \end{cases}$$

Remark 2.2 *'s–maj' is a variant of W.A. Howard's relation 'maj' from [6] which is due to [2]. For more details see [8].*

Let $A(\underline{a})$ be a formula of $\mathrm{G}_n\mathrm{A}^\omega$ ($\underline{a}$ are all free variables of A) and $\exists\underline{x}\forall\underline{y}A_D(\underline{x},\underline{y},\underline{a})$ its Gödel functional interpretation (see e.g. [25] for details on Gödel's functional interpretation). We say that a tuple of closed terms $\underline{t}$ realizes the **monotone** functional interpretation of $A(\underline{a})$ if[7]

$$(*)\ \exists\underline{x}\big(\underline{t}\ \text{s-maj}\ \underline{x} \wedge \forall\underline{a},\underline{y}\, A_D(\underline{x}\,\underline{a},\underline{y},\underline{a})\big)$$

(Monotone functional interpretation which directly extracts a tuple $\underline{t}$ satisfying $(*)$ from a proof of $A(\underline{a})$ was introduced in [9]. See also [12] for details.)

We next define what it means for a formula to be 'monotone'. In order to motivate the somewhat technical definition lets consider the simple case of a Σ^0_2-formula $A \equiv \exists y\forall x A_0(y,x)$. A is monotone if

$$\tilde{y} \geq y \wedge \tilde{x} \leq x \to \big(A_0(x,y) \to A_0(\tilde{x},\tilde{y})\big).$$

Innermost existential quantifiers and outmost universal quantifiers are not supposed to be monotone. Hence we get the following

Definition 2.3 ([13]) *Let $A \in \mathcal{L}(G_nA^\omega)$ be a formula having the form*

$$A \equiv \forall u^1 \forall v \leq_\tau tu \exists y^0_1 \forall x^0_1 \ldots \exists y^0_k \forall x^0_k \exists w^\gamma A_0(u,v,y_1,x_1,\ldots,y_k,x_k,w),$$

where A_0 is quantifier–free and contains only $u,v,\underline{y},\underline{x},w$ free, $t \in G_nR^\omega$ and τ,γ are arbitrary finite types.

1) A is called (arithmetically) **monotone** *if*

$$Mon(A) :\equiv \begin{cases} \forall u^1\forall v \leq_\tau tu \forall x_1,\tilde{x}_1,\ldots,x_k,\tilde{x}_k,y_1,\tilde{y}_1,\ldots y_k,\tilde{y}_k \\ \qquad \Big(\bigwedge\limits_{i=1}^{k}(\tilde{x}_i \leq_0 x_i \wedge \tilde{y}_i \geq_0 y_i) \wedge \exists w^\gamma A_0(u,v,y_1,x_1,\ldots,y_k,x_k,w) \\ \qquad\qquad \to \exists w^\gamma A_0(u,v,\tilde{y}_1,\tilde{x}_1,\ldots,\tilde{y}_k,\tilde{x}_k,w)\Big). \end{cases}$$

2) The **Herbrand normal form** *A^H of A is defined to be*

$$A^H :\equiv \forall u^1 \forall v \leq_\tau tu \forall h_1^{\rho_1},\ldots,h_k^{\rho_k} \exists y^0_1,\ldots,y^0_k,w^\gamma \underbrace{A_0(u,v,y_1,h_1y_1,\ldots,y_k,h_ky_1\ldots y_k,w)}_{A_0^H:\equiv},\ \text{where } \rho_i = 0\underbrace{(0)\ldots(0)}_{i}.$$

Remark 2.4 *In definition 2.3 (and theorems 2.5,2.7 below) one may also have tuples '$\exists\underline{w}$' instead of '$\exists w^\gamma$' in A where $\underline{w} = w_1^{\gamma_1},\ldots,w_l^{\gamma_l}$ and γ_i is arbitrary. Also instead of $\forall u^1$ we may have $\forall\underline{u}$ where $\underline{u} = u_1^{\rho_1},\ldots,u_q^{\rho_q}$ with $deg(\rho_i) \leq 1$ for $1 \leq i \leq q$. In particular we can consider an innermost existential number quantifier $\exists y^0_{k+1}$ as part of $\exists\underline{w}$ and an outermost universal number quantifier $\forall x^0_0$ as part of $\forall\underline{u}$. So for $\forall x^0_0$ and $\exists y^0_{k+1}$ no monotonicity is required in definition 2.3.1).*

[7] Here $\underline{t}$ s-maj $\underline{x}$ means $\bigwedge\limits_i(t_i \text{ s-maj } x_i)$.

Theorem 2.5 ([13]) *Let $n \geq 1$ and $\Psi_1, \ldots, \Psi_k \in G_nR^\omega$. Then*

$$G_nA^\omega + Mon(A) \vdash \forall u^1 \forall v \leq_\tau tu \forall h_1, \ldots, h_k \Big(\bigwedge_{i=1}^{k} (h_i \text{ monotone}) \\ \to \exists y_1 \leq_0 \Psi_1 u\underline{h} \ldots \exists y_k \leq_0 \Psi_k u\underline{h} \exists w^\gamma A_0^H \Big) \to A,$$

where (h_i monotone) $:\equiv \forall x_1, \ldots, x_i, y_1, \ldots, y_i \big(\bigwedge_{j=1}^{i} (x_j \geq_0 y_j) \to h_i\underline{x} \geq_0 h_i\underline{y} \big)$.

Definition 2.6 (Bounded choice) $b\text{–}AC := \bigcup_{\delta,\rho \in \mathbf{T}} \Big\{ (b\text{–}AC^{\delta,\rho}) \Big\}$ *denotes the schema of bounded choice*

$$(b\text{–}AC^{\delta,\rho}) \ : \ \forall Z^{\rho\delta} \big(\forall x^\delta \exists y \leq_\rho Zx \ A(x,y,Z) \to \exists Y \leq_{\rho\delta} Z \forall x A(x, Yx, Z) \big).$$

In general $\mathrm{G}_n\mathrm{A}^\omega \vdash A^H$ does not imply $\mathrm{G}_n\mathrm{A}^\omega \vdash A$ (see [13] for a detailed discussion of this phenomenon), which is in contrast to the first-order case where the derivability of A^H follows from that of A by Herbrand's theorem (see [21]). If however A is monotone then this rule is valid also for $\mathrm{G}_n\mathrm{A}^\omega$ (but for very different reasons):

Theorem 2.7 ([13]) *Let A be as in thm.2.5 and Δ be a set of sentences $\forall x^\delta \exists y \leq_\rho sx \forall z^\eta G_0(x,y,z)$ where s is a closed term of G_nA^ω and G_0 a quantifier-free formula, and let A' denote the negative translation[8] of A. Then the following rule holds:*

$$\begin{cases} G_nA^\omega + AC\text{–}qf + \Delta \vdash A^H \wedge Mon(A) \Rightarrow \\ G_nA^\omega + \tilde{\Delta} \ \vdash A \ \textit{and} \\ \textit{by monotone functional interpretation one can extract a tuple } \underline{\Psi} \in \ G_nR^\omega \textit{ such that} \\ G_nA_i^\omega + \tilde{\Delta} \vdash \underline{\Psi} \textit{ satisfies the monotone functional interpretation of } A', \end{cases}$$

where $\tilde{\Delta} := \{\exists Y \leq_{\rho\delta} s \forall x^\delta, z^\eta G_0(x, Yx, z) : \forall x^\delta \exists y \leq_\rho sx \forall z^\eta G_0(x,y,z) \in \Delta\}$. (In particular the second conclusion can be proved in $G_nA_i^\omega + \Delta +$ b-AC).
The weakened conclusion $G_nA^\omega + \tilde{\Delta} + Mon(A) \vdash A$ follows already from $G_nA^\omega +$ AC-qf $+\Delta \vdash A^H$.[9]

3 Making arithmetical comprehension monotone

In this section we consider the arithmetical content of instances Π_k^0-CA(ξuv) of Π_k^0-CA which are used in given proofs of sentences $\forall u^1 \forall v \leq_\tau tu \, B_{ar}(u,v)$ as discussed in the introduction.

Definition 3.1

$$\Pi_k^0\text{-}CA(f) :\equiv \exists g^1 \forall x^0 \big(gx =_0 0 \leftrightarrow \forall u_1^0 \exists u_2^0 \ldots \exists^{(d)} u_k^0 \big(f(x, \underline{u}) =_0 0 \big) \big).^{10}$$

[8]Here we can use Gödel's [4] translation or any other of the various negative translations. For a systematical treatment of negative translations see [17].

[9]This last assertion is not stated in the formulation of the theorem in [13] but does follow immediately from its proof.

[10]Whether one has here '$\exists u_k^0$' or '$\forall u_k^0$' depends of course on whether k is even or odd.

Remark 3.2 *There is no need here to incorporate closure under number parameters in the definition of* Π^0_k*-CA(f), i.e. by defining*

$$\Pi^0_k\text{-}CA(f) :\equiv \forall l^0 \exists g^1 \forall x^0 \big(gx =_0 0 \leftrightarrow \forall u^0_1 \exists u^0_2 \ldots \exists^{(d)} u^0_k \big(f(l,x,\underline{u}) =_0 0\big)\big),$$

since the latter can be reduced to the former (relative to G_nA^ω *for* $n \geq 2$*) by coding* l, x *together and applying comprehension without number parameters to this pair.*

In order to be able to apply the method of elimination of Skolem functions for monotone formulas from section 2 we follow this strategy:

Construct an arithmetical principle $A_{ar}(f)$ such that for suitable $\xi_1, \xi_2 \in \mathrm{G}_n\mathrm{R}^\omega$:

1) $\mathrm{G}_n\mathrm{A}^\omega \vdash Mon(\forall f\, A_{ar}(f))$,
2) $\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{0,0}\text{-qf} \vdash \forall f \big(A^S_{ar}(\xi_1 f) \to \Pi^0_k\text{-CA}(f)\big)$ and
3) $\mathrm{G}_n\mathrm{A}^\omega \vdash \forall f \big(\Pi^0_k\text{-CA}(\xi_2 f) \to A_{ar}(f)\big)$.

Because of 2) the use of Π^0_k-CA(ξuv) in a given proof of a monotone sentence $\forall u^1 \forall v \leq_\tau tu\, B_{ar}(u,v)$ can be reduced to the use of $A^S_{ar}(\xi' uv)$ (where $\xi' uv := \xi_1(\xi uv)$) which in turn (by 1) and theorem 2.7) can be reduced to the use of $A_{ar}(\xi' uv)$. Because of 3) nothing is lost by this reduction.
It will turn out that the correct principle $A_{ar}(f)$ is a 'monotone version' $\Pi^0_k\text{-TND}^{mon}(f)$ of the tertium-non-datur principle for Π^0_k-formulas.

Definition 3.3 *In the following* $m := \frac{k}{2}$ *if* k *is even (resp.* $m := \frac{k-1}{2}$ *if* k *is odd).*

1) *The* Π^0_k*-tertium-non-datur axiom is given by the following formula (where* f *is a function variable of appropriate type)*[11]

$$\underline{\Pi^0_k\text{-}TND(f)} :\equiv$$
$$\begin{cases} \forall x^0 \big(\forall y^0_1 \exists z^0_1 \ldots \forall y^0_m \exists z^0_m (\forall y^0_{m+1}) \big(f(x,y_1,z_1,\ldots,y_m,z_m,(y_{m+1})) =_0 0\big) \\ \qquad \vee \exists u^0_1 \forall v^0_1 \ldots \exists u^0_m \forall v^0_m (\exists u^0_{m+1}) \big(f(x,u_1,v_1,\ldots,u_m,v_m,(u_{m+1})) \neq 0\big)\big), \end{cases}$$

2) *We also need the following prenex normal form of* Π^0_k*-TND(f):*

$$\underline{\Pi^0_k\text{-}TND(f)^{pr}} :\equiv$$
$$\begin{cases} \forall x^0 \exists u^0_1 \forall y^0_1 \exists z^0_1 \forall v^0_1 \ldots \exists u^0_m \forall y^0_m \exists z^0_m \forall v^0_m (\exists u^0_{m+1} \forall y^0_{m+1}) \\ \qquad \big(f(x,y_1,z_1,\ldots,y_m,z_m,(y_{m+1})) =_0 0 \vee f(x,u_1,v_1,\ldots,u_m,v_m,(u_{m+1})) \neq 0\big), \end{cases}$$

3) *The Skolem normal form of* Π^0_k*-TND(f)*^{pr} *is given by*

$$\underline{\big(\Pi^0_k\text{-}TND(f)^{pr}\big)^S} :\equiv$$
$$\begin{cases} \exists h_1,\ldots,h_m,(h_{m+1}),g_1,\ldots,g_m \forall x^0, y^0_1, v^0_1, \ldots, y^0_m, v^0_m, (y_{m+1}) \\ \big(f(x,y_1,g_1(x,y_1),\ldots,y_m,g_m(x,y_1,\ldots,y_m,v_1,\ldots,v_{m-1}),(y_{m+1})) =_0 0 \vee \\ f(x,h_1x,v_1,\ldots,h_m(x,y_1,\ldots,y_{m-1},v_1,\ldots,v_{m-1}),v_m,(h_{m+1}(x,y_1,\ldots,y_m,v_1,\ldots,v_m))) \neq 0\big). \end{cases}$$

[11] Here and in the following the quantifiers $\forall y^0_{m+1}, \exists u^0_{m+1}$ are only present if k is odd.

Remark 3.4 *For $n \geq 2$ we have coding of finite tuples (of fixed length) available in G_nA^ω. Hence quantifier-blocks can be contracted to a single quantifier. Since in all of our results we assume that (at least) $n \geq 2$, it is no restriction in the definition above to consider only single quantifiers.*

Lemma 3.5 *For every $k \in \mathbb{N}$ the following implication can be proved in G_1A^ω:*

$$\forall f \big(\big(\Pi^0_k\text{-}TND(f)^{pr} \big)^S \to \Pi^0_k\text{-}CA(f) \big).$$

Proof:
For notational simplicity we confine ourselves to the case $k = 4$ which well shows the general pattern of the proof for arbitrary k:
$(\Pi^0_4\text{-TND}(f)^{pr})^S$ yields the existence of functions g_1, g_2, h_1, h_2 such that

$$(1)\ \forall x, y_1, v_1, y_2 \big(f(x, y_1, g_1(x, y_1), y_2, g_2(x, y_1, y_2, v_1)) = 0 \vee \forall v_2 (f(x, h_1x, v_1, h_2(x, y_1, v_1), v_2) \neq 0) \big).$$

(1) in turn yields

$$(2)\ \forall x, y_1, v_1 \big(\forall y_2 \exists z_2 f(x, y_1, g_1(x, y_1), y_2, z_2) = 0 \vee \forall v_2 (f(x, h_1x, v_1, h_2(x, y_1, v_1), v_2) \neq 0) \big),$$

$$(3)\ \forall x, y_1, v_1 \big(\forall y_2 \exists z_2 f(x, y_1, g_1(x, y_1), y_2, z_2) = 0 \vee \exists u_2 \forall v_2 (f(x, h_1x, v_1, u_2, v_2) \neq 0) \big),$$

$$(4)\ \forall x, y_1 \big(\forall y_2 \exists z_2 f(x, y_1, g_1(x, y_1), y_2, z_2) = 0 \vee \forall v_1 \exists u_2 \forall v_2 (f(x, h_1x, v_1, u_2, v_2) \neq 0) \big),$$

$$(5)\ \forall x, y_1 \big(\exists z_1 \forall y_2 \exists z_2 f(x, y_1, z_1, y_2, z_2) = 0 \vee \forall v_1 \exists u_2 \forall v_2 (f(x, h_1x, v_1, u_2, v_2) \neq 0) \big)$$

and finally

$$(6)\ \forall x \big(\forall y_1 \exists z_1 \forall y_2 \exists z_2 f(x, y_1, z_1, y_2, z_2) = 0 \vee \forall v_1 \exists u_2 \forall v_2 (f(x, h_1x, v_1, u_2, v_2) \neq 0) \big).$$

(1) applied to $y_1 := h_1x, v_1 := g_1(x, h_1x), y_2 := h_2(x, h_1x, g_1(x, h_1x))$ gives

$$(*) :\equiv$$
$$\forall x^0 \Big(f\big(x, h_1x, g_1(x, h_1x), h_2(x, h_1x, g_1(x, h_1x)), g_2(x, h_1x, h_2(x, h_1x, g_1(x, h_1x)), g_1(x, h_1x))\big) = 0$$
$$\vee \forall v_2 \big(f\big(x, h_1x, g_1(x, h_1x), h_2(x, h_1x, g_1(x, h_1x)), v_2\big) \neq 0 \big) \Big).$$

We now show $(+) :\equiv$

$$\forall x^0 \Big(f\big(x, h_1x, g_1(x, h_1x), h_2(x, h_1x, g_1(x, h_1x)), g_2(x, h_1x, h_2(x, h_1x, g_1(x, h_1x)), g_1(x, h_1x))\big) = 0$$
$$\leftrightarrow \forall y_1 \exists z_1 \forall y_2 \exists z_2 \big(f(x, y_1, z_1, y_2, z_2) = 0 \big) \Big).$$

(+) yields the claim of the lemma with

$$gx := \Phi x h_1 h_2 g_1 g_2 :=$$
$$f\big(x, h_1x, g_1(x, h_1x), h_2(x, h_1x, g_1(x, h_1x)), g_2(x, h_1x, h_2(x, h_1x, g_1(x, h_1x)), g_1(x, h_1x))\big).$$

Proof of (+):

'→': $\Phi x f h_1 h_2 g_1 g_2 = 0$ implies

$$\neg\forall v_2\big(f(x, h_1x, g_1(x, h_1x), h_2(x, h_1x, g_1(x, h_1x)), v_2) \neq 0\big).$$

Hence by (2) (putting $y_1 := h_1x, v_1 := g_1(x, h_1x)$)

$$\forall y_2 \exists z_2 \big(f(x, h_1x, g_1(x, h_1x), y_2, z_2) = 0\big)$$

and therefore

$$\exists z_1 \forall y_2 \exists z_2 \big(f(x, h_1x, z_1, y_2, z_2) = 0\big),$$

i.e.

$$\neg\forall v_1 \exists u_2 \forall v_2 \big(f(x, h_1x, v_1, u_2, v_2) \neq 0\big).$$

By (6) this implies

$$\forall y_1 \exists z_1 \forall y_2 \exists z_2 (f(x, y_1, z_1, y_2, z_2) = 0).$$

'←': $\Phi x f h_1 h_2 g_1 g_2 \neq 0$ implies by (∗)

$$\forall v_2\big(f(x, h_1x, g_1(x, h_1x), h_2(x, h_1x, g_1(x, h_1x)), v_2) \neq 0\big)$$

and therefore

$$\exists u_2 \forall v_2 \big(f(x, h_1x, g_1(x, h_1x), u_2, v_2) \neq 0\big),$$

i.e.

$$\neg\forall y_2 \exists z_2 \big(f(x, h_1x, g_1(x, h_1x), y_2, z_2) = 0\big).$$

By (4) this yields (putting $y_1 := h_1x$)

$$\forall v_1 \exists u_2 \forall v_2 \big(f(x, h_1x, v_1, u_2, v_2) \neq 0\big)$$

and therefore

$$\exists u_1 \forall v_1 \exists u_2 \forall v_2 \big(f(x, u_1, v_1, u_2, v_2) \neq 0\big),$$

which concludes the proof of (+) and hence of the lemma.

Definition 3.6 *For a Π^0_k-formula $A(\underline{a}) \equiv \forall x^0_1 \exists x^0_2 \ldots \exists^{(d)} x^0_k A_0(\underline{a}, x_1, x_2, \ldots, x_k)$ of G_nA^ω (where $\underline{a}$ are all free variables of A which may have arbitrary type) we define*
$\tilde{A}(\underline{a}) :\equiv \forall x^0_1 \exists x^0_2 \ldots \exists^{(d)} x^0_k \forall \tilde{x}_1 \leq x_1 \exists \tilde{x}_2 \leq x_2 \ldots \exists^{(d)} \tilde{x}_k \leq x_k A_0(\underline{a}, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k)$.

In the following we need a variant Mon^* of Mon where monotonicity is required for **all** number quantifiers (compare this with remark 2.4):

Definition 3.7 *Let $A(\underline{a}) :\equiv \forall x^0_1 \exists y^0_1 \ldots \forall x^0_k \exists y^0_k A_0(\underline{a}, x_1, y_1, \ldots, x_k, y_k)$.*[12] *Then*

$$Mon^*\big(A(\underline{a})\big) :\equiv \forall x_1, \tilde{x}_1, y_1, \tilde{y}_1, \ldots, x_k, \tilde{x}_k, y_k, \tilde{y}_k$$
$$\big(\bigwedge_{i=1}^{k} (\tilde{x}_i \leq_0 x_i \wedge \tilde{y}_i \geq_0 y_i) \wedge A_0(\underline{a}, x_1, y_1, \ldots, x_k, y_k) \rightarrow A_0(\underline{a}, \tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_k, \tilde{y}_k)\big).$$

Lemma 3.8 *For $\tilde{A}(\underline{a})$ as in the previous definition we have*

$$G_nA^\omega \vdash Mon^*(\tilde{A}(\underline{a})).$$

[12] Here the quantifiers $\forall x^0_1$ and $\exists y^0_k$ may be empty ('dummy') quantifiers.

Proof: Trivial.

The Π^0_k-collection principle is the schema

$$\underline{\Pi^0_k\text{-CP}}: \ \forall x \leq_0 a \exists y^0 A(x,y) \to \exists z^0 \forall x \leq_0 a \exists y \leq_0 z\, A(x,y),$$

for all Π^0_k-formulas $A(x,y)$.

Convention 3.9 *In Π^0_k-CP (and other axiom schemas which we will consider below) $A(x,y)$ may contain arbitrary parameters (besides x,y) of the language we consider. E.g. if we write $G_nA^\omega + \Pi^0_k$-CP then instances of Π^0_k-CP may contain parameters of arbitrary type. In EA $+\Pi^0_k$-CP however (where EA denotes first-order elementary recursive arithmetic) instances of Π^0_k-CP of course contain only number parameters.*

Π^0_k-CP is equivalent over many systems (e.g. $\mathrm{G}_n\mathrm{A}^\omega$ for $n \geq 3$) to the axiom schema of finite choice for Π^0_k-formulas

$$\underline{\Pi^0_k\text{-FAC}}: \ \forall x \leq_0 a \exists y^0 A(x,y) \to \exists z^0 \forall x \leq_0 a\, A(x,(z)_x),$$

for all Π^0_k-formulas $A(x,y)$ (with the convention stated above).

In the presence of function variables as in $\mathrm{G}_n\mathrm{A}^\omega$ the schema Π^0_k-CP can be expressed as a single second-order axiom $\forall f \Pi^0_k\text{-CP}(f)$, where

$$\underline{\Pi^0_k\text{-CP}(f)} :\equiv \begin{cases} \forall l^0, a^0 \Big(\forall x \leq_0 a \exists y^0 \forall u^0_1 \exists u^0_2 \ldots \exists^{(d)} u^0_k \big(f(l,a,x,y,\underline{u}) =_0 0\big) \\ \to \exists z^0 \forall x \leq_0 a \exists y \leq_0 z \forall u^0_1 \exists u^0_2 \ldots \exists^{(d)} u^0_k \big(f(l,a,x,y,\underline{u}) =_0 0\big)\Big). \end{cases}$$

By incorporating the universal closure w.r.t. to **arithmetical** parameters $\forall l^0, a^0$ in Π^0_k-CP(f), we achieve that the universal closure of every instance of Π^0_k-CP which contains only number parameters can be written as a sentence Π^0_k-CP(ξ) in $\mathrm{G}_n\mathrm{A}^\omega$ where ξ is a closed term (essentially the characteristic function of the quantifier-free matrix of the Π^0_k-formula $A(x,y)$) which will be of importance below.

The same is true for the principle of Σ^0_k-induction Σ^0_k-IA(f) which we need below:

$$\underline{\Sigma^0_k\text{-IA}(f)} :\equiv \begin{cases} \forall l^0 \Big(\exists u^0_1 \forall u^0_2 \ldots \forall^{(d)} u^0_k \big(f(l,0,\underline{u}) =_0 0\big) \wedge \\ \quad \forall x^0 \big(\exists u^0_1 \forall u^0_2 \ldots \forall^{(d)} u^0_k \big(f(l,x,\underline{u}) =_0 0\big) \to \exists u^0_1 \forall u^0_2 \ldots \forall^{(d)} u^0_k \big(f(l,x',\underline{u}) =_0 0\big)\big) \\ \quad \to \forall x^0 \exists u^0_1 \forall u^0_2 \ldots \forall^{(d)} u^0_k \big(f(l,x,\underline{u}) =_0 0\big)\Big). \end{cases}$$

Lemma 3.10 *Let $A(\underline{a}), \tilde{A}(\underline{a})$ be as in definition 3.6. Then for suitable $\xi_1, \ldots \xi_l, \tilde{\xi}_1, \ldots, \tilde{\xi}_{\tilde{l}} \in G_nR^\omega$ the following holds:*

$$G_nA^\omega \vdash \bigwedge_{i=1}^{l} \Pi^0_{k-2}\text{-}CP(\xi_i \underline{a}) \to \big(A(\underline{a}) \to \tilde{A}(\underline{a})\big)$$

and

$$G_nA^\omega \vdash \bigwedge_{i=1}^{\tilde{l}} \Pi^0_{k-3}\text{-}CP(\tilde{\xi}_i \underline{a}) \to \big(\tilde{A}(\underline{a}) \to A(\underline{a})\big)$$

(Here and in the following we use the convention that Π^0_k-S is empty (i.e. $\equiv (0=0)$ for an axiom schema S if $k<0$).

Proof: Induction on k: For $k=0,1$ the lemma is trivial. So let $k \geq 1$.
$k \mapsto k+1$: Consider

$$A(\underline{a}) \equiv \forall x_1^0 \exists x_2^0 \ldots \exists^{(d)} x_{k+1}^0 A_0(\underline{a}, x_1, x_2, \ldots, x_{k+1}) \in \Pi^0_{k+1}.$$

By the induction hypothesis applied to the Π^0_k-formula

$$\forall x_2 \exists x_3 \ldots \forall^{(d)} x_{k+1} \neg A_0(\underline{a}, x_1, \ldots, x_{k+1})$$

we have instances Π^0_{k-2}-CP$(\xi_i \underline{a})$ (note that instances of Π^0_{k-3}-CP can be considered as instance of Π^0_{k-2}-CP as well) such that $\bigwedge\limits_i \Pi^0_{k-2}$-CP$(\xi_i \underline{a})$ implies (relative to $\mathrm{G}_n\mathrm{A}^\omega$

$$\begin{aligned}
&\exists x_2 \forall x_3 \ldots \exists^{(d)} x_{k+1} A_0 \leftrightarrow \\
&\exists x_2 \forall x_3 \ldots \exists^{(d)} x_{k+1} \exists \tilde{x}_2 \leq x_2 \forall \tilde{x}_3 \leq x_3 \ldots \exists^{(d)} \tilde{x}_{k+1} \leq x_{k+1} A_0(\underline{a}, x_1, \tilde{x}_2, \ldots, \tilde{x}_{k+1}).
\end{aligned}$$

Hence

$$\begin{aligned}
&A(\underline{a}) \\
&\leftrightarrow \forall x_1 \exists x_2 \ldots \exists^{(d)} x_{k+1} \exists \tilde{x}_2 \leq x_2 \ldots \exists^{(d)} \tilde{x}_{k+1} \leq x_{k+1} A_0(\underline{a}, x_1, \tilde{x}_2, \ldots, \tilde{x}_{k+1}) \\
&\leftrightarrow \forall x_1 \forall \tilde{x}_1 \leq x_1 \exists x_2 \ldots \exists^{(d)} x_{k+1} \exists \tilde{x}_2 \leq x_2 \ldots \exists^{(d)} \tilde{x}_{k+1} \leq x_{k+1} A_0(\underline{a}, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{k+1}) \\
&\underset{(logic)}{\overset{\Pi^0_{k-1}\text{-}CP(\widehat{\xi \underline{a}})}{\rightleftarrows}} \forall x_1 \exists x_2 \forall \tilde{x}_1 \leq x_1 \exists \widehat{x}_2 \leq x_2 \forall x_3 \ldots \exists^{(d)} x_{k+1} \exists \tilde{x}_2 \leq \widehat{x}_2 \ldots \exists^{(d)} \tilde{x}_{k+1} \leq x_{k+1} A_0(\underline{a}, \underline{\tilde{x}}) \\
&\leftrightarrow \forall x_1 \exists x_2 \forall \tilde{x}_1 \leq x_1 \forall x_3 \ldots \exists^{(d)} x_{k+1} \exists \tilde{x}_2 \leq x_2 \ldots \exists^{(d)} \tilde{x}_{k+1} \leq x_{k+1} A_0(\underline{a}, \underline{\tilde{x}}) \\
&\leftrightarrow \forall x_1 \exists x_2 \forall x_3 \forall \tilde{x}_1 \leq x_1 \exists x_4 \ldots \exists^{(d)} x_{k+1} \exists \tilde{x}_2 \leq x_2 \ldots \exists^{(d)} \tilde{x}_{k+1} \leq x_{k+1} A_0(\underline{a}, \underline{\tilde{x}}).
\end{aligned}$$

In the same way as we shifted $\forall \tilde{x}_1 \leq x_1$ over $\exists x_2$ we now move $\forall \tilde{x}_1 \leq x_1$ over $\exists x_4$, then permute $\forall \tilde{x}_1 \leq x_1$ with $\forall x_5$, move over $\exists x_6$ and so on until we obtain $\tilde{A}(\underline{a})$. This requires only Π^0_{k-3}-instances (or simpler ones) of CP which can be considered a fortiori as instances Π^0_{k-2}-CP$(\zeta_j \underline{a})$. Putting things together we have shown that (relative to $\mathrm{G}_n\mathrm{A}^\omega$):

$$\Pi^0_{k-1}\text{-CP}(\widehat{\xi \underline{a}}) \wedge \bigwedge_i \Pi^0_{k-2}\text{-CP}(\xi_i \underline{a}) \wedge \bigwedge_j \Pi^0_{k-2}\text{-CP}(\zeta_j \underline{a}) \to \big(A(\underline{a}) \to \tilde{A}(\underline{a})\big)$$

and

$$\bigwedge_i \Pi^0_{k-2}\text{-CP}(\xi_i \underline{a}) \wedge \bigwedge_j \Pi^0_{k-2}\text{-CP}(\zeta_j \underline{a}) \to \big(\tilde{A}(\underline{a}) \to A(\underline{a})\big),$$

which concludes the proof of the lemma.

Since in our main results we assume $n \geq 2$ or $n \geq 3$ for the level n of $\mathrm{G}_n\mathrm{A}^\omega$ we also use for simplicity $\mathrm{G}_2\mathrm{A}^\omega$ in the following definition and lemmas although some of them can be carried out even in $\mathrm{G}_1\mathrm{A}^\omega$.

Definition 3.11 (and lemma) *For $m \in \mathbb{N}$ let $\Phi \in G_2R^\omega$ be such that*

$$G_2A^\omega \vdash \forall f^{(0)\dots(0)}, x^0, y_1^0, z_1^0, \dots, y_m^0, z_m^0, (y_{m+1})$$
$$\big(\Phi f x y_1 z_1 \dots y_m z_m (y_{m+1}) =_0 0 \leftrightarrow \forall \tilde{y}_1 \le y_1 \exists \tilde{z}_1 \le z_1 \dots \forall \tilde{y}_m \le y_m \exists \tilde{z}_m \le z_m (\forall \tilde{y}_{m+1} \le y_{m+1})$$
$$\big(f(x, \tilde{y}_1, \tilde{z}_1, \dots, \tilde{y}_m, \tilde{z}_m, (\tilde{y}_{m+1})) =_0 0\big)\big).$$

We denote Φf by f'.

Lemma 3.12 *Let $k \ge 1$. There are (effectively) finitely many terms $\xi_1, \dots, \xi_l \in G_2R^\omega$ such that*

$$G_2A^\omega \vdash \forall f\Big(\big(\bigwedge_{i=1}^{l} \Pi^0_{k-2}\text{-}CP(\xi_i f)\big) \to \big(\Pi^0_k\text{-}CA(f) \leftrightarrow \Pi^0_k\text{-}CA(f')\big)\Big).$$

Proof: The lemma follows from lemma 3.10.

Definition 3.13 *The 'monotone' tertium-non-datur is given by*

$$\underline{\Pi^0_k\text{-}TND^{mon}}(f) :\equiv$$
$$\begin{cases} \forall x^0 \exists u_1^0 \forall y_1^0 \exists z_1^0 \forall v_1^0 \dots \exists u_m^0 \forall y_m^0 \exists z_m^0 \forall v_m^0 (\exists u_{m+1}^0 \forall y_{m+1}^0) \forall \tilde{x} \le x \\ \qquad \big(f'(\tilde{x}, y_1, z_1, \dots, y_m, z_m, (y_{m+1})) =_0 0 \vee f'(\tilde{x}, u_1, v_1, \dots, u_m, v_m, (u_{m+1})) \ne 0\big), \end{cases}$$

Lemma 3.14 *1) $G_2A^\omega \vdash \forall f\big(\big(\Pi^0_k\text{-}TND^{mon}(f)\big)^S \to \big(\Pi^0_k\text{-}TND(f')^{pr}\big)^S\big)$.*

2) $G_2A^\omega \vdash \forall f\big(Mon^*\big(\Pi^0_k\text{-}TND^{mon}(f)\big)\big)$.

Proof: 1) follows by putting $\tilde{x} := x$.
2) Follows immediately from the definition of Π^0_k-TND$^{mon}(f)$.

Proposition 3.15 $G_2A^\omega \vdash \forall f\big(\big(\Pi^0_k\text{-}TND^{mon}(f)\big)^S \to \Pi^0_k\text{-}CA(f')\big)$.

Proof: Lemmas 3.5 and 3.14.1.

Lemma 3.16 *One can construct a $\xi \in G_2R^\omega$ such that*

$$G_2A^\omega + AC^{0,0}\text{-}qf \vdash \forall f\big(\Pi^0_k\text{-}CA(\xi f) \to \Pi^0_k\text{-}CP(f)\big).$$

Proof:
Using Π^0_k-CA(ξf) for a suitable $\xi \in$ G$_2$R$^\omega$ one can reduce Π^0_k-CP(f) to Π^0_0-CP which is provable in G$_2$A$^\omega$+ AC^{0,0}-qf.

Proposition 3.17 *For a suitable $\xi \in G_2R^\omega$ one has*

$$G_2A^\omega + AC^{0,0}\text{-}qf \vdash \forall f\Big(\big(\Pi^0_k\text{-}TND^{mon}(\xi f)\big)^S \to \Pi^0_k\text{-}CA(f)\Big).$$

Proof: Induction on k: $k = 0, 1$: easy. Let $k > 1$ and lets assume that the proposition holds for all $m < k$. Π^0_{k-2}-CP$(\xi_i f)$ denote the instances of Π^0_{k-2}-collection from lemma 3.12 which are needed to show

$$\Pi^0_k\text{-CA}(f) \leftrightarrow \Pi^0_k\text{-CA}(f').$$

Let $\widehat{\xi} \in \mathrm{G_2R^\omega}$ be (using lemma 3.16) such that[13]

$$(1)\mathrm{G_2A^\omega} + \mathrm{AC^{0,0}}\text{-qf} \vdash \Pi^0_{k-2}\text{-CA}(\widehat{\xi} f) \to \big(\Pi^0_k\text{-CA}(f) \leftrightarrow \Pi^0_k\text{-CA}(f')\big).$$

By the induction hypothesis we have

$$(2)\ \mathrm{G_2A^\omega} + \mathrm{AC^{0,0}}\text{-qf} \vdash \forall f\Big(\big(\Pi^0_{k-2}\text{-TND}^{mon}(\tilde{\xi} f)\big)^S \to \Pi^0_{k-2}\text{-CA}(f)\Big)$$

for a suitable $\tilde{\xi} \in \mathrm{G_2R^\omega}$. So by proposition 3.15

$$(3)\ \mathrm{G_2A^\omega} + \mathrm{AC^{0,0}}\text{-qf} \vdash \big(\Pi^0_k\text{-TND}^{mon}(f)\big)^S \wedge \big(\Pi^0_{k-2}\text{-TND}^{mon}(\tilde{\xi}(\widehat{\xi} f))\big)^S \to \Pi^0_k\text{-CA}(f).$$

Introducing dummy quantifiers, $\big(\Pi^0_{k-2}\text{-TND}^{mon}(\tilde{\xi}(\widehat{\xi} f))\big)^S$ can be reduced to $\big(\Pi^0_k\text{-TND}^{mon}(\xi^* f)\big)^S$ for a suitable $\xi^* \in \mathrm{G_2R^\omega}$. Furthermore

$$(4)\ \big(\Pi^0_k\text{-TND}^{mon}(h)\big)^S \to \big(\Pi^0_k\text{-TND}^{mon}(f)\big)^S \wedge \big(\Pi^0_k\text{-TND}^{mon}(g)\big)^S$$

for

$$h(x, \underline{y}, \underline{z}) = \begin{cases} f(\tilde{x}, \underline{y}, \underline{z}) \text{ if } x = 2\tilde{x} \\ g(\tilde{x}, \underline{y}, \underline{z}) \text{ if } x = 2\tilde{x} + 1. \end{cases}$$

Hence

$$(5)\ \big(\Pi^0_k\text{-TND}^{mon}(\xi f)\big)^S \to \big(\Pi^0_k\text{-TND}^{mon}(f)\big)^S \wedge \big(\Pi^0_k\text{-TND}^{mon}(\xi^* f)\big)^S$$

for a suitable $\xi \in \mathrm{G_2R^\omega}$. By (3) and (5) we have

$$\mathrm{G_2A^\omega} + \mathrm{AC^{0,0}}\text{-qf} \vdash \big(\Pi^0_k\text{-TND}^{mon}(\xi f)\big)^S \to \Pi^0_k\text{-CA}(f).$$

Lemma 3.18 *Let $k \geq 1$ and $A \in \Sigma^0_{k-1}$. Then*

$$G_3A^\omega + \Sigma^0_k\text{-}IA \vdash \forall x^0 \exists u^0 \forall \tilde{x} \leq_0 x\big(\forall y^0 A(\tilde{x}, y) \vee \exists \tilde{u} \leq u \neg A(\tilde{x}, \tilde{u})\big).$$

Proof: Assume

$$(+)\ \forall u^0 \exists \tilde{x} \leq x\big(\exists y \neg A(\tilde{x}, y) \wedge \forall \tilde{u} \leq u A(\tilde{x}, \tilde{u})\big).$$

We show by induction on n:

$$(*)\ \forall n \exists u, \tilde{x}\big(\overbrace{lth\,\tilde{x} = n+1 \wedge \bigwedge_{\substack{i,j \leq n \\ i \neq j}} \big((\tilde{x})_i \neq (\tilde{x})_j \wedge \bigwedge_{i \leq n} \big((\tilde{x})_i \leq x\big) \wedge \forall i \leq n \exists \tilde{u} \leq u \neg A((\tilde{x})_i, \tilde{u})}^{G(n,u,\tilde{x})}\big)$$

[13] Note that two instances $\Pi^0_k\text{-CA}(\xi_1 f) \wedge \Pi^0_k\text{-CA}(\xi_2 f)$ can be coded together into one instance $\Pi^0_k\text{-CA}(\xi_3 f)$ in $\mathrm{G_2A^\omega}$.

(For $n = x + 1$ this obviously is contradictory and so $\neg(+)$ is proved).
$n = 0$: $(+)$ applied to $u := 0$ yields an $x_0 \leq x$ such that $A(x_0, 0)$ and $\exists y_0 \neg A(x_0, y_0)$. $(*)$ is now satisfied by taking $\tilde{x} := \langle x_0 \rangle, u := y_0$.
$n \to n + 1$: Let $u, \tilde{x}$ be such that $(*)$ is satisfied for n. By $(+)$ there exists an $x_{n+1} \leq x$ such that $\exists y_{n+1} \neg A(x_{n+1}, y_{n+1})$ and $\forall \tilde{u} \leq u A(x_{n+1}, \tilde{u})$. By $(*)$ we have $\forall i \leq n \exists \tilde{u} \leq u \neg A((\tilde{x})_i, \tilde{u})$. Hence $\forall i \leq n\big((\tilde{x})_i \neq x_{n+1}\big)$ and so $\widehat{u} := \max(u, y_{n+1}), \widehat{x} := \tilde{x} * \langle x_{n+1} \rangle$ satisfy $G(n+1, \widehat{u}, \widehat{x})$.
It remains to show that $\exists u, \tilde{x} G(n, u, \tilde{x})$ is equivalent to a Σ^0_k-formula:
Using Σ^0_{k-1}-CP, $\exists \tilde{u} \leq u \neg A((\tilde{x})_i, \tilde{u})$ can be shown to be equivalent to a Π^0_{k-1}-formula. Since Σ^0_{k-1}-CP follows from Σ^0_k-IA, the whole proof can be carried out in $\mathrm{G_3A}^\omega + \Sigma^0_k$-IA.

In contrast to Π^0_k-TND(f) its monotone version Π^0_k-TND$^{mon}(f)$ does not hold logically. However it can be proved using Σ^0_k-induction. More precisely the following proposition holds:

Proposition 3.19 *Let $k \geq 1$. There are finitely many instances Σ^0_k-IA$(\xi_i f)$ such that*

$$G_3A^\omega \vdash \forall f \Big(\big(\bigwedge_{i=1}^{l} \Sigma^0_k\text{-}IA(\xi_i f) \big) \to \Pi^0_k\text{-}TND^{mon}(f) \Big).$$

Proof: By (the proof of) lemma 3.18 there are instances Σ^0_k-IA$(\xi_i f)$ which prove (relatively to $\mathrm{G_3A}^\omega$)

$$(*) \begin{cases} \forall x \exists u_1 \forall \tilde{x} \leq x \big(\forall y_1 \exists z_1 \ldots \forall y_m \exists z_m (\forall y_{m+1})(f'(\tilde{x}, y_1, z_1, \ldots, y_m, z_m, (y_{m+1})) = 0) \\ \qquad \vee \exists \tilde{u} \leq u_1 \forall v_1 \ldots \exists u_m \forall v_m (\exists u_{m+1})(f'(\tilde{x}, \tilde{u}, v_1, \ldots, u_m, v_m, (u_{m+1})) \neq 0) \big) \end{cases}$$

and therefore by the definition of f' (which makes
$\exists \tilde{u} \leq u_1 \forall v_1 \ldots \exists u_m \forall v_m (\exists u_{m+1})(f'(\tilde{x}, \tilde{u}, v_1, \ldots, u_m, v_m, (u_{m+1})) \neq 0)$ monotone w.r.t. $\exists \tilde{u}$)

$$\begin{cases} \forall x \exists u_1 \forall \tilde{x} \leq x \big(\forall y_1 \exists z_1 \ldots \forall y_m \exists z_m (\forall y_{m+1})(f'(\tilde{x}, y_1, z_1, \ldots, y_m, z_m, (y_{m+1})) = 0) \\ \qquad \vee \forall v_1 \ldots \exists u_m \forall v_m (\exists u_{m+1})(f'(\tilde{x}, u_1, v_1, \ldots, u_m, v_m, (u_{m+1})) \neq 0) \big), \end{cases}$$

which is equivalent to

$$(**) \begin{cases} \forall x \exists u_1 \forall y_1 \forall \tilde{x} \leq x \exists z_1 \big(\forall y_2 \ldots \forall y_m \exists z_m (\forall y_{m+1})(f'(\tilde{x}, y_1, z_1, \ldots, y_m, z_m, (y_{m+1})) = 0) \\ \qquad \vee \forall v_1 \ldots \exists u_m \forall v_m (\exists u_{m+1})(f'(\tilde{x}, u_1, v_1, \ldots, u_m, v_m, (u_{m+1})) \neq 0) \big). \end{cases}$$

By a suitable instance of Π^0_{k-1}-CP and the monotonicity of $(**)$ w.r.t. $\exists z_1$ one can 'shift' $\forall \tilde{x} \leq x$ over $\exists z_1$. Now one continues in this way until one obtains Π^0_k-TND$^{mon}(f)$ which needs only suitable instances of Π^0_l-CP with $l < k-1$ which can be considered as instances of Π^0_{k-1}-CP. All the instances of Π^0_{k-1}-CP used follow from suitable instances of Σ^0_k-IA.

Corollary 3.20 *$G_3A^\omega \vdash \forall f \big(\Pi^0_k\text{-}CA(\xi f) \to \Pi^0_k\text{-}TND^{mon}(f) \big)$ for a suitable $\xi \in G_3R^\omega$.*

4 Conservation results for Π^0_k-AC(f) and Δ^0_k-CA(f,g)

We are now ready to determine the arithmetical content of instances Π^0_k-CA(ξuv) and even Π^0_k-AC(ξuv) and Δ^0_{k+1}-CA(ξuv) in proofs of monotone sentences (and without monotonicity assumption if the logical complexity is restricted). It turns out that this content is given by certain instances of Π^0_k-TND^{mon}.

Definition 4.1

$$\Pi^0_k\text{-}AC(f) :\equiv \begin{cases} \forall l^0 \big(\forall x^0 \exists y^0 \forall u^0_1 \exists u^0_2 \ldots \exists^{(d)} u^0_k \big(f(l,x,y,\underline{u}) =_0 0\big) \\ \qquad \to \exists g^1 \forall x^0 \forall u^0_1 \exists u^0_2 \ldots \exists^{(d)} u^0_k \big(f(l,x,gx,\underline{u}) =_0 0\big)\big) \end{cases}$$

$$\Delta^0_k\text{-}CA(f,g) :\equiv \begin{cases} \forall l^0 \Big(\forall x^0 \big([\forall u^0_1 \exists u^0_2 \ldots \exists^{(d)} u^0_k \big(f(l,x,\underline{u}) =_0 0\big) \leftrightarrow \\ \qquad \exists v^0_1 \forall v^0_2 \ldots \forall^{(d)} v^0_k \big(g(l,x,\underline{v}) =_0 0\big)]\big) \\ \qquad \to \exists h^1 \forall x^0 \big(hx =_0 0 \leftrightarrow \forall u_1 \exists u_2 \ldots \exists^{(d)} u_k (f(l,x,\underline{u}) =_0 0)\big)\Big) \end{cases}$$

$$\Delta^0_k\text{-}CA(f) :\equiv \Delta^0_k\text{-}CA(j^1_1 f, j^1_2 f) \text{ for the projection functions } j^1_i \in G_2R^\omega.$$

Lemma 4.2 *Let $k \in \mathbb{N}$. Then for suitable $\xi_1, \xi_2 \in G_2R^\omega$:*

1) $G_2A^\omega + AC^{0,0}\text{-}qf \vdash \forall f\big(\Pi^0_k\text{-}CA(\xi_1 f) \to \Pi^0_k\text{-}AC(f)\big)$.

2) $G_2A^\omega + AC^{0,0}\text{-}qf \vdash \forall f\big(\Pi^0_k\text{-}CA(\xi_2 f) \to \Delta^0_{k+1}\text{-}CA(f)\big)$.

Proof: Obvious.

Below we also need a certain 'non-standard' axiom F^-

$$F^- :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0, z^1, n^0 \big(\bigwedge_{i <_0 n} (zi \leq_0 yki) \to \Phi k(\overline{z,n}) \leq_0 \Phi k(y_0 k)\big),$$

where, for $z^{\rho 0}$, $(\overline{z,n})(k^0) :=_\rho zk$, if $k <_0 n$ and $:= 0^\rho$, otherwise.
F^- does not hold in the full set-theoretic type-structure but can be eliminated from proofs of monotone sentences in our theories. This axiom was introduced and studied in [12] and implies the principle of uniform Σ^0_1-boundedness which was mentioned in the introduction and which will be generalized in section 5 below.

Proposition 4.3 *Let $n \geq 2$, $k \geq 0$ and $B :\equiv \forall u^1 \forall v \leq_\tau tu \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0$ be a sentence in $\mathcal{L}(G_nA^\omega)$, where B_0 is quantifier-free and $t \in G_nR^\omega$. Let $\xi_1, \xi_2 \in G_nR^\omega$ (of suitable types) and Δ a set of sentences having the form $\forall x^\delta \exists y \leq_\rho sx \forall z^\eta A_0$ (A_0 quantifier-free, $s \in G_nR^\omega$). Then for a* *suitable* $\xi \in G_nR^\omega$ *the following holds:*

$$\begin{cases} \textit{If} \\ G_nA^\omega + \Delta + AC\text{-}qf \vdash \\ \qquad \forall u^1 \forall v \leq_\tau tu\big(\Delta^0_{k+1}\text{-}CA(\xi_1 uv) \wedge \Pi^0_k\text{-}AC(\xi_2 uv) \rightarrow \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0\big) \\ \textit{then} \\ G_nA^\omega + \tilde{\Delta} + Mon(B) \vdash \forall u^1 \forall v \leq_\tau tu\big(\Pi^0_k\text{-}TND^{mon}(\xi uv) \rightarrow \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0\big) \\ \textit{and in particular} \\ G_{\max(3,n)}A^\omega + \Sigma^0_k\text{-}IA + \tilde{\Delta} + Mon(B) \vdash \forall u^1 \forall v \leq_\tau tu \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0. \end{cases}$$

In the assumption of the rule the theory $G_nA^\omega + \Delta +$ *AC-qf can be strengthened to*[14] $(G_nA^\omega + \Delta + AC\text{-}qf) \oplus F^-$. *Then in the first conclusion* G_nA^ω *must be replaced by* $G_{\max(3,n)}A^\omega$.

Proof: By lemma 4.2, proposition 3.17 and the fact that two instances of Π^0_k-CA can be coded together into a single instance of Π^0_k-CA, there is a $\xi \in \mathrm{G}_n\mathrm{R}^\omega$ such that

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{0,0}\text{-qf} \vdash \forall u^1 \forall v \leq_\tau tu\Big(\big(\Pi^0_k\text{-TND}^{mon}(\xi uv)\big)^S \rightarrow \Delta^0_{k+1}\text{-CA}(\xi_1 uv) \wedge \Pi^0_k\text{-AC}(\xi_2 uv)\Big).$$

So the assumption of the rule implies

$$(1)\ \mathrm{G}_n\mathrm{A}^\omega + \text{AC-qf} + \Delta \vdash \forall u^1 \forall v \leq_\tau tu\Big(\big(\Pi^0_k\text{-TND}^{mon}(\xi uv)\big)^S \rightarrow \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0\Big).$$

By lemma 3.14.2) the prenexation[15]

$$A^{pr} :\equiv \forall u^1 \forall v \leq_\tau tu \exists x \forall u_1 \exists y_1 \forall z_1 \exists v_1 \ldots \exists a_1 \forall b_1 \ldots \exists w^\gamma \big(\mathrm{TND}^{mon}_0(\xi uv) \rightarrow B_0\big)$$

of[16]

$$A :\equiv \forall u^1 \forall v \leq_\tau tu\big(\Pi^0_k\text{-TND}^{mon}(\xi uv) \rightarrow \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0\big)$$

is monotone if B is:

$$\mathrm{G}_n\mathrm{A}^\omega \vdash Mon(B) \rightarrow Mon(A^{pr}).$$

Now (1) implies

$$\mathrm{G}_n\mathrm{A}^\omega + \text{AC-qf} + \Delta \vdash \big(A^{pr}\big)^H$$

and therefore using theorem 2.7

$$\mathrm{G}_n\mathrm{A}^\omega + \tilde{\Delta} + Mon(B) \vdash A^{pr} \text{ i.e.}$$

$$\mathrm{G}_n\mathrm{A}^\omega + \tilde{\Delta} + Mon(B) \vdash A.$$

The second part of the claim in the proposition now follows from proposition 3.19.

[14] Here $\oplus$ means that F^- must not be used in the proof of the premise of an application of the quantifier–free rule of extensionality QF–ER. $\mathrm{G}_n\mathrm{A}^\omega$ satisfies the deduction theorem w.r.t $\oplus$ but not w.r.t $+$.

[15] Note that A^{pr} is not completely in prenex normal form because of the universal quantifiers hidden in $v \leq_\tau tu$. However it has the form required in theorem 2.7 used below.

[16] TND^{mon}_0 denotes the quantifier-free matrix of (some prenex normal form of) Π^0_k-TND^{mon}.

The proof above can be combined with the elimination procedure for F^- given in [12](thm.4.21) yielding the claim about adding F^-.

The following corollary in particular states (for $\Delta = \emptyset$, $\gamma = 0$ and '$\forall v \leq tu$' non-existent) that the provably recursive function (al)s of type ≤ 2 of fixed instances of Δ^0_{k+1}-CA and Π^0_k-AC (relative to the base system $\mathrm{G}_\infty\mathrm{A}^\omega$+ AC-qf) are definable in the fragment T_{k-1} of Gödel's T:

Corollary 4.4 *Let $k \geq 1, \gamma \leq 2$ and $\xi_1, \xi_2 \in G_nR^\omega$. Then the following rule holds*

$$\begin{cases} G_\infty A^\omega + \Delta + \ AC\text{-}qf \vdash \forall u^1 \forall v \leq_\tau tu\big(\Delta^0_{k+1}\text{-}CA(\xi_1 uv) \wedge \Pi^0_k\text{-}AC(\xi_2 uv) \to \exists w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists \Phi \in T_{k-1} \text{ such that} \\ \mathrm{PA}^\omega_i + \tilde{\Delta} \vdash \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \Phi u\, B_0(u,v,w). \end{cases}$$

Again we may strengthen the theory in the assumption of the rule above by $\oplus F^-$.

Proof: The corollary follows from proposition 4.3 by observing that the condition $Mon(\forall u^1 \forall v \leq_\tau tu \exists w^\gamma B_0)$ is empty and using the fact that $\mathrm{G}_\infty\mathrm{A}^\omega + \tilde{\Delta} + \Sigma^0_k$-IA has a monotone functional interpretation as developed in [9] (via negative translation) in $\mathrm{PA}^\omega_i + \tilde{\Delta}$ by terms $\in T_{k-1}$. The latter follows from the proof that the negative translation of Σ^0_k-IA has a functional interpretation in T_{k-1} (provable in (a subsystem of) PA^ω_i) as given in [20] and the fact that every (closed) term of T_{k-1} can be majorized (in the sense of definition 2.1) by a suitable term in T_{k-1} which follows from Howard's proof of this fact for full T as given in [6].

Corollary 4.5 *Let $n \geq 3$ and A be a Π^1_1-sentence.*

$$\begin{aligned} &\text{If } E\text{-}G_nA^\omega + \ AC^{1,0}\text{-}qf + \Delta^0_{k+1}\text{-}CA^- + \Pi^0_k\text{-}AC^- + WKL \ \vdash A \\ &\text{then } G_nA^\omega + \Sigma^0_k\text{-}IA + Mon(A) \vdash A. \end{aligned}$$

Proof:
Using the deduction theorem for E-$\mathrm{G}_n\mathrm{A}^\omega$, the fact that E-$\mathrm{G}_3\mathrm{A}^\omega$+ $\mathrm{AC}^{1,0}$-qf $+F^-$ proves WKL (see [12]) and the existence of characteristic terms $\in \mathrm{G}_n\mathrm{R}^\omega$ for quantifier-free formulas of E-$\mathrm{G}_n\mathrm{A}^\omega$ the assumption implies

$$\text{E-G}_n\text{A}^\omega\text{+AC}^{1,0}\text{-qf} + F^- \vdash \bigwedge_{i=1}^{l} \big(\Delta^0_{k+1}\text{-CA}(\xi_i)\big) \wedge \bigwedge_{j=1}^{\tilde{l}} \big(\Pi^0_k\text{-AC}(\tilde{\xi}_j)\big) \to A$$

for certain terms $\xi_i, \tilde{\xi}_j \in \mathrm{G}_n\mathrm{R}^\omega$ (corresponding to the universal closures of the instances of Δ^0_{k+1}-CA^- and Π^0_k-AC^- used in the proof).
For suitable $\xi, \tilde{\xi} \in \mathrm{G}_n\mathrm{R}^\omega$ we have

$$\text{G}_n\text{A}^\omega \vdash \Delta^0_{k+1}\text{-CA}(\xi) \to \bigwedge_{i=1}^{l} \big(\Delta^0_{k+1}\text{-CA}(\xi_i)\big)$$

and

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \Pi^0_k\text{-AC}(\tilde{\xi}) \to \bigwedge_{j=1}^{\tilde{l}} \big(\Pi^0_k\text{-AC}(\tilde{\xi}_j)\big).$$

Together with elimination of extensionality (see e.g. [17]) we obtain

$$\big(\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{1,0}\text{-qf}\big) \oplus F^- \vdash \Delta^0_{k+1}\text{-CA}(\xi) \wedge \Pi^0_k\text{-AC}(\tilde{\xi}) \to A.$$

The conclusion now follows from proposition 4.3.

Lemma 4.6 *Let $\forall u^1 \forall v \leq_\tau tu\, A(u,v)$ be a sentence with $A(u,v) \in \Sigma^0_{k+1}$. Then one can construct a sentence $\forall u^1 \forall v \leq_\tau tu \tilde{A}(u,v)$ with $\tilde{A}(u,v) \in \Sigma^0_{k+1}$ such that*

1) $G_nA^\omega \vdash Mon\big(\forall u^1 \forall v \leq_\tau tu \tilde{A}(u,v)\big)$,

2) $G_nA^\omega \vdash \forall u^1 \forall v \leq_\tau tu\big(\bigwedge_{i=1}^{l} \Pi^0_{k-2}\text{-}CP(\xi_i uv) \to \big(A(u,v) \to \tilde{A}(u,v)\big)\big)$,

3) $G_nA^\omega \vdash \forall u^1 \forall v \leq_\tau tu\big(\bigwedge_{i=1}^{\tilde{l}} \Pi^0_{k-1}\text{-}CP(\tilde{\xi}_i uv) \to \big(\tilde{A}(u,v) \to A(u,v)\big)\big)$,

where $\xi_i, \tilde{\xi}_j \in G_nR^\omega$ are suitable terms.

Proof: Lemmas 3.8,3.10.

Corollary 4.7 *Let $n \geq 3$, $\forall u^1 \forall v \leq_\tau tu\, A(u,v)$ be a sentence in G_nA^ω with $A(u,v) \in \Sigma^0_{k+1}$, $t \in G_nR^\omega$ and $\xi_1, \xi_2 \in G_nR^\omega$ of suitable types. Then the following rule holds:*

$$\begin{cases} \text{If } G_nA^\omega + \Delta + AC\text{-}qf \vdash \forall u^1 \forall v \leq_\tau tu\big(\Delta^0_{k+1}\text{-}CA\,(\xi_1 uv) \wedge \Pi^0_k\text{-}AC(\xi_2 uv) \to A(u,v)\big) \\ \text{then } G_nA^\omega + \Sigma^0_k\text{-}IA + \tilde{\Delta} \vdash \forall u^1 \forall v \leq_\tau tu\, A(u,v). \end{cases}$$

We may strengthen the theory in the assumption of the rule above by $\oplus F^-$.

Proof:
Let $\tilde{A}$ be as in lemma 4.6. Π^0_{k-2}-CP$(\xi_i uv)$ follows from a corresponding instance Π^0_{k-2}-AC$(\widehat{\xi}_i uv)$ of Π^0_{k-2}-AC which can be considered as an instance Π^0_k-AC$(\widehat{\xi}_i uv)$ of Π^0_k-AC. All these instances Π^0_k-AC$(\widehat{\xi}_i uv)$ $(i = 1, \ldots, l)$ can be combined with Π^0_k-AC$(\xi_2 uv)$ into a single instance Π^0_k-AC$(\widehat{\xi}_2 uv)$. Hence the assumption of the corollary yields

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \text{AC-qf} \vdash \forall u^1 \forall v \leq_\tau tu\big(\Delta^0_{k+1}\text{-CA}\,(\xi_1 uv) \wedge \Pi^0_k\text{-AC}\,(\widehat{\xi}_2 uv) \to \tilde{A}(u,v)\big).$$

The conclusion now follows from proposition 4.3, lemma 4.6 and the fact that
$\mathrm{G}_n\mathrm{A}^\omega + \Sigma^0_k$-IA $\vdash \Pi^0_{k-1}$-CP.

Corollary 4.8 *For $n \geq 3$, E-G_nA^ω+ $AC^{1,0}$-qf $+\Delta^0_{k+1}$-CA^- + Π^0_k-AC^- + WKL is conservative w.r.t. Π^0_{k+2}-sentences over $G_nA^\omega + \Sigma^0_k$-IA^-.*

Proof: The corollary follows from the proofs of corollary 4.5 and corollary 4.7.

Remark 4.9 Corollary 4.8 is optimal in the following sense. For every k there is a sentence $A \in \Pi^0_{k+3}$ such that

$$\mathrm{G_3A^\omega} + \Pi^0_k\text{-AC}^- \vdash A, \text{ but } \mathrm{G_3A^\omega} + \Sigma^0_k\text{-IA} \not\vdash A.$$

Proof: There is a first-order instance A (i.e. without parameters of types > 0) of Π^0_k-FAC which does not follow from Σ^0_k-IA relative to e.g. $\mathrm{G_3A^\omega}$ (see [19]). It is clear that $\mathrm{G_3A^\omega} + \Pi^0_k\text{-AC}^- \vdash A$. Since the universal closure of A can be shown to be equivalent to a Π^0_{k+3}-sentence in $\mathrm{G_3A^\omega} + \Sigma^0_k\text{-IA}^-$ (and hence in $\mathrm{G_3A^\omega} + \Pi^0_k\text{-AC}^-$), the claim follows.

Corollary 4.10 *Let $\forall u^1 \forall v \leq_\tau tu\, A(u,v)$ be a sentence with $A(u,v) \in \Sigma^0_{k+2}$. Then for $n \geq 3$ the following rule holds:*

$$\begin{cases} \text{If } G_nA^\omega + \Delta + AC\text{-}qf \vdash \forall u^1 \forall v \leq_\tau tu\big(\Delta^0_{k+1}\text{-}CA\,(\xi_1 uv) \wedge \Pi^0_k\text{-}AC(\xi_2 uv) \to A(u,v)\big) \\ \text{then } G_nA^\omega + \Pi^0_k\text{-}CP + \tilde{\Delta} \vdash \forall u^1 \forall v \leq_\tau tu\, A(u,v). \end{cases}$$

We may strengthen the theory in the assumption of the rule above by $\oplus F^-$.

Proof: The corollary follows analogously to the proof of corollary 4.7 using lemma 4.6 for $k+1$ instead of k and the well-known fact (see e.g. [19]) that $\mathrm{G}_n\mathrm{A}^\omega + \Pi^0_k\text{-CP} \vdash \Sigma^0_k\text{-IA}$.

Corollary 4.11 *For $n \geq 3$, $E\text{-}G_nA^\omega + AC^{1,0}\text{-}qf + \Delta^0_{k+1}\text{-}CA^- + \Pi^0_k\text{-}AC^- + WKL$ is conservative w.r.t. Π^0_{k+3}-sentences over $G_nA^\omega + \Pi^0_k\text{-}CP^-$.*

Proof: The corollary follows from corollary 4.10 analogously to the proof of corollary 4.8.

Let EA be Kalmar-elementary arithmetic EA (with number quantifiers) and let us consider the variant $\mathrm{G}_n\mathrm{A}^\omega_-$ of $\mathrm{G}_n\mathrm{A}^\omega$ where the arbitrary true universal axioms 9) from its definition in [12] are replaced by the schema of quantifier-free induction (with arbitrary parameters)[17] only. The results above also hold for $\mathrm{G}_n\mathrm{A}^\omega_-$ since no other universal axioms from 9) were used. EA can be considered as a subsystem of $\mathrm{G_3A^\omega_-}$ and the latter is conservative over the former. Hence we obtain the following corollaries for EA:

Corollary 4.12 *Let A be an arbitrary sentence of EA. Then the following rule holds:*

$$\mathrm{EA} + \Pi^0_k\text{-}CP \vdash A \;\Rightarrow\; \mathrm{EA} + \Sigma^0_k\text{-}IA + Mon(A) \vdash A.$$

In particular we have the following

Corollary 4.13 *Let $A, \tilde{A}$ be sentences from EA such that*

1) *EA $+\Pi^0_k$-CP $\vdash A \to \tilde{A}$,*

2) *EA $+\Sigma^0_k$-IA $\vdash \tilde{A} \to A$ and*

3) *EA $+\Sigma^0_k$-IA $\vdash Mon(\tilde{A})$.*

Then EA $+\Pi^0_k$-CP $\vdash A$ implies EA $+\Sigma^0_k$-IA $\vdash A$.

Combined with lemma 4.6 we finally obtain

Corollary 4.14 (Paris-Kirby [18], H. Friedman)
EA $+\Pi^0_k$-CP is Π^0_{k+2}-conservative over EA $+\Sigma^0_k$-IA.

[17] Or equivalently the second-order axiom of quantifier-free induction.

5 Generalized principles of uniform boundedness and their arithmetical content

In the following we define a generalization of the principle of uniform Σ^0_1-boundedness Σ^0_1-UB$^-$ which was studied in [12],[14],[15]:

$$\mathbf{\Sigma^0_1\text{–}UB^-} :\equiv \begin{cases} \forall y^{1(0)}\big(\forall k^0 \forall x \leq_1 yk \exists z^0\ A(x,y,k,z) \to \exists \chi^1 \forall k^0, x^1, n^0 \\ \qquad \big(\bigwedge\limits_{i<_0 n} (xi \leq_0 yki) \to \exists z \leq_0 \chi k\ A((\overline{x,n}), y, k, z)\big)\big), \end{cases}$$

where $A \equiv \exists l^0 A_0(l)$ is a purely existential formula.
Σ^0_1-UB$^-$ follows from F^- relative to G$_n$A$^\omega$+ AC^{1,0}-qf (for $n \geq 2$).
In G$_2$A$^\omega$ + Σ^0_1–UB$^-$ and hence in G$_2$A$^\omega$ + F^-+AC^{1,0}–qf one can give very short and perspicuous proofs of various important analytical theorems like

- Every pointwise continuous function $f : [0,1]^d \to \mathbb{R}$ is uniformly continuous
- The attainment of the maximum value of $f \in C([0,1]^d, \mathbb{R})$ on $[0,1]^d$
- The sequential form of the Heine–Borel covering property for $[0,1]^d$
- Dini's theorem
- The existence of a uniformly continuous inverse function for every strictly increasing continuous function $f : [0,1] \to \mathbb{R}$.

Since F^- does not contribute to the growth of extractable bounds one can extract polynomial bounds from proofs in G$_2$A$^\omega$ + Σ^0_1–UB$^-$+ AC-qf.

Whereas the straightforward generalization of Σ^0_1-UB$^-$ to Π^0_k-formulas is not consistent with G$_n$A$^\omega$ (see [15]), the following restricted form is (although it does – like Σ^0_1-UB$^-$ – not hold in the full set-theoretic type structure):

Definition 5.1 *Let* $\rho = 0(0)(0)\big(1(0)\big)(1)$*,* $k \geq 0$*.*

$$\Pi^0_k\text{-}UB^-{\restriction}(g) :\equiv \begin{cases} \forall \Phi^\rho, y^{1(0)}, a^0 \big(\forall k^0 \forall x \leq_1 yk \exists z^0 A(g, \Phi(x,y,k,z), k, z, a) \to \\ \exists \chi^1 \forall k^0 \forall x \leq_1 yk \forall l^0 \exists z \leq_0 \chi k\, A(g, \Phi((\overline{x,l}), y, k, z), k, z, a)\big), \end{cases}$$

where $A(g, v^0, k^0, z^0, a^0) :\equiv \forall u^0_1 \exists u^0_2 \ldots \exists^{(d)} u^0_k \big(g(v,k,z,a,\underline{u}) =_0 0\big) \in \Pi^0_k$*.*

Remark 5.2 $G_nA^\omega \vdash \Pi^0_0\text{-}UB^-{\restriction}(t) \to \Sigma^0_1\text{-}UB^-$*, where* $t \in G_1R^\omega$ *such that* $t(v,k,z,a) =_0 v$*.*

In [15] we have shown that every single (sequence of) instance(s) of the Bolzano-Weierstraß principle for bounded sequences in $\mathbb{R}^d$ and of the Ascoli-lemma (in the sense of [23]) follows from suitable instances of Π^0_1-UB$^-{\restriction}$ and used this to calibrate precisely the contribution of such instances to the growth of extractable bounds. This indicates the mathematical relevance of our generalized principles of uniform boundedness.

Proposition 5.3 *Let $n \geq 2, k \geq 0$. For suitable $\xi \in G_nR^\omega$ we have*

$$G_nA^\omega + AC^{1,0}\text{-}qf \vdash F^- + \Pi^0_k\text{-}CA(\xi g) \to \Pi^0_k\text{-}UB^-\restriction(g),$$

where g is a free (function) variable.

Proof:
For a suitable $\xi \in \mathrm{G_2R}^\omega$, Π^0_k-CA(ξg) yields the existence of a function h such that

$$\forall v^0, k^0, z^0, a^0 \big(hvkza =_0 0 \leftrightarrow A(g, v, k, z, a)\big),$$

where A is as in definition 5.1. Using h, the assumption of Π^0_k-UB$^-\restriction(g)$ can be expressed as

$$\forall k^0 \forall x \leq_1 yk \exists z^0 \big(h(\Phi(x, y, k, z), k, z, a) =_0 0\big).$$

By Σ^0_1-UB$^-$, which follows from F^- and AC^{1,0}-qf relative to G$_n$A$^\omega$ (see [12]), this yields

$$\exists \chi^1 \forall k^0 \forall x \leq_1 yk \forall l^0 \exists z \leq_0 \chi k \big(h(\Phi((\overline{x,l}), y, k, z), k, z, a) =_0 0\big)$$

and hence

$$\exists \chi^1 \forall k^0 \forall x \leq_1 yk \forall l^0 \exists z \leq_0 \chi k \, A\big(g, \Phi((\overline{x,l}), y, k, z), k, z, a\big).$$

Using proposition 5.3 we can strengthen proposition 4.3 and corollary 4.4 to

Theorem 5.4 *Let $n \geq 3$, $k \geq 0$ and $B :\equiv \forall u^1 \forall v \leq_\tau tu \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0$ be a sentence in $\mathcal{L}(G_nA^\omega)$, where B_0 is quantifier-free and $t \in G_nR^\omega$. Let $\xi_1, \xi_2, \xi_3 \in G_nR^\omega$ (of suitable types) and Δ a set of sentences having the form $\forall x^\delta \exists y \leq_\rho sx \forall z^\eta A_0$ (A_0 quantifier-free, $s \in G_nR^\omega$). Then for a suitable $\xi \in G_nR^\omega$ the following holds:*

$$\begin{cases} \textit{If} \\ G_nA^\omega + \Delta + AC\text{-}qf \vdash \\ \quad \forall u^1 \forall v \leq_\tau tu\big(\Delta^0_{k+1}\text{-}CA(\xi_1 uv) \wedge \Pi^0_k\text{-}AC(\xi_2 uv) \wedge \Pi^0_k\text{-}UB^-\restriction(\xi_3 uv) \to \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0\big) \\ \textit{then} \\ G_nA^\omega + \tilde{\Delta} + Mon(B) \vdash \forall u^1 \forall v \leq_\tau tu\big(\Pi^0_k\text{-}TND^{mon}(\xi uv) \to \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0\big) \\ \textit{and in particular} \\ G_nA^\omega + \Sigma^0_k\text{-}IA + \tilde{\Delta} + Mon(B) \vdash \forall u^1 \forall v \leq_\tau tu \exists a^0_1 \forall b^0_1 \ldots \exists a^0_l \forall b^0_l \exists w^\gamma B_0. \end{cases}$$

In the assumption of the rule the theory $G_nA^\omega + \Delta +$ AC-qf can be strengthened to $(G_nA^\omega + \Delta + AC\text{-}qf) \oplus F^-$.

The following corollary implies (for $\Delta = \emptyset$, $\gamma = 0$ and '$\forall v \leq tu$' being a dummy quantifier) that the provably recursive function(al)s of type ≤ 2 of fixed instances of Π^0_k-UB$^-\restriction$ (relative to the base system G$_\infty$A$^\omega$+ AC-qf) are definable in the fragment T_{k-1} of Gödel's T:

Corollary 5.5 *Let $k \geq 1, \gamma \leq 2$ and $\xi_1, \xi_2, \xi_3 \in G_n R^\omega$. Then the following rule holds*

$$\begin{cases} G_\infty A^\omega + \Delta + AC\text{-}qf \vdash \\ \quad \forall u^1 \forall v \leq_\tau tu\big(\Delta^0_{k+1}\text{-}CA(\xi_1 uv) \wedge \Pi^0_k\text{-}AC(\xi_2 uv) \wedge \Pi^0_k\text{-}UB^- {\restriction}(\xi_3 uv) \to \exists w^\gamma B_0(u,v,w)\big) \\ \Rightarrow \exists \Phi \in T_{k-1} \text{ such that} \\ \mathrm{PA}^\omega_i + \tilde{\Delta} \vdash \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \Phi u\, B_0(u,v,w). \end{cases}$$

Again we may strengthen the theory in the assumption of the rule above by $\oplus F^-$.

We now show that Π^0_k-CA(f) in fact is implied by suitable instances of Π^0_k-UB$^-{\restriction}$:

Proposition 5.6 *Let $n \geq 2, k \geq 1$. For suitable $\xi_1, \ldots, \xi_l \in G_2 R^\omega$ we have*

$$G_n A^\omega \vdash \bigwedge_{i=1}^{l} \Pi^0_k\text{-}UB^-{\restriction}(\xi_i f) \to \Pi^0_k\text{-}CA(f),$$

where f is a free (function) variable.

Proof: Induction on k. $k=1$: Π^0_1-CA(f) is logically equivalent to

$$(1)\ \exists g \leq_1 1 \forall x^0, y^0 \exists z^0 \big((gx =_0 0 \to f(x,y) =_0 0) \wedge (f(x,z) =_0 0 \to gx =_0 0)\big)$$

and hence to

$$(2)\ \neg\forall g \leq_1 1 \exists x^0, y^0 \forall z^0 \neg\big((gx =_0 0 \to f(x,y) =_0 0) \wedge (f(x,z) =_0 0 \to gx =_0 0)\big).$$

For a suitable $\xi_1 \in \mathrm{G_2R}^\omega$, Π^0_1-UB$^-{\restriction}(\xi_1 f)$ yields the equivalence of (2) and

$$(3)\ \neg\exists n^0 \forall g \leq_1 1 \exists x, y \leq n \forall z^0 \neg\big((gx =_0 0 \to f(x,y) =_0 0) \wedge (f(x,z) =_0 0 \to gx =_0 0)\big)$$

i.e.

$$(4)\ \forall n^0 \exists g \leq_1 1 \forall x \leq n\big((gx =_0 0 \to \forall y \leq n f(x,y) =_0 0) \wedge (\forall z(f(x,z) =_0 0) \to gx =_0 0)\big).$$

Define

$$gx := \begin{cases} 0^0 \text{ if } \forall y \leq n(f(x,y) = 0) \\ 1^0 \text{ otherwise.} \end{cases}$$

Let $k \geq 1$. $k \mapsto k+1$:
Π^0_{k+1}-CA(f) is equivalent to

$$(*) \begin{cases} \exists g \leq_1 1 \forall x^0, y^0 \exists z^0 \Big(\big(gx =_0 0 \to \exists u^0_1 \forall u^0_2 \ldots \forall^{(d)} u^0_k (f(x,y,\underline{u}) =_0 0)\big) \wedge \\ \qquad \big(\exists u^0_1 \forall u^0_2 \ldots \forall^{(d)} u^0_k (f(x,z,\underline{u}) =_0 0) \to gx = 0\big)\Big). \end{cases}$$

By induction hypothesis there exists an instance Π^0_k-UB$^-{\restriction}(\xi_2 f)$ (which can be considered as an instance Π^0_{k+1}-UB$^-{\restriction}(\xi_2 f)$) which implies (relative to $\mathrm{G}_n\mathrm{A}^\omega$) Π^0_k-CA(f) and hence the existence of an h such that

$$\forall x, a\big(h(x,a) =_0 0 \leftrightarrow \exists u_1 \forall u_2 \ldots \forall^{(d)} u_k (f(x,a,\underline{u}) =_0 0)\big).$$

By Π^0_{k+1}-UB$^-\restriction(\xi_3 f)$ (for a suitable ξ_3) applied to the negation of $(*)$, Π^0_{k+1}-CA(f) is equivalent to

$$(**)\begin{cases} \forall n\exists g\leq_1 1\forall x\leq n\Big(\big(gx=_0 0\to\forall y\leq n\exists u^0_1\forall u^0_2\ldots\forall^{(d)}u^0_k f(x,y,\underline{u})=_0 0\big)\wedge \\ \qquad\qquad \big(\forall z\exists u^0_1\forall u^0_2\ldots\forall^{(d)}u^0_k f(x,z,\underline{u})=0\to gx=_0 0\big)\Big), \end{cases}$$

which is satisfied by

$$gx:=\begin{cases} 0^0 \text{ if } \forall y\leq n(h(x,y)=0) \\ 1^0 \text{ otherwise.}\end{cases}$$

Corollary 5.7 *For $n\geq 2, k\geq 1$ the following holds:*

1) $G_nA^\omega\vdash\forall g\Pi^0_1$-$UB^-\restriction(g)\to\forall\tilde{g}\Pi^0_k$-$CA(\tilde{g})$.

2) $G_nA^\omega\vdash\forall g\Pi^0_1$-$UB^-\restriction(g)\leftrightarrow\forall\tilde{g}\Pi^0_k$-$UB^-\restriction(\tilde{g})$.

Proof: 1) By proposition 5.6 $\forall g\Pi^0_1$-UB$^-\restriction(g)$ implies $\forall f\Pi^0_1$-CA(f) and hence $\forall f\Pi^0_k$-CA(f) (by iteration).
2) follows from 1) and the proof of proposition 5.3.

Let $\mathrm{B}_{0,1}$ be the type-0-bar recursor constant of equality rank 1, i.e. $\mathrm{B}_{0,1}$ is characterized by the axioms

$$(\mathrm{BR}_{0,1})\ :\ \begin{cases} x^2(\overline{y^1,n^0})<n\to\mathrm{B}_{0,1}xzuny=_1 zny \\ x(\overline{y,n})\geq n\to\mathrm{B}_{0,1}xzuny=_1 u\big(\lambda D^0.\mathrm{B}_{0,1}xzun'(\overline{y,n}*D)\big)ny, \end{cases}$$

where u is of type $1(1)(0)(1(0))$ and

$$(\overline{y,n}*D)(k^0)=_0\begin{cases} yk, \text{ if } k<n \\ D, \text{ if } k=n \\ 0^0, \text{ otherwise.}\end{cases}$$

Definition 5.8 *The schema of dependent choice of type 0 for arithmetical formulas is given by*

$$\Pi^0_\infty\text{-}(DC^0):\equiv\forall x^0\exists y^0A(x,y)\to\forall x^0\exists z^1\big(z0=_0 x\wedge\forall z^0_1A(zz_1,z(z_1'))\big),$$

where $A\in\Pi^0_\infty$ with arbitrary parameters.

Proposition 5.9 *Let $n\geq 3, k\geq 1$, $B_0(u,v,w)$ be a quantifier-free formula of G_nA^ω containing only u,v,w free, $t^{\tau 1}\in G_nR^\omega,\gamma\leq 2$. Then the following rule holds:*

$$\begin{cases} G_nA^\omega+\Delta+AC\text{-}qf\vdash\forall g\Pi^0_k\text{-}UB^-\restriction(g)\to\forall u^1\forall v\leq_\tau tu\exists w^\gamma B_0(u,v,w) \\ \Rightarrow\exists\Phi\in G_nR^\omega[\mathrm{B}_{0,1}] \text{ such that} \\ G_nA^\omega+\tilde{\Delta}+(\mathrm{BR}_{0,1})+\Pi^0_\infty\text{-}(DC^0)\vdash\forall u^1\forall v\leq_\tau tu\exists w\leq_\gamma\Phi u\,B_0(u,v,w). \end{cases}$$

Φ can be written as a closed term $\tilde{\Phi}$ of T (i.e. it is a primitive recursive functional in the sense of Gödel) such that $\mathrm{PA}^\omega + BR_{0,1} \vdash \Phi =_{\gamma 1} \tilde{\Phi}$.
Moreover if $\delta, \eta \leq 2$ and $\rho \leq 1$ for the types in Δ and if $\mathcal{S}^\omega \models \Delta$ and $\tau \leq 1$, then

$$\mathcal{S}^\omega \models \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \tilde{\Phi} u \, B_0(u,v,w).$$

Proof: By proposition 5.3 and corollary 5.7 one has

$$\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC}^{1,0}\text{-qf} + \forall g \, \Pi^0_1\text{-CA}(g) \vdash F^- \to \forall \tilde{g} \, \Pi^0_k\text{-UB}^- {\restriction}(\tilde{g}).$$

Hence the assumption of the rule to be proved yields

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC}\text{-qf} + \forall g \, \Pi^0_1\text{-CA}(g) \vdash F^- \to \forall u^1 \forall v \leq_\tau tu \exists w^\gamma B_0(u,v,w).$$

From the work of Spector [24] it follows that $\mathrm{G}_n\mathrm{A}^\omega +$ AC-qf $+ \forall g \, \Pi^0_1\text{-CA}(g)$ has (via negative translation) a Gödel functional interpretation in $\mathrm{G}_n\mathrm{A}^\omega_i + (\mathrm{BR}_{0,1})$ by terms $\in \mathrm{G}_n\mathrm{R}^\omega[\mathrm{B}_{0,1}]$. In [2] it is shown that the type structure $\mathcal{M}^\omega$ of the so-called strongly majorizable functionals forms a model of full bar recursion. From the proof of this fact (restricted to type-0-bar recursion) one obtains the construction of a term $\mathrm{B}^*_{0,1} \in \mathrm{G}_n\mathrm{R}^\omega[\mathrm{B}_{0,1}]$ such that

$$\mathrm{G}_n\mathrm{A}^\omega + (\mathrm{BR}_{0,1}) + \Pi^0_\infty\text{-}(DC^0) \vdash \mathrm{B}^*_{0,1} \text{ s-maj } \mathrm{B}_{0,1},$$

where 's-maj' is the corresponding syntactic notion of strong majorization as defined in definition 2.1. Therefore the proof of the fact that (the negative translation of) $\mathrm{G}_n\mathrm{A}^\omega +$ AC-qf $+\Delta$ has a monotone functional interpretation (in the sense of [9]) in $\mathrm{G}_n\mathrm{A}^\omega_i$ by terms in $\mathrm{G}_n\mathrm{R}^\omega$ (see [12]) extends to $\mathrm{G}_n\mathrm{A}^\omega + \Delta +$ AC-qf $+\forall g \, \Pi^0_1\text{-CA}(g)$ yielding a monotone functional interpretation (via negative translation) in $\mathrm{G}_n\mathrm{A}^\omega + \tilde{\Delta} + (\mathrm{BR}_{0,1}) + \Pi^0_\infty\text{-}(DC^0)$ by terms in $\mathrm{G}_n\mathrm{R}^\omega[\mathrm{B}_{0,1}]$. This has the consequence that as in the case of $\mathrm{G}_n\mathrm{A}^\omega + \Delta +$ AC-qf (see the proof of theorem 4.21 in [12]) we can eliminate F^- from the proof of $\forall u \forall v \leq tu \exists w \, B_0$ and extract a uniform bound Φ on '$\exists w$' which now of course is only in $\mathrm{G}_n\mathrm{R}^\omega[\mathrm{B}_{0,1}]$ (instead of $\mathrm{G}_n\mathrm{R}^\omega$) and its verification can be carried out in $\mathrm{G}_n\mathrm{A}^\omega + \tilde{\Delta} + (\mathrm{BR}_{0,1}) + \Pi^0_\infty\text{-}(DC^0)$.
By [16] (proposition 4.2) it follows (since $deg(\gamma 1) = 2$) that Φ can be written as a primitive recursive functional $\tilde{\Phi}$ such that $\mathrm{PA}^\omega + BR_{0,1} \vdash \Phi =_{\gamma 1} \tilde{\Phi}$.
The final claim follows using again the model $\mathcal{M}^\omega$. Since $M_0 = S_0, M_1 = S_1$ and $M_2 \subset S_2$, the assumption $\mathcal{S}^\omega \models \Delta$ implies $\mathcal{M}^\omega \models \Delta$ and therefore (since $\mathcal{M}^\omega \models b$-AC, see [8]) $\mathcal{M}^\omega \models \tilde{\Delta}$. From [2] it follows that $\mathcal{M}^\omega \models \mathrm{PA}^\omega + BR_{0,1} + \Pi^0_\infty\text{-}(DC^0)$. Therefore

$$\mathcal{M}^\omega \models \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \tilde{\Phi} u \, B_0(u,v,w),$$

and hence (since $\tau \leq 1$, $\gamma \leq 2$)

$$\mathcal{S}^\omega \models \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \tilde{\Phi} u \, B_0(u,v,w).$$

Corollary 5.10 *The provably recursive function(al)s of type ≤ 2 of $\forall g \Pi^0_k$-$UB^- {\restriction}(g)$ (relative to G_nA^ω+ AC-qf) are definable in T.*

Remark 5.11 *Because of corollary 5.7.1), PA is a subsystem of G_nA^ω+AC-qf$\oplus\forall g\Pi^0_k$-$UB^- {\restriction}(g)$. Hence corollary 5.10 is optimal.*

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