Things that can and things that can't be done in PRA

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Abstract

It is well-known by now that large parts of (non-constructive) mathematical reasoning can be carried out in systems $\mathcal{T}$ which are conservative over primitive recursive arithmetic **PRA** (and even much weaker systems). On the other hand there are principles **S** of elementary analysis (like the Bolzano-Weierstraß principle, the existence of a limit superior for bounded sequences etc.) which are known to be equivalent to arithmetical comprehension (relative to $\mathcal{T}$) and therefore go far beyond the strength of **PRA** (when added to $\mathcal{T}$).

In this paper we determine precisely the arithmetical and computational strength (in terms of optimal conservation results and subrecursive characterizations of provably recursive functions) of weaker function parameter-free schematic versions $\mathbf{S}^-$ of **S**, thereby exhibiting different levels of strength between these principles as well as a sharp borderline between fragments of analysis which are still conservative over **PRA** and extensions which just go beyond the strength of **PRA**.

1 Introduction

It is well-known by now, that substantial parts of mathematics (and in particular analysis) can be carried out in systems $\mathcal{T}$ which are conservative over primitive recursive arithmetic PRA. In particular, a lot of results in this direction follow from the work done on the program of so-called reverse mathematics although not using the reverse direction explicitly (see [26] for a comprehensive treatment of reverse mathematics). Formalization of analysis

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in sub-systems of second order arithmetic as in reverse mathematics, however, requires a complicated encoding of analytic notions which sometimes (e.g. in the case of continuous functions between Polish spaces) entails a constructive enrichment of the data. This can be overcome to a large extent by the use of more flexible systems of arithmetic in all finite types. Such systems which are on the one hand mathematically very strong but on the other hand are still conservative over PRA (and even much weaker systems) have been developed by the author in a series of papers (see e.g. [8],[9] and – for a general survey – [15]).

These facts are of interest for mainly two reasons

1) If a Π^0_2-sentence A is provable in $\mathcal{T}$ and the conservation of $\mathcal{T}$ over PRA has been established proof-theoretically, then one can extract a primitive recursive program which realizes A from a given proof. Typically the resulting program will have a quite restricted complexity or rate of growth (compared to merely being primitive recursive). In fact in a series of papers we have shown that in many cases even a polynomial bound is guaranteed (see [9],[11],[14] among others). The applicability of these facts is increased further by the fact that certain principles Δ which may not be reducible to PRA can nevertheless be added to $\mathcal{T}$ provided they are of a certain logical form (this includes as a special case Π^0_1-lemmas). This point, which has been emphazised by G. Kreisel since the 50's, shows that conservation results as mentioned above should be considered merely as guidelines which in concrete applications have to be adpated to the case at hand.

2) One can argue that PRA formalizes what has been called finitistic reasoning (see e.g. [27]). If the conservation of $\mathcal{T}$ over PRA has been established finitistically (which is possible for mathematically strong systems $\mathcal{T}$ (see [23],[8]), then all the mathematics which can be carried out in $\mathcal{T}$ has a finitistic justification (see [25],[26] and [15] for discussions of this point).

In this paper we exhibit a sharp boundary between finistically reducible parts of analysis and extensions which provably go beyond the strength of PRA.
More precisely we study the (proof-theoretical and numerical) strength of function parameter-free schematic forms of[1]

- the convergence (with modulus of convergence) of bounded monotone sequences $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ principle (PCM)

- the Bolzano-Weierstraß principle (BW) for $(a_n)_{n\in\mathbb{N}} \subset [0,1]^d$

[1]For precise formalizations of these principles in systems based on number and function variables see [12] on which the present paper partially relies. We slightly deviate from the notation used in [12] by writing (PCM),(PCM$_{ar}$) instead of (PCM2),(PCM1).

- the Ascoli-Arzela principle for bounded sequences $(f_n)_{n\in\mathbb{N}} \subset C[0,1]$ of equicontinuous functions (A-A)
- the existence of the limit superior principle for $(a_n)_{n\in\mathbb{N}} \subset [0,1]$ (Limsup).

Let us discuss what we mean by 'function parameter-free schematic form' in more detail for BW:
'Schematic' means that an instance BW(t) of BW is given by a term t of the underlying system which defines a sequence in $[0,1]^d$. We allow number parameters k in t, i.e. we consider sequences $\forall k \in \mathbb{N}$ BW($t[k]$) of instances of BW, but not function parameters.
Allowing function parameters to occur in BW would make the schema equivalent to the single second-order sentence

$$(*)\ \forall (a_n) \subset [0,1]^d\ \mathrm{BW}(a_n).$$

It is well-known by the work on program of reverse mathematics that $(*)$ is equivalent to the schema of arithmetical comprehension (relative to weak fragments of second-order arithmetic).
On the other hand, the restriction of BW to function parameter-free instances – in short: BW^- – is much weaker since the iterated use of BW is now no longer possible.

We calibrate precisely the strength of PCM^-, BW^-, A-A^- and Limsup^- relative second-order extensions of primitive recursive arithmetic PRA (thereby completing research started in [12]).
Whereas in [12] we were mainly concerned with upper bounds on the complexity and growth of the provable recursive function(al)s caused by these principles, this paper focusses on lower bounds thereby showing that the results from [12] were optimal. In order to make the formlation of the lower bound results as strong as possible we use very restricted second-order extensions of PRA as base systems (whereas the upper bounds in [12] even hold for systems in all finite types) and state all results in terms of these systems to avoid to have to introduce too many systems (however, see remark 3.8 below).
It turns out that the results depend heavily on what type of extension of PRA we choose: One option is straightforward: extend PRA by variables and quantifiers for numbers x^0 and objects f^ρ of type-level 1, i.e. $\rho = 0(0)\cdots(0)$, where $\rho(0)$ is the type of functions from $\mathbb{N}$ into objects of type ρ (note that modulo λ-abstraction objects of type $0\overbrace{(0)\ldots(0)}^{n}$ are just n-ary number theoretic functions).[2] We have the axioms and rules of many-sorted classical predicate logic as well as symbols and defining equations for all primitive recursive

[2] So we could have used also variables and quantifiers for n-ary functions instead and treat sequences of functions as $f_n := \lambda m.f(n,m)$. However the use of variables $f^{0(0)\cdots(0)}$ is more convenient since it avoids the use of the λ-operator in many cases.

functionals of type level ≤ 2 in the sense of Kleene [7] (i.e. ordinary primitive recursion uniformly in function parameters, for details see e.g. [6](II.1) or [22]; we do not include higher type primitive recursion in the sense of [5]). We also have a schema of quantifier-free induction (w.r.t. to this extended language) and λ-abstraction for number variables, i.e.

$$(\lambda \underline{y}.t[\underline{y}])\underline{x} = t[\underline{x}],\ \underline{x}, \underline{y} \text{ tuples of the same length.}$$

So PRA^2 essentially is the second-order fragment of the (restricted) finite type system $\widehat{\mathrm{PA}}^{\omega}\restriction$ from [3]. It is clear that the resulting system PRA^2 is conservative over PRA.
We often write 1 instead of 0(0).

Another option is to impose a restriction on the type-2-functionals which are allowed. We include functionals of arbitrary Grzegorczyk level in the sense of [9][3] (including all elementary recursive functionals) but not the iteration functional

$$(It)\ \Phi_{it}(0, y, f) = y,\ \Phi_{it}(x+1, y, f) = f(x, \Phi_{it}(x, y, f)),$$

although it is primitive recursive in the sense of Kleene (and not only in the extended sense of Gödel [5], '=' is equality between natural numbers). We call the resulting system PRA^2_{-}.
On easily shows that PRA^2 is a definitorial extension of $\mathrm{PRA}^2_{-} + (It)$.
EA^2 is the restriction of PRA^2_{-} to elementary recursive function(al)s only (see [21] for a definition of 'elementary recursive functional').

Remark 1.1 *In contrast to the class of primitive recursive functions, there exists no Grzegorczyk hierarchy for primitive recursive functionals which would include all of them: if Φ_{it} would occur at a certain level of such a hierarchy, then this hierarchy would collapse to this level since all primitive recursive functions can be obtained from the initial functions and Φ_{it} by substitution.*

The schema of quantifier-free choice for numbers is given by

$$\mathrm{AC}^{0,0}\text{-qf}:\ \forall x^0 \exists y^0 A_0(x, y) \to \exists f \forall x A_0(x, fx),$$

where A_0 is a quantifier-free formula.[4] We also consider the binary König's lemma as formulated in [28]:

$$\mathrm{WKL} :\equiv \forall f^1(T(f) \wedge \forall x^0 \exists n^0(lth(n) =_0 x \wedge f(n) =_0 0) \to \exists b \leq_1 1 \forall x^0(f(\overline{b}x) =_0 0)),$$

[3]This means that we allow all the type-2-functionals Φ_n from [9] plus a bounded search operator and bounded recursion – uniformly in function parameters – on the ground type (see [9]).

[4]Throughout this paper $A_0, B_0, C_0, \ldots$ denote quantifier-free formulas.

where $b \leq_1 1 :\equiv \forall n(bn \leq 1)$ and

$$T(f) :\equiv \forall n^0, m^0(f(n*m) = 0 \to f(n) = 0) \wedge \forall n^0, x^0(f(n*\langle x\rangle) = 0 \to x \leq 1)$$

(here $lth, *, \overline{b}x, \langle\cdot\rangle$ refer to the elementary recursive coding of finite sequences of numbers from [9]).

One easily shows that the schema of Σ^0_1-induction is derivable in PRA^2+ AC^{0,0}-qf (but not in PRA$^2_-$+ AC^{0,0}-qf). The schema of recursive comprehension is already provable in PRA$^2_-$+ AC^{0,0}-qf. So PRA^2+ AC^{0,0}-qf (resp. PRA^2+ AC^{0,0}-qf + WKL) is a function variable version of the system RCA$_0$ (resp. WKL$_0$) used in reverse mathematics, which uses set variables instead of function variables.

Let E-PRA$^\omega_-$ be the system which results if we add variables and quantifiers of all finite types plus the combinators $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$ and the extensionality axioms E_ρ to PRA$^2_-$, higher type equality being defined extensionally:
$s =_{0\tau_k\ldots\tau_1} t :\equiv \forall x_1^{\tau_1}, \ldots, x_k^{\tau_k}(sx_1\ldots x_k =_0 tx_1\ldots x_k)$.

The main results of this paper are[5]

Theorem 1.2 *1)* PRA$^2_-$+PCM$^-$ *contains* PRA+Σ^0_1-IA.

2) PRA$^2_-$+AC^{0,0}-qf+WKL+PCM$^-$+BW$^-$+A-A$^-$ *is* Π^0_3*-(but not* Π^0_4*-)conservative over* PRA+Σ^0_1-IA *and hence* Π^0_2*-conservative over* PRA.
This also holds for E-PRA$^\omega_-$ *instead of* PRA$^2_-$.

Together with the well-known fact (due to [16],[18],[19]) that the provable recursive functions of PRA+Σ^0_1-IA are just the primitive recursive functions we obtain

Corollary 1.3
The provably recursive functions of PRA$^2_-$+AC^{0,0}-qf+WKL+PCM$^-$+BW$^-$+A-A$^-$ *are exactly the primitive recursive ones.*

Theorem 1.4 *1)* PRA$^2_-$+Limsup$^-$ *contains* PRA+Σ^0_2-IA.

2) PRA$^2_-$+AC^{0,0}-qf+WKL+PCM$^-$+BW$^-$+A-A$^-$+Limsup$^-$ *is* Π^0_4*-conservative over* PRA+Σ^0_2-IA. *This also holds for* E-PRA$^\omega_-$ *instead of* PRA$^2_-$.

[5]Here and in the following we denote the (conservative) extension of PRA by first-order predicate logic also by PRA.

With Parsons' characterization of the provable recursive functions of PRA+Σ^0_2-IA ([19],[20]) this yields

Corollary 1.5 *The provably recursive functions of* PRA^2_{-}+$\mathrm{AC}^{0,0}$-qf+WKL+PCM^-+BW^-+$\mathrm{A\text{-}A}^-$+Limsup^- *are exactly the* $\alpha(<\omega^{(\omega^\omega)})$*- recursive ones,*[6] *i.e. the functions definable in the fragment* T_1 *of Gödel's* T ([5]) *with recursion of level* ≤ 1 *only, which includes the Ackermann function.*
This results also holds for EA^2 *instead of* PRA^2_{-}.

For PRA^2 instead of PRA^2_{-} we have the following results:

Theorem 1.6 PRA^2+ PCM^- *is closed under the function parameter-free rule* Σ^0_2-IR^- *of* Σ^0_2*-induction, where*

$$\Sigma^0_2\text{-IR}^- :\ \frac{\exists y^0 \forall z^0 A_0(0,y,z)\ ,\ \exists y \forall z A_0(x,y,z) \to \exists y \forall z A_0(x',y,z)}{\exists y \forall z A_0(x,y,z)}$$

with A_0 *quantifier-free and without function parameters.*

Corollary 1.7 *Every* $\alpha(<\omega^{(\omega^\omega)})$*-recursive (i.e.* T_1*-definable) function (including the Ackermann function) is provably recursive in* PRA^2+ PCM^-.

Together with the fact that PRA^2+$\mathrm{AC}^{0,0}$-qf+WKL is Π^0_2-conservative over PRA (see [23] and for more general results [8]) this yields

Corollary 1.8 PRA^2+$\mathrm{AC}^{0,0}$-qf+WKL $\not\vdash$ PCM^- *(this holds a fortiori for* BW^-, $\mathrm{A\text{-}A}^-$ *and* Limsup^- *instead of* PCM^-).

Theorem 1.9 *Let* P *be* PCM^-, BW^- *or* $\mathrm{A\text{-}A}^-$. *Then* PRA^2+ $\mathrm{AC}^{0,0}$-qf $+P$ *contains* PRA +Π^0_2-IA (=PRA +Σ^0_2-IA).

So relative to PRA^2+ $\mathrm{AC}^{0,0}$-qf, the principles PCM^-, BW^- and $\mathrm{A\text{-}A}^-$ are not conservative over PRA.
Relative to PRA^2_{-} (+$\mathrm{AC}^{0,0}$-qf +WKL) these principles are conservative over PRA but the principle Limsup^- is not.

[6] Here α-recursive is meant in the sense of [17], i.e. unnested. In contrast to this the notion of α-recursiveness as used e.g. in [2],[22] corresponds to nested recursion.

2 Preliminaries

We first indicate how to represent real numbers and the basic arithmetical operations and relations on them in EA^2.
The results of this section a fortiori hold for PRA^2_- instead of EA^2.
Our representation of $\mathbb{R}$ relies on the following **representation of** $\mathbb{Q}$: Let j be the Cantor pairing function. Rational numbers are represented as codes $j(n,m)$ of pairs (n,m) of natural numbers n, m. $j(n,m)$ represents

$$\text{the rational number } \frac{\frac{n}{2}}{m+1}, \text{ if } n \text{ is even, and the negative rational } -\frac{\frac{n+1}{2}}{m+1} \text{ if } n \text{ is odd.}$$

j is surjective and every natural number is a code of a uniquely determined rational number.
On the codes of $\mathbb{Q}$, i.e. on $\mathbb{N}$, we define an equivalence relation by

$$n_1 =_{\mathbb{Q}} n_2 :\equiv \frac{\frac{j_1 n_1}{2}}{j_2 n_1 + 1} = \frac{\frac{j_1 n_2}{2}}{j_2 n_2 + 1} \text{ if } j_1 n_1, j_1 n_2 \text{ both are even}$$

and analogously in the remaining cases, where $\frac{a}{b} = \frac{c}{d}$ is defined to hold iff $ad =_0 cb$ (for $bd > 0$).
On $\mathbb{N}$ one easily defines $=_{\mathbb{Q}}$-extensional functions $|\cdot|_{\mathbb{Q}}, +_{\mathbb{Q}}, -_{\mathbb{Q}}, \cdot_{\mathbb{Q}} :_{\mathbb{Q}}, \max_{\mathbb{Q}}, \min_{\mathbb{Q}} \in \mathrm{EA}^2$ and (quantifier-free) relations) $<_{\mathbb{Q}}, \leq_{\mathbb{Q}}$ which represent the corresponding functions and relations on $\mathbb{Q}$. We often omit the index $\mathbb{Q}$ if this does not cause any confusion.

Notational convention: For better readability we usually write e.g. $\frac{1}{k+1}$ instead of its 'canonical' code $j(2,k)$ in $\mathbb{N}$. So e.g. we write $x^0 \leq_{\mathbb{Q}} \frac{1}{k+1}$ for $x \leq_{\mathbb{Q}} j(2,k)$.

Real numbers are represented as Cauchy sequences $(q_n)_{n\in\mathbb{N}}$ of rational numbers with fixed rate of convergence

$$\forall n \forall m, \tilde{m} \geq n(|q_m - q_{\tilde{m}}| \leq \frac{1}{n+1}).$$

By the coding of rational numbers as natural numbers, **sequences of rationals** are just functions f^1 (and every function f^1 can be conceived as a sequence of rational numbers in a unique way). In particular representatives of real numbers are functions f^1 modulo this coding. Using the following functional we achieve that **every** function can be viewed of as an representative of a uniquely determined Cauchy sequence of rationals with modulus $1/(k+1)$ and therefore can be conceived as an representative of a uniquely determined real number.

Definition 2.1 *The functional* $\lambda f^1.\widehat{f} \in \mathrm{EA}^2$ *is defined such that*

$$\widehat{f}n = \begin{cases} fn, \ \text{if} \ \forall k, m, \tilde{m} \leq_0 n(m, \tilde{m} \geq_0 k \to |fm -_{\mathbb{Q}} f\tilde{m}| \leq_{\mathbb{Q}} \frac{1}{k+1}) \\ f(n_0 \dot{-} 1) \ \text{for} \ n_0 := \min l \leq_0 n[\exists k, m, \tilde{m} \leq_0 l(m, \tilde{m} \geq_0 k \wedge |fm -_{\mathbb{Q}} f\tilde{m}| >_{\mathbb{Q}} \frac{1}{k+1})], \\ \text{otherwise.} \end{cases}$$

One easily proves in EA^2 that

1) if f^1 represents a Cauchy sequence of rational numbers with modulus $1/(k+1)$, then $\forall n^0(fn =_0 \widehat{f}n)$,

2) for every f^1 the function $\widehat{f}$ represents a Cauchy sequence of rational numbers with modulus $1/(k+1)$.

To improve readability we write (x_n) instead of fn and $(\widehat{x}_n)$ instead of $\widehat{f}n$.

Definition 2.2 *1)* $(x_n) =_{\mathbb{R}} (\tilde{x}_n) :\equiv \forall k^0(|\widehat{x}_k -_{\mathbb{Q}} \widehat{\tilde{x}}_k| \leq_{\mathbb{Q}} \frac{3}{k+1})$;

2) $(x_n) <_{\mathbb{R}} (\tilde{x}_n) :\equiv \exists k^0(\widehat{\tilde{x}}_k - \widehat{x}_k >_{\mathbb{Q}} \frac{3}{k+1})$;

3) $(x_n) \leq_{\mathbb{R}} (\tilde{x}_n) :\equiv \neg(\widehat{\tilde{x}}_n) <_{\mathbb{R}} (\widehat{x}_n)$;

4) $(x_n) +_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2n+1} +_{\mathbb{Q}} \widehat{\tilde{x}}_{2n+1})$;

5) $(x_n) -_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2n+1} -_{\mathbb{Q}} \widehat{\tilde{x}}_{2n+1})$;

6) $|(x_n)|_{\mathbb{R}} := (|\widehat{x}_n|_{\mathbb{Q}})$;

7) $(x_n) \cdot_{\mathbb{R}} (\tilde{x}_n) := (\widehat{x}_{2(n+1)k} \cdot_{\mathbb{Q}} \widehat{\tilde{x}}_{2(n+1)k})$, *where* $k := \lceil \max_{\mathbb{Q}}(|x_0|_{\mathbb{Q}} + 1, |\tilde{x_0}|_{\mathbb{Q}} + 1) \rceil$;

8) For (x_n) *and* l^0 *we define*

$$(x_n)^{-1} := \begin{cases} (\max_{\mathbb{Q}}(\widehat{x}_{(n+1)(l+1)^2}, \frac{1}{l+1})^{-1}), \ \text{if} \ \widehat{x}_{2(l+1)} >_{\mathbb{Q}} 0 \\ (\min_{\mathbb{Q}}(\widehat{x}_{(n+1)(l+1)^2}, \frac{-1}{l+1})^{-1}), \ \text{otherwise}; \end{cases}$$

9) $\max_{\mathbb{R}}((x_n), (\tilde{x}_n)) := (\max_{\mathbb{Q}}(\widehat{x}_n, \widehat{\tilde{x}}_n))$, $\min_{\mathbb{R}}((x_n), (\tilde{x}_n)) := (\min_{\mathbb{Q}}(\widehat{x}_n, \widehat{\tilde{x}}_n))$.

Sequences of real numbers are coded as sequences $f^{1(0)}$ of codes of real numbers.

The principles PCM and PCM_{ar} of convergence for bounded monotone sequences are given by[7]

$$\mathrm{PCM}_{ar}(f^{1(0)}) :\equiv$$
$$\forall n(0 \leq_{\mathbb{R}} f(n+1) \leq_{\mathbb{R}} f(n)) \to \forall k \exists n \forall m, \tilde{m} \geq n(|fm -_{\mathbb{R}} f\tilde{m}| \leq \frac{1}{k+1}),$$

$$\mathrm{PCM}(f^{1(0)}) :\equiv$$
$$\forall n(0 \leq_{\mathbb{R}} f(n+1) \leq_{\mathbb{R}} f(n)) \to \exists g \forall k \forall m, \tilde{m} \geq gk(|fm -_{\mathbb{R}} f\tilde{m}| \leq \frac{1}{k+1}).$$

Relative to PRA^2_-, PCM is equivalent to the principle stating the convergence of f with a modulus of convergence (PCM_{ar} does not imply in PRA^2_- the existence of a limit since reals have to be given as Cauchy sequences with given rate of convergence). For monotone sequences the existence of a modulus of convergence can be obtained from the existence of a limit by the use of $\mathrm{AC}^{0,0}$-qf. So relative to $\mathrm{PRA}^2_- + \mathrm{AC}^{0,0}$-qf we don't have to distinguish between our formulation of PCM, the existence of a limit of f and the existence of a limit together with a modulus of convergence.

PCM^- and PCM^-_{ar} denote the function parameter-free schematic versions of $\mathrm{PCM}(f)$ and $\mathrm{PCM}_{ar}(f)$ (in the sense discussed in the introduction).

Let $\mathrm{BW}(f)$ be the statement

$$(f^{1(0)} \text{ codes a sequence } \subset [0,1]^d \Rightarrow \text{ this sequence has a limit point in } [0,1]^d)$$

(for details see [12]). In [12] we also discuss the (relative to PRA^2_- slightly stronger) principle $\mathrm{BW}^+(f)$ expressing that f possesses a convergent subsequence (with modulus of convergence). All the results of this paper are valid for both versions $\mathrm{BW}(f)$ and $\mathrm{BW}^+(f)$ and so we don't distinguish between them and denote their function parameter-free schematic forms both by BW^-. Likewise for the Arzela-Ascoli lemma (see [12] for the precise formulations of A-A(f) and A-A$^+(f)$).

We define the limit superior according to the so-called ε-definition, i.e. $a \in [-1,1]$ is the limit superior of $(x_n) \subset [-1,1]$ if[8]

$$(*)\ \forall k(\forall m \exists n > m(|a - x_n| \leq \frac{1}{k+1}) \wedge \exists l \forall j > l(x_j \leq a + \frac{1}{k+1})).$$

[7] The restriction to decreasing sequences and the special lower bound 0 is of course inessential.
[8] Again the restriction to the particular bound 1 is inessential.

(∗) implies (relative to PRA^2_-) that a is the maximal limit point of (x_n). The reverse direction follows with the use of BW (we don't know whether it can be proved in PRA^2_-).

Limsup(f) is the principle stating

$$(f \text{ codes a sequence} \subset [-1,1] \Rightarrow \text{this sequence has a } \limsup \text{ in the sense of } (*)).$$

Limsup^- is the corresponding function parameter-free schematic version.

3 Things that can be done in (a conservative extension of) PRA resp. in PRA $+\Sigma^0_2$-IA

In this section we draw some consequences of our results from [12] and [13] and formulate them in a way which fits into the present framework.

Theorem 3.1 *Every* Π^0_3*-theorem of* $\mathrm{PRA}^2_- + \mathrm{AC}^{0,0}$-qf+WKL+$\mathrm{PCM}^-$+$\mathrm{BW}^-$+A-$\mathrm{A}^-$ *is provable in* PRA+Σ^0_1-IA.

Proof: From the proofs of propositions 5.5 and 5.6 from [12] and proposition 5.5.2) below it follows that there exist instances Π^0_1-CA(ξ_j) which prove, relative to E-$\mathrm{G}_\infty\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$-qf+$F^-$ all universal closures $\tilde{G}_i$ of the instances G_i of PCM^-, BW^- and A-A^- which are used in the proof of the Π^0_3-sentence $A \equiv \forall x \exists y \forall z\, A_0(x,y,z) \in$ PRA. The instances Π^0_1-CA(ξ_j) can be coded together into a single instance Π^0_1-CA(ξ) (see again the proof of proposition 5.5 from [12]). Since furthermore $\mathrm{PRA}^2_- \subset$ E-$\mathrm{G}_\infty\mathrm{A}^\omega$ and – by [9] (section 4) – WKL can be derived in E-$\mathrm{G}_\infty\mathrm{A}^\omega$+$\mathrm{AC}^{1,0}$-qf +$F^-$,[9] we obtain

$$\text{E-G}_\infty\text{A}^\omega + \text{AC}^{1,0}\text{-qf} + F^- \vdash \Pi^0_1\text{-CA}(\xi) \to A.$$

Corollary 4.7 from [13] (combined with the elimination of extensionality procedure as used in the proof of corollary 4.5 in [13]) yields that

$$\text{G}_\infty\text{A}^\omega + \Sigma^0_1\text{-IA} \vdash A,$$

and hence (since $\mathrm{G}_\infty\mathrm{A}^\omega + \Sigma^0_1$-IA can easily be seen to be conservative over PRA+Σ^0_1-IA)[10]

$$\text{PRA} + \Sigma^0_1\text{-IA} \vdash A.$$

□

[9]In the proof of theorem 4.27 from [9], $\mathrm{AC}^{0,1}$-qf is only needed to derive the strong sequential version WKL_{seq} of WKL.

[10]We work here in the variant of $\mathrm{G}_\infty\mathrm{A}^\omega$ where the universal axioms 9) are replaced by the schema of quantifier-free induction. The system E-$\mathrm{G}_\infty\mathrm{A}^\omega$ corresponding to this variant is equivalent to the system E-PRA^ω_- from the introduction.

Remark 3.2 *1) In section 4 below we will show that already* PRA^2_-+PCM$^-$ *contains* PRA+Σ^0_1-IA.

2) Already PRA^2_-+ AC^{0,0}-qf+PCM$^-$ *is not* Π^0_4*-conservative over* PRA+Σ^0_1-IA: *From proposition 5.5 below it follows that* PRA^2_-+ AC^{0,0}-qf+PCM$^-$ *proves* Π^0_1-CA$^-$ *and therefore every function parameter-free instance of the principle of* Π^0_1*-collection principle* Π^0_1-CP. *Hence* PRA+Π^0_1-CP *is a subsystem of* PRA^2_-+ AC^{0,0}-qf+PCM$^-$. *However from* [18] *we know that there exists an instance of* Π^0_1-CP *which cannot be proved in* PRA +Σ^0_1-IA. *The claim now follows from the fact that (the universal closure of) every instance of* Π^0_1-CP *can be shown to be equivalent to a* Π^0_4*-sentence in* PRA+Σ^0_1-IA.

Corollary 3.3 *Let* $A \equiv \forall x \exists y A_0(x,y)$ *be a* Π^0_2*-sentence in* $\mathcal{L}(\mathrm{PRA})$. *Then the following rule holds:*

$$\begin{cases} \mathrm{PRA}^2_- + \mathrm{AC}^{0,0}\text{-qf} + \mathrm{WKL} + \mathrm{PCM}^- + \mathrm{BW}^- + \text{A-A}^- \vdash \forall x \exists y A_0(x,y) \\ \Rightarrow \textit{ one can extract a primitive recursive function } p \textit{ such that} \\ \mathrm{PRA} \vdash A_0(x, px). \end{cases}$$

Proof: The corollary follows from theorem 3.1 and the well-known fact that PRA+Σ^0_1-IA is Π^0_2-conservative over PRA. □

We are now ready to prove the second part of theorem 1.4 from the introduction (the first part will be proved in section 6 below):

Theorem 3.4 *Every* Π^0_4*-theorem of* PRA^2_-+AC^{0,0}-qf+WKL+PCM$^-$+BW$^-$+A-A$^-$+Limsup$^-$ *is provable in* PRA +Σ^0_2-IA.

Proof: One easily shows (relative to PRA^2_-+AC^{0,0}-qf) that Limsup$^-$ follows from Π^0_2-CA$^-$: for sequences $(q_n) \subset [0,1]$ of rational numbers this is particularly straightforward (the general case can be reduced to this one): by Π^0_2-CA define f such that for $i < 2^j$

$$f(i,j) = 0 \leftrightarrow \infty\text{-many elements of } (q_n) \text{ are in } [\tfrac{i}{2^j}, \tfrac{i+1}{2^j}].$$

Let $g(j) :=$ maximal $i < 2^j[f(i,j) = 0]$ (By Π^0_1-collection Π^0_1-CP$^{(-)}$, which is derivable from Π^0_2-CA$^{(-)}$ and AC^{0,0}-qf, it follows that such an i always exists). Then (a_n) defined by $a_n := \frac{g(n)}{2^n}$ is a Cauchy sequence which converges (with rate 2^n) to the *limsup* (in the sense of $(*)$) of (q_n).
The theorem now follows from [13](corollary 4.7) similar to the use of this corollary in the proof of theorem 3.1 above. □

Remark 3.5 *In section 5 below we will show that already* PRA^2_{-}+Limsup^- *contains* PRA+Σ^0_2-IA.

Definition 3.6 *By* T_n *we denote the fragment of Gödel's calculus* T *of primitive recursive functionals in all finite types where one only has recursor constants* R_ρ *with* $deg(\rho) \leq n$ *(see* [20] *for details).*

Corollary 3.7 *Let* $A \equiv \forall x \exists y A_0(x, y)$ *be a* Π^0_2*-sentence in* $\mathcal{L}(\mathrm{PRA})$*. Then the following rule holds:*

$$\begin{cases} \mathrm{PRA}^2_{-}+\mathrm{AC}^{0,0}\text{-qf}+\mathrm{WKL}+\mathrm{PCM}^-+\mathrm{BW}^-+\text{A-A}^-+\mathrm{Limsup}^- \vdash \forall x \exists y A_0(x,y) \\ \Rightarrow \text{ one can extract a closed term } \Phi^1 \text{ of } T_1 \text{ such that} \\ T_1 \vdash A_0(x, \Phi x). \end{cases}$$

Proof: The corollary follows from theorem 3.4 and Parsons' result from [20] that PRA+Σ^0_{n+1}-IA has (via negative translation) a Gödel functional interpretation in T_n. $\Box$

Remark 3.8 *Our results in* [12] *and* [13] *actually can be used to obtain stronger forms of the corollaries 3.3 and 3.7 since in* [12],[13] *we*

1) *allowed finite type extensions of the systems in the corollaries 3.3 and 3.7,*
2) *considered more general conclusions* $A \equiv \forall u^1 \forall v \leq_\rho tu \exists z^\tau A_0(x, y, z)$ *(where* ρ *is an arbitrary type and* $\tau \leq 2$*) and showed how to extract uniform bounds* $\Phi \in T_0$ *(resp.* $\in T_1$ *in the case of corollary 3.7) such that* $\forall u^1 \forall v \leq_\rho tu \exists z \leq_\tau \Phi u A_0(x, y, z)$*,*
3) *allowed the instances of* PCM, BW, A-A, Limsup *to depend on the parameters* u, v *of the conclusion and*
4) *allowed more general analytical axioms* Δ *(than only* F^-*).*

One may ask for handy formal systems $\mathcal{T}_1$ and $\mathcal{T}_2$ which allow to **derive** PCM^-, BW^-, A-A$^-$ and – in the case of $\mathcal{T}_2$ – also Limsup^- from simple axioms and which satisfy the conservation results above. This can be achieved most economically by means of non-standard systems which are based on generalized principles of uniform boundedness as introduced in [13]: Let $(\Pi^0_k\text{-UB}^-{\restriction})^-$ be the function parameter-free schematic version of the principle $\forall g(\Pi^0_k$-$\mathrm{UB}^-{\restriction}(g))$ from [13]. Define

$$\mathcal{T}_k := \text{ E-PRA}^\omega_{-}+\mathrm{AC}^{1,0}\text{-qf}+ (\Pi^0_k\text{-UB}^-{\restriction})^-,$$

then one can show – using results from [12] and [13] (and a reasoning similar to remark 3.2.2 above) – that

- $\mathcal{T}_1$ proves WKL, PCM$^-$, BW$^-$, A-A$^-$, Δ^0_2-CA$^-$, Π^0_1-AC$^-$ and is Π^0_3-conservative (but not Π^0_4-conservative) over PRA+Σ^0_1-IA,
- $\mathcal{T}_2$ proves WKL, PCM$^-$, BW$^-$, A-A$^-$, Limsup$^-$, Δ^0_3-CA$^-$, Π^0_2-AC$^-$ and is Π^0_4-conservative (but not Π^0_5-conservative) over PRA+Σ^0_2-IA.

Both systems are non-standard in the sense that the full set-theoretic type structure $\mathcal{S}^\omega$ is not a model of them. However all analytical consequences of $\mathcal{T}_i$ (i.e. their theorems containing only free and bound variables of type ≤ 1) do hold in $\mathcal{S}^\omega$. We will discuss this further in [15].

4 Some proof theory of $\mathbf{PRA}^2 + \Pi^0_1$-$\mathbf{AC}^-$

We consider the following schemata:

$$\begin{aligned}
&\Pi^0_1\text{-CA}^- &&: \exists f^1 \forall x^0 (fx = 0 \leftrightarrow \forall y A_0(x,y)),\\
&\Pi^0_1\text{-}\widehat{\text{AC}}^- &&: \exists f^1 \forall x^0, z^0 (\neg A_0(x, fx) \vee A_0(x,z)),\\
&\Pi^0_1\text{-AC}^- &&: \forall x^0 \exists y^0 \forall z^0 A_0(x,y,z) \to \exists f^1 \forall x, z A_0(x, fx, z),
\end{aligned}$$

where A_0 is quantifier–free and has **no function parameters**.
Π^0_1-CA(g) is the form of Π^0_1-CA$^-$ where $A_0(x,y)$ is replaced by $g(x,y) = 0$. Similarly for Π^0_1-$\widehat{\text{AC}}(g)$ and Π^0_1-AC(g). One easily verifies the following

Lemma 4.1

1) PRA^2 *proves the implications* Π^0_1-AC$^- \to \Pi^0_1$-$\widehat{\text{AC}}^- \to \Pi^0_1$-CA$^-$.

2) PRA^2+$AC^{0,0}$-*qf proves* Π^0_1-CA$^- \leftrightarrow \Pi^0_1$-$\widehat{\text{AC}}^- \leftrightarrow \Pi^0_1$-AC$^-$.

This lemma also holds for EA^2 *and* PRA$^2_-$ *instead of* PRA^2.

Proposition 4.2 *1)* PRA^2+Π^0_1-$\widehat{\text{AC}}^-$ *is closed under* Σ^0_2-IR$^-$ *(i.e. the induction rule for* Σ^0_2*-formulas without function parameters) and hence contains the first-order system* PRA+Σ^0_2-IR.

2) PRA$^2 + \Pi^0_1$-$\widehat{\text{AC}}^-$ *proves every* Π^0_3*-theorem of* PRA $+\Pi^0_2$-IA.

3) Every function which is definable in T_1 *(i.e. every* $\alpha(< \omega^{(\omega^\omega)})$*-recursive function) is provably recursive in* PRA$^2 + \Pi^0_1$-$\widehat{\text{AC}}^-$. *In particular* PRA$^2 + \Pi^0_1$-$\widehat{\text{AC}}^-$ *(and a fortiori* PRA$^2 + \Pi^0_1$-AC$^-$*) proves the totality of the Ackermann function.*

Proof: 1) Let $A \equiv \exists y^0 \forall z^0 A_0(a^0, x^0, y^0, z^0)$ be a Σ^0_2–formula which contains only a, x free. Suppose that $\mathrm{PRA}^2 + \Pi^0_1\text{-}\widehat{\mathrm{AC}}^-$ proves:

$$\exists y \forall z A_0(a, 0, y, z) \wedge \forall x(\exists y \forall z A_0(a, x, y, z) \to \exists y \forall z A_0(a, x', y, z)).$$

For notational simplicity we assume that only one instance of Π^0_1–$\widehat{\mathrm{AC}}^-$ without parameters is used (the universal closure of every instance of Π^0_1–$\widehat{\mathrm{AC}}^-$ with a number parameter a can be reduced to a parameter-free one by coding a and x together) and let this instance be $\exists f \forall a, b(\underbrace{\neg G_0(a, fa) \vee G_0(a, b)}_{\tilde{G}_0 :\equiv})$.

Then (by the deduction theorem for PRA^2)

$$(1)\ \mathrm{PRA}^2 \vdash \exists f \forall a, b\tilde{G}_0 \to \exists y \forall z A_0(a, 0, y, z) \text{ and}$$

$$(2)\ \mathrm{PRA}^2 \vdash \exists f \forall a, b\tilde{G}_0 \to \forall x(\exists y \forall z A_0(a, x, y, z) \to \exists y \forall z A_0(a, x', y, z)).$$

Since

$$\forall g(\forall a, x, y, z(\overbrace{\neg A_0(a, x, y, gaxy) \vee A_0(a, x, y, z)}^{\tilde{A}_0(a,x,y,z,g) :\equiv})$$
$$\to \forall a, x, y(\tilde{g}axy = 0 \leftrightarrow \forall z A_0(a, x, y, z))),$$

where

$$\tilde{g}axy := \begin{cases} 1, \text{if } \neg A_0(a, x, y, gaxy) \\ 0, \text{otherwise} \end{cases}$$

is primitive recursive in g, one has

$$(1)^*\ \mathrm{PRA}^2 \vdash \forall f, g(\forall a, b\tilde{G}_0 \wedge \forall a, x, y, z\tilde{A}_0 \to \exists y_0(\tilde{g}(a, 0, y_0) = 0))$$

$$(2)^* \begin{cases} \mathrm{PRA}^2 \vdash \\ \forall f, g(\forall a, b\tilde{G}_0 \wedge \forall a, x, y, z\tilde{A}_0 \to \forall x(\exists y_1(\tilde{g}axy_1 = 0) \to \exists y_2(\tilde{g}ax'y_2 = 0))). \end{cases}$$

Using (negative translation followed by) functional interpretation combined with normalization (and the fact that the finite type extension of PRA^2 obtained by adding variables and quantifiers for all finite types as well as the Π, Σ-combinators is conservative over PRA^2) or alternatively cut-elimination as in [22]) one obtains closed terms Φ_1, Φ_2 of PRA^2 such that

$$(3)\ \mathrm{PRA}^2 \vdash \begin{cases} \forall f, g(\forall a, b\tilde{G}_0 \wedge \forall a, x, y, z\tilde{A}_0 \to \tilde{g}(a, 0, \Phi_1 fga) = 0 \\ \quad \wedge \forall x, y_1(\tilde{g}(a, x, y_1) = 0 \to \tilde{g}(a, x', \Phi_2(f, g, a, x, y_1) = 0)). \end{cases}$$

Using ordinary (Kleene–) primitive recursion we define in PRA^2 a functional Φ by

$$\begin{cases} \Phi fga0 =_0 \Phi_1 fga \\ \Phi fgax' =_0 \Phi_2(f,g,a,x,\Phi fgax). \end{cases}$$

Using only quantifier-free induction, (3) yields

$$\mathrm{PRA}^2 \vdash \forall f,g(\forall a,b\tilde{G}_0 \wedge \forall a,x,y,z\tilde{A}_0 \rightarrow \forall x(\tilde{g}(a,x,\Phi fgax)=0)),$$

hence $\mathrm{PRA}^2 \vdash \forall f,g(\forall a,b\tilde{G}_0 \wedge \forall a,x,y,z\tilde{A}_0 \rightarrow \forall x\exists y\forall z A_0(a,x,y,z))$
and therefore $\mathrm{PRA}^2 + \Pi^0_1\text{-}\widehat{\mathrm{AC}}^- \vdash \forall x\exists y\forall z A_0(a,x,y,z)$.
2) follows from 1) using the result from [20] that PRA+Σ^0_2-IR proves every Π^0_3-theorem of PRA+Π^0_2-IA and the fact that $\mathrm{PRA}^2 + \Sigma^0_2\text{-IR}^- \supseteq$ PRA +Σ^0_2-IR.
3) follows from 2) and the fact (see e.g. [19]) that the provably recursive functions of PRA+Π^0_2–IA are just the functions definable in T_1 (i.e. the $\alpha(<\omega^{(\omega^\omega)})$-recursive functions) which includes the Ackermann function.

□

Corollary to the proof of proposition 4.2.1: Even when the premises of Σ^0_2-IR$^-$ are proved in $\mathrm{PRA}^2 + \Pi^0_1\text{-}\widehat{\mathrm{AC}}^-$+AC^{0,0}-qf, the conclusion of Σ^0_2-IR$^-$ is provable in $\mathrm{PRA}^2 + \Pi^0_1$-$\widehat{\mathrm{AC}}^-$.

Remark 4.3 *The only part of the proof of proposition 4.2 which cannot be carried out with* PRA^2_- *instead of* PRA^2 *is the definition of* Φ.

Proposition 4.4 PRA^2+ AC^{0,0}-qf +Π^0_1-CA$^-$ *contains* PRA +Π^0_2-IA (=PRA +Σ^0_2-IA).

Proof: One easily shows that PRA^2+ AC^{0,0}-qf proves the second-order axiom of Σ^0_1-induction

$$\forall f(\exists y(f(0,y)=0 \wedge \forall x(\exists y(f(x,y)=0) \rightarrow \exists y(f(x',y)=0)) \rightarrow \forall x\exists y(f(x,y)=0)).$$

Together with Π^0_1-CA$^-$ this yields every function parameter-free instance of Σ^0_2-induction. The proposition now follows from the fact that Π^0_2-IA and Σ^0_2-IA are equivalent over PRA (see e.g. [23]). □

5 Where the convergence of bounded monotone sequences of real numbers goes beyond PRA

We now determine the pointwise relationship of PCM_{ar} and PCM to Σ^0_1-IA and Π^0_1-$\widehat{\mathrm{AC}}$ respectively and use this to calibrate the strength of PCM$^-$ relative to PRA^2.

We first show a result which in particular implies that, relatively to EA^2, the principle (PCM_{ar}) is equivalent to the axiom of Σ^0_1-induction

Σ^0_1-IA : $\forall g^{000}(\exists y^0(g0y =_0 0) \wedge \forall x^0(\exists y^0(gxy =_0 0) \to \exists y^0(gx'y =_0 0)) \to \forall x^0 \exists y^0(gxy =_0 0))$.

Remark 5.1 *This axiom is (relative to* EA^2*) equivalent to the schema of induction for all* Σ^0_1*-formulas in* $\mathcal{L}(\mathrm{EA}^2)$ *: Let* $\exists y^0 A_0(\underline{f}, \underline{x}, y)$ *be a* Σ^0_1*–formula (containing only* $\underline{f}, \underline{x}$ *as free function and number variables). There exists a term* $t_{A_0} \in \mathrm{EA}^2$ *such that*

$$\mathrm{EA}^2 \vdash \forall \underline{f}, \underline{x}(\exists y^0 A_0(\underline{f}, \underline{x}, y) \leftrightarrow \exists y^0(t_{A_0}\underline{f}\,\underline{x}y =_0 0)).$$

Proposition 5.2 *One can construct functionals* $\Psi_1, \Psi_2 \in \mathrm{EA}^2$ *such that:*
1) EA^2 *proves*

$$\forall a^{1(0)}\Big(\forall k^0[\exists y^0(\Psi_1 ak0y =_0 0) \wedge \forall x^0(\exists y^0(\Psi_1 akxy =_0 0) \to \exists y^0(\Psi_1 akx'y =_0 0)) \to \\ \forall x^0 \exists y^0(\Psi_1 akxy =_0 0)] \to [\forall n^0(0 \le_{\mathbb{R}} a(n+1) \le_{\mathbb{R}} an) \\ \to \forall k^0 \exists n^0 \forall m, \tilde{m} \ge_0 n(|am -_{\mathbb{R}} a\tilde{m}| \le_{\mathbb{R}} \tfrac{1}{k+1})]\Big).$$

2) EA^2 *proves*

$$\forall g^{000}\Big([\forall n^0(0 \le_{\mathbb{Q}} \Psi_2 g(n+1) \le_{\mathbb{Q}} \Psi_2 gn \le_{\mathbb{Q}} 1) \to \\ \forall k^0 \exists n^0 \forall m, \tilde{m} \ge_0 n(|\Psi_2 gm -_{\mathbb{Q}} \Psi_2 g\tilde{m}| \le_{\mathbb{Q}} \tfrac{1}{k+1})] \\ \to [\exists y^0(g0y =_0 0) \wedge \forall x^0(\exists y^0(gxy =_0 0) \to \exists y^0(gx'y =_0 0)) \to \forall x^0 \exists y^0(gxy =_0 0)]\Big).$$

Proof: 1) Assume that $\forall n^0(0 \le_{\mathbb{R}} a(n+1) \le_{\mathbb{R}} an)$ and $\exists k \forall n \exists m > n(|am -_{\mathbb{R}} an| >_{\mathbb{R}} \frac{1}{k+1})$. By Σ^0_1–IA one proves that

$$(+) \quad \begin{cases} \forall n^0 \exists i^0(lth(i) = n \wedge \forall j <_0 n \dot{-} 1((i)_j < (i)_{j+1} \\ \qquad \wedge (a((i)_j) \widehat{-_{\mathbb{R}}} a((i)_{j+1}))(3(k+1)) >_{\mathbb{Q}} \frac{2}{3(k+1)})). \end{cases}$$

Let $C \in \mathbb{N}$ be such that $C \ge_{\mathbb{R}} a(0)$. For $n := 3C(k+1)+1$, (+) yields an $i \in \mathbb{N}$ such that

$$\forall j < 3C(k+1)((i)_j < (i)_{j+1}) \text{ and} \\ \forall j < 3C(k+1)(a((i)_j) -_{\mathbb{R}} a((i)_{j+1}) >_{\mathbb{R}} \tfrac{1}{3(k+1)}).$$

Hence $a((i)_0) -_{\mathbb{R}} a((i)_{3C(k+1)}) >_{\mathbb{R}} C$ which contradicts the assumption $\forall n(0 \leq_{\mathbb{R}} a_n \leq_{\mathbb{R}} C)$. Define

$$\Psi_1 akni :=_0 \begin{cases} 0, \text{ if } lth(i) = n \wedge \forall j <_0 n \dot{-} 1((i)_j < (i)_{j+1} \\ \qquad \wedge (a((i)_j) \widehat{-_{\mathbb{R}}} a((i)_{j+1}))(3(k+1)) >_{\mathbb{Q}} \frac{2}{3(k+1)}) \\ 1, \text{otherwise.} \end{cases}$$

2) Define $\Psi_2 \in \mathrm{EA}^2$ such that $\Psi_2 gn =_{\mathbb{Q}} 1 -_{\mathbb{Q}} \sum_{i=1}^{n} \frac{\chi gni}{i(i+1)}$, where $\chi \in \mathrm{EA}^2$ such that

$$\chi gni =_0 \begin{cases} 1, \text{ if } \exists l \leq_0 n(gil =_0 0) \\ 0, \text{ otherwise.} \end{cases}$$

Using $\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1$ (which is provable in EA^2) it follows that

$$\forall n^0 (0 \leq_{\mathbb{Q}} \Psi_2 g(n+1) \leq_{\mathbb{Q}} \Psi_2 gn \leq_{\mathbb{Q}} 1).$$

By the assumption there exists an n_x for every $\mathbb{N} \ni x > 0$ such that

$$\forall m, \tilde{m} \geq n_x (|\Psi_2 gm -_{\mathbb{Q}} \Psi_2 g\tilde{m}| < \frac{1}{x(x+1)}).$$

Claim: $\forall \tilde{x}(0 < \tilde{x} \leq_0 x \to (\exists y(g\tilde{x}y = 0) \leftrightarrow \exists y \leq n_x(g\tilde{x}y = 0)))$:
Assume that $\exists l^0(g\tilde{x}l = 0) \wedge \forall l \leq n_x(g\tilde{x}l \neq 0)$ for some $\tilde{x} > 0$ with $\tilde{x} \leq x$.
Subclaim: Let l_0 be minimal such that $g\tilde{x}l_0 = 0$. Then $l_0 > n_x$ and

$$\Psi_2 g(\max(l_0, \tilde{x})) \leq_{\mathbb{Q}} \Psi_2 g(\max(l_0, \tilde{x}) - 1) -_{\mathbb{Q}} \frac{1}{\tilde{x}(\tilde{x}+1)}.$$

Proof of the subclaim: i) $\sum_{i=1}^{\max(l_0,\tilde{x})} \frac{\chi g(\max(l_0,\tilde{x}))i}{i(i+1)}$ contains $\frac{1}{\tilde{x}(\tilde{x}+1)}$ as an element of the sum, since $g\tilde{x}l_0 = 0$ and therefore $\chi g(\max(l_0, \tilde{x}))\tilde{x} = 1$.
ii) $\sum_{i=1}^{\max(l_0,\tilde{x})-1} \frac{\chi g(\max(l_0,\tilde{x})-1)i}{i(i+1)}$ does not contain $\frac{1}{\tilde{x}(\tilde{x}+1)}$ as an element of the sum:
Case 1. $\tilde{x} \geq l_0$: Then $\max(l_0, \tilde{x}) - 1 = \tilde{x} - 1 < \tilde{x}$.
Case 2. $l_0 > \tilde{x}$: Then $\max(l_0, \tilde{x}) - 1 = l_0 - 1$. Since l_0 is the minimal l such that $g\tilde{x}l = 0$, it follows that

$$\forall i \leq \max(l_0, \tilde{x}) - 1(g\tilde{x}i \neq 0) \text{ and thus } \chi g(\max(l_0, \tilde{x}) - 1)\tilde{x} = 0,$$

which finishes case 2.
Because of

$$\chi g(\max(l_0,\tilde{x})-1)i \neq 0 \rightarrow \chi g(\max(l_0,\tilde{x}))i \neq 0,$$

i) and ii) yield

$$\sum_{i=1}^{\max(l_0,\tilde{x})} \frac{\chi g(\max(l_0,\tilde{x}))i}{i(i+1)} \geq \sum_{i=1}^{\max(l_0,\tilde{x})-1} \frac{\chi g(\max(l_0,\tilde{x})-1)i}{i(i+1)} + \frac{1}{\tilde{x}(\tilde{x}+1)},$$

which concludes the proof of the subclaim.
The subclaim implies

$$\max(l_0,\tilde{x})-1 \geq n_x \wedge |\Psi_2 g(\max(l_0,\tilde{x})) -_{\mathbb{Q}} \Psi_2 g(\max(l_0,\tilde{x})-1)| \geq \frac{1}{x(x+1)}.$$

However this contradicts the construction of n_x and therefore concludes the proof of the claim.
Assume

$$(a)\ \exists y_0(g0y_0 = 0).$$

Define $\Phi \in \mathrm{EA}^2$ such that

$$\Phi g\tilde{x}y = \begin{cases} \min \tilde{y} \leq_0 y[g\tilde{x}\tilde{y} =_0 0], \text{ if } \exists \tilde{y} \leq_0 y(g\tilde{x}\tilde{y} =_0 0) \\ 0^0, \text{otherwise.} \end{cases}$$

By the claim above and (a) we obtain for $y := \max(n_x, y_0)$:

$$(b)\ \forall \tilde{x} \leq_0 x(\exists \tilde{y}(g\tilde{x}\tilde{y} =_0 0) \leftrightarrow g\tilde{x}(\Phi g\tilde{x}y) =_0 0).$$

QF–IA applied to $A_0(\tilde{x}) :\equiv (g\tilde{x}(\Phi g\tilde{x}y) =_0 0)$ yields

$$g0(\Phi g0y) = 0 \wedge \forall \tilde{x} < x(g\tilde{x}(\Phi g\tilde{x}y) = 0 \rightarrow g\tilde{x}'(\Phi g\tilde{x}'y) = 0) \rightarrow gx(\Phi gxy) = 0.$$

From this and $(a), (b)$ we obtain

$$\exists y_0(g0y_0 = 0) \wedge \forall \tilde{x} < x(\exists \tilde{y}(g\tilde{x}\tilde{y} = 0) \rightarrow \exists \tilde{y}(g\tilde{x}'\tilde{y} = 0)) \rightarrow \exists \tilde{y}(gx\tilde{y} = 0).$$

□

Corollary 5.3

$$\mathrm{EA}^2 \vdash \Sigma^0_1\text{-IA} \leftrightarrow \mathrm{PCM}_{ar}.$$

Remark 5.4 *1) Proposition 5.2 provides much more information than corollary 5.3. In particular one can compute (in* EA^2*) uniformly in* g *a (*EA*-provable) decreasing sequence of positive rational numbers such that the Cauchy property of this sequence implies induction for the* Σ^0_1*–formula* $A(x) :\equiv \exists y(gxy = 0)$*. The converse is not that explicit but* Ψ_1 *provides an* **arithmetical family** $A_k(x) :\equiv \exists y(\Psi_1 akxy = 0)$ *of* Σ^0_1*–formulas such that the induction principle for these formulas implies the Cauchy property of the decreasing sequence of positive reals* a*.*

2) The proof of proposition 5.2.2) could be simplified a bit by using $\sum_{i=0}^{\infty} 2^{-i}$ *instead of* $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$*. However* $a_n :=_{\mathbb{R}} \sum_{i=1}^{n} \frac{1}{i(i+1)}$ *as a sequence of real numbers (but not as rational numbers) can be defined already at the second level of the Grzegorczyk hierarchy so that the implication* $\mathrm{PCM}_{ar} \to \Sigma^0_1\text{-IA}$ *holds even in* G_2A^ω *(see* [14]*).*

We now show that $\Pi^0_1\text{-}\widehat{\mathrm{AC}}(a)$ can be reduced to $\mathrm{PCM}(\xi a)$ (for a suitable $\xi \in \mathrm{EA}^2$) relative to EA^2 and that $\mathrm{PCM}(a)$ can be reduced to $\Pi^0_1\text{-AC}(\zeta a)$.

Proposition 5.5 *1) For a suitable closed term* Φ *of* EA^2 *we have*

$$\mathrm{EA}^2 \vdash \forall f^1(\Pi^0_1\text{-AC}(\Phi f) \to \mathrm{PCM}(f)).$$

2) $\mathrm{EA}^2 \vdash \forall f^{1(0)}(\mathrm{PCM}(\lambda n^0.\Psi_2 f' n) \to \Pi^0_1\text{-}\widehat{\mathrm{AC}}(f))$,[11]
where $\Psi_2 \in \mathrm{EA}^2$ *is the functional from prop. 5.2.2) such that* $\Psi_2 fn =_{\mathbb{Q}} 1 -_{\mathbb{Q}} \sum\limits_{i=1}^{n} \frac{\chi fni}{i(i+1)}$
and $\chi \in \mathrm{EA}^2$ *such that*

$$\chi fni =_0 \begin{cases} 1^0, \textit{if } \exists l \leq_0 n(fil =_0 0) \\ 0^0, \textit{otherwise}, \quad \textit{and} \end{cases}$$

$$f' := \lambda x, y.\overline{sg}(fxy).$$

Proof: 1) Let $\Psi_1 \in \mathrm{EA}^2$ be as in proposition 5.2.1. By $\Pi^0_1\text{-CA}(\tilde{\Psi}_1 f)$, where $\tilde{\Psi}_1 fxy = \Psi_1(f, j_1 x, j_2 x, y)$, there exists a function g such that

$$\forall k^0, x^0(gkx = 0 \leftrightarrow \exists y(\Psi_1(f,k,x,y) = 0)).$$

Hence (by applying QF-IA to '$gkx = 0$') one gets

$$\forall k^0(\exists y^0(\Psi_1 fk0y =_0 0) \land \forall x^0(\exists y^0(\Psi_1 fkxy =_0 0) \to \exists y^0(\Psi_1 fkx'y =_0 0)) \to \forall x^0 \exists y^0(\Psi_1 fkxy =_0 0))$$

[11]Strictly speaking we refer here to the embedding $\lambda k.\Psi_2 f'n$ of $\mathbb{Q}$ into $\mathbb{R}$ instead of $\Psi_2 f'n$.

and therefore (by proposition 5.2.1) $\mathrm{PCM}_{ar}(f)$. For a suitable $\tilde{\Phi} \in \mathrm{EA}^2$, Π^0_1-AC$(\tilde{\Phi}f)$ allows to derive PCM(f) from $\mathrm{PCM}_{ar}(f)$. Π^0_1-CA$(\tilde{\Psi}_1 f)$ follows from Π^0_1-AC$(\hat{\Psi}_1 f)$ for a suitable $\hat{\Psi}_1 \in \mathrm{EA}^2$. Finally both instances Π^0_1-AC$(\tilde{\Phi}f)$ and Π^0_1-AC$(\hat{\Psi}_1 f)$ can be coded together into a single instance Π^0_1-AC(Φf) for a suitable $\Phi \in \mathrm{EA}^2$ (using that the universal closure w.r.t. arithmetical parameters is incorporated into the definition of Π^0_1-AC(f)). Hence

$$\mathrm{EA}^2 \vdash \forall f^1(\Pi^0_1\text{-AC}(\Phi f) \to \mathrm{PCM}(f)).$$

2) From the proof of prop.5.2.2) we know

$$(1)\ \forall n^0(0 \leq_{\mathbb{Q}} \Psi_2 f'(n+1) \leq_{\mathbb{Q}} \Psi_2 f' n)$$

and

$$(2) \begin{cases} \forall x >_0 0 \forall n\Big((\forall m, \tilde{m} \geq n \to |\Psi_2 f' m -_{\mathbb{Q}} \Psi_2 f' \tilde{m}| <_{\mathbb{Q}} \frac{1}{x(x+1)}) \to \\ \forall \tilde{x}(0 <_0 \tilde{x} \leq_0 x \to (\exists y(f'\tilde{x}y = 0) \leftrightarrow \exists y \leq_0 n(f'\tilde{x}y = 0)))\Big) \end{cases}$$

By (1) and (PCM)$(\lambda n^0.\Psi_2 f' n)$ there exists a function h^1 such that

$$\forall x >_0 0 \forall m, \tilde{m} \geq_0 hx(|\Psi_2 f' m -_{\mathbb{Q}} \Psi_2 f' \tilde{m}| <_{\mathbb{Q}} \frac{1}{x(x+1)}).$$

Hence by (2)

$$\forall x >_0 0(\exists y(f'xy = 0) \leftrightarrow \exists y \leq_0 hx(f'xy = 0)).$$

Furthermore, (classical) logic yields $\exists z_0(f0z_0 \neq_0 0 \vee \forall y(f0y = 0))$. One now easily concludes that Π^0_1-$\widehat{\mathrm{AC}}(f)$. $\Box$

Remark 5.6 *Proposition 5.5 in particular yields that relatively to* EA^2 *the principle* PCM:$\equiv \forall f$ PCM(f) *implies the schema of arithmetical comprehension. For RCA_0 instead of* EA^2 *this implication is stated in* [4]*. A proof (which is different from our proof) can be found in* [24].

Lemma 4.1.2) and proposition 5.5 imply

Corollary 5.7 EA^2+$\mathrm{AC}^{0,0}$-qf $\vdash \Pi^0_1$-$\mathrm{CA}^- \leftrightarrow \mathrm{PCM}^-$ *and* $\mathrm{EA}^2 \vdash \mathrm{PCM}^- \to \Pi^0_1$-$\widehat{\mathrm{AC}}^-$. *Analogously for* PRA^2_-, PRA^2 *instead of* EA^2.

Theorem 3.1, remark 3.2.2), lemma 4.1, corollary 5.7 and the fact that $\mathrm{PRA}^2_- + \Pi^0_1$-$\mathrm{CA}^-$ proves every functions parameter free instance of Σ^0_1-IA yield theorem 1.2 from the introduction.
The corollary to the proof of proposition 4.2.1 and corollary 5.7 imply theorem 1.6 from the

introduction.

Proposition 4.2.2 and proposition 5.5.2 together yield (using the fact that finitely many instances of Π^0_1-$\widehat{\mathrm{AC}}^-$ can be coded into a single function **and** number parameter-free instance)

Theorem 5.8 *Let* $A \in \Pi^0_3$*-theorem of* PRA $+\Pi^0_2$-IA*. Then one can construct a primitive recursive sequence* $(q_n)_{n\in\mathbb{N}}$ *of (codes of) rational numbers such that*

$$\mathrm{PRA} \vdash \forall n, m(n \geq_0 m \to 0 \leq_{\mathbb{Q}} q_n \leq_{\mathbb{Q}} q_m \leq_{\mathbb{Q}} 1)$$

and

$$\mathrm{PRA}^2 + \mathrm{PCM}(q_n) \vdash A.$$

Corollary 5.9 PRA^2+ PCM^- *proves every* Π^0_3*-theorem of* PRA+Π^0_2-IA*. In particular:* PRA^2+ PCM^- *proves the totality of the Ackermann function (and more general of every* $\alpha(< \omega^{(\omega^\omega)})$*-recursive function, i.e. of every function definable in* T_1*).*

We are now able to prove theorem 1.9 from the introduction:

Theorem 5.10 *Let* P *be* PCM^-, BW^- *or* A-A^-*. Then* PRA^2+ $\mathrm{AC}^{0,0}$-qf $+P$ *contains* PRA $+\Pi^0_2$-IA (=PRA $+\Sigma^0_2$-IA).

Proof: For PCM^- this follows from proposition 4.4 and corollary 5.7. BW^- and A-A^- imply PCM^- relative to PRA^2+ $\mathrm{AC}^{0,0}$-qf. □

6 Where the existence of the limit superior of bounded sequences goes beyond PRA

Theorem 6.1 PRA^2_-+ Limsup^- *contains* PRA $+\Sigma^0_2$-IA.

Proof: Let f be a function $\mathbb{N} \to \mathbb{N}$ and define $q^f_n := \frac{1}{f(n)+1}$. Then obviously $(q_n)_{n\in\mathbb{N}} \subset [0,1] \cap \mathbb{Q}$. Let $a := \limsup\limits_{n\to\infty} q^f_n$, where lim sup is defined as in $(*)$ at the end of section 2.

Claim 1: For $k \in \mathbb{N}, k > 0$ we have

$$a \geq_{\mathbb{R}} \frac{1}{k} \leftrightarrow a >_{\mathbb{R}} \frac{1}{k+1} \leftrightarrow \forall n \exists m \geq n(f(m) < k).$$

Proof of claim 1: $\stackrel{1}{\to}$ is trivial.
$\stackrel{2}{\to}$: Assume $\exists n \forall m \geq n(f(m) \geq k)$. Then $\exists n \forall m \geq n(q^f_m \leq_{\mathbb{Q}} \frac{1}{k+1})$ and hence, since a is a

limit point of (q^f_m), $a \leq_{\mathbb{R}} \frac{1}{k+1}$.
$\stackrel{2}{\leftarrow}$: $\forall n \exists m \geq n(f(m) < k)$ implies $\forall n \exists m \geq n(f(m) \leq k-1)$ and therefore

$$(1)\ \forall n \exists m \geq n(q^f_m \geq_{\mathbb{Q}} \frac{1}{k} =_{\mathbb{Q}} \frac{1}{k+1} + \frac{1}{k(k+1)}).$$

Since a is the maximal limit point of $(q^f_n)_{n\in\mathbb{N}}$, we have

$$(2)\ \exists n \forall m \geq n(q^f_m <_{\mathbb{R}} a + \frac{1}{k(k+1)}).$$

(1) and (2) yield that $a >_{\mathbb{R}} \frac{1}{k+1}$.
$\stackrel{1}{\leftarrow}$: We have already shown that $a >_{\mathbb{R}} \frac{1}{k+1}$ implies $\forall n \exists m \geq n(f(m) \leq k-1)$ and so $\forall n \exists m \geq n(q^f_m \geq_{\mathbb{Q}} \frac{1}{k})$ and hence $a \geq_{\mathbb{R}} \frac{1}{k}$, since a is the maximal limit point of $(q^f_n)_{n\in\mathbb{N}}$.
Claim 2: Relative to PRA^2_- we have

$$\begin{cases} \forall a^1, k^0 (a =_{\mathbb{R}} \limsup\limits_{n\to\infty} q^f_n \wedge \forall n \exists m \geq n(f(m) < k) \\ \qquad \to \exists k_0 \leq k(k_0 \text{ minimal such that } \forall n \exists m \geq n(f(m) < k_0)). \end{cases}$$

Proof of claim 2: Assume $a =_{\mathbb{R}} \limsup\limits_{n\to\infty} q^f_n$ and $\forall n \exists m \geq n(f(m) < k)$. Then, by claim 1, $a \geq_{\mathbb{R}} \frac{1}{k}$. We now show that there exists a k_0 such that $0 < k_0 \leq k$ and $a =_{\mathbb{R}} \frac{1}{k_0}$ (it is clear that k_0 is minimal such that $\forall n \exists m \geq n(f(m) < k_0)$ since otherwise (by claim 1) $a \geq_{\mathbb{R}} \frac{1}{k_0-1}$). Let k_0, $0 < k_0 \leq k$, be such that $|\frac{1}{k_0} -_{\mathbb{Q}} a(2k(k+1))|$ is minimal. Then $\frac{1}{k_0+1} <_{\mathbb{R}} a$ and, if $k_0 - 1 > 0$, $a <_{\mathbb{R}} \frac{1}{k_0-1}$, since

$$\frac{1}{2k(k+1)} \leq \frac{1}{2}\left(\frac{1}{k_0} - \frac{1}{k_0+1}\right) \stackrel{\text{if} k_0-1>0}{<} \frac{1}{2}\left(\frac{1}{k_0-1} - \frac{1}{k_0}\right)$$

and $|a -_{\mathbb{R}} a(2k(k+1))| <_{\mathbb{R}} \frac{1}{2k(k+1)}$.
Claim 1 now implies that $a =_{\mathbb{R}} \frac{1}{k_0}$.
Claim 3: Relative to PRA^2_- we have

$$\begin{cases} \forall a^1, k^0 (a =_{\mathbb{R}} \limsup\limits_{n\to\infty} q^f_n \wedge \forall n \exists m \geq n(f(m) = k) \\ \qquad \to \exists k_0 \leq k(k_0 \text{ minimal such that } \forall n \exists m \geq n(f(m) = k_0)). \end{cases}$$

Proof of claim 3: Assume that $\exists a^1(a =_{\mathbb{R}} \limsup_{n\to\infty} q_n^f)$. Then

$$\begin{aligned}
&\exists k \forall n \exists m \geq n(fm = k) \ \Rightarrow \\
&\exists k \forall n \exists m \geq n(fm < k+1) \ \stackrel{\text{Claim2}}{\Rightarrow} \\
&\exists k(\ k \text{ least such that } \forall n \exists m \geq n(fm < k+1)) \ \Rightarrow \\
&\exists k(\ k \text{ least such that } \forall n \exists m \geq n(fm = k)).
\end{aligned}$$

Claim 4: Let $R(l^0, k^0, m^0)$ be a primitive recursive predicate. Then there exists a primitive recursive function f such that

$$\text{PRA } \vdash \forall l, k \forall \tilde{k} \leq k(\forall n \exists m \geq n\, R(l, \tilde{k}, m) \leftrightarrow \forall n \exists m \geq n(flkm = \tilde{k})).$$

Proof of Claim 4: Define (using the Cantor pairing function j and its projections j_i)

$$\tilde{t}lkm := \begin{cases} j_1 m, \text{ if } R(l, j_1 m, j_2 m) \\ k+1, \text{otherwise.} \end{cases}$$

We show (for all l and all $\tilde{k} \leq k$)

$$\forall n \exists m \geq n(\tilde{t}lkm = \tilde{k}) \leftrightarrow \forall n \exists m \geq n\, R(l, \tilde{k}, m).$$

'$\rightarrow$': Let $n_0 := \max_{i \leq n} j(\tilde{k}, i)$ and $m > n_0$ such that $\tilde{t}lkm = \tilde{k}$. Then $j_1 m = \tilde{k}$, $R(l, \tilde{k}, j_2 m)$ and $j_2 m > n$, since $m = j(\tilde{k}, j_2 m) > n_0$. Hence $\exists m \geq n\, R(l, \tilde{k}, m)$.
'$\leftarrow$': Let $m \geq n$ be such that $R(l, \tilde{k}, m)$. Then $\tilde{t}(l, k, j(\tilde{k}, m)) = \tilde{k}$. Since $j(\tilde{k}, m) \geq m \geq n$, we get $\exists m \geq n(\tilde{t}lkm = \tilde{k})$.
Claim 5: Let $R(k, n, m)$ be primitive recursive and
$\tilde{R}(k, n, m) :\equiv R(k, n, m) \wedge \forall \tilde{m} < m \neg R(k, n, \tilde{m})$. Then PRA $+\Sigma_1^0$-IA proves

$$\forall k(\forall n \exists m\, R(k, n, m) \leftrightarrow \forall n \exists m \geq n(lth(j_2 m) = j_1 m + 1 \wedge \forall \tilde{n} \leq j_1 m\, \tilde{R}(k, \tilde{n}, (j_2 m)_{\tilde{n}}))).$$

Proof of Claim 5:
'$\rightarrow$': Assume $\forall n \exists m\, R(k, n, m)$ and hence $\forall n \exists m\, \tilde{R}(k, n, m)$. By the principle of finite choice for Σ_1^0-formulas (which follows from Σ_1^0-IA, see [18]) we obtain
$\exists \tilde{m}(lth(\tilde{m}) = n + 1 \wedge \forall \tilde{n} \leq n\, \tilde{R}(k, \tilde{n}, (\tilde{m})_{\tilde{n}}))$. So $m := j(n, \tilde{m})$ satisfies the right-hand side of the equivalence.
'$\leftarrow$': Assume

$$(+)\ \forall n \exists m \geq n(lth(j_2 m) = j_1 m + 1 \wedge \forall \tilde{n} \leq j_1 m\, \tilde{R}(k, \tilde{n}, (j_2 m)_{\tilde{n}}))$$

and suppose that $\exists n\forall m\neg R(k,n,m)$ and hence $\exists n\forall m\neg\tilde{R}(k,n,m)$. By the least number principle for Π^0_1-formulas (which easily follows from Σ^0_1-IA) we get a least such n, call it n_0. Hence

$$\forall n < n_0\exists m\,\tilde{R}(k,n,m).$$

Again by finite Σ^0_1-choice we obtain

$$(++)\ \exists m_0(lth(m_0) = n_0 \wedge \forall n < n_0\tilde{R}(k,n,(m_0)_n)).$$

By $(+)$ there exists an $m > j(n_0 \dot{-} 1, m_0)$ such that

$$(+++)\ lth(j_2m) = j_1m + 1 \wedge \forall\tilde{n} \leq j_1m\,\tilde{R}(k,\tilde{n},(j_2m)_{\tilde{n}}).$$

Then either $j_1m \geq n_0$ or $j_1m < n_0 \wedge j_2m > m_0$. The first case yields a contradiction to $\forall m\neg\,\tilde{R}(k,n_0,m)$ and the second case contradicts the fact that (by $\tilde{R}$-definition) $(++)$ and $(+++)$ imply

$$\forall\tilde{n} < lth(j_2m)((j_2m)_{\tilde{n}} = (m_0)_{\tilde{n}}),$$

since for our coding of finite sequences we have

$$lth(a) \leq lth(b) \wedge \forall i < lth(a)((a)_i = (b)_i) \rightarrow b \geq a.$$

We now finish the proof of the theorem. By the claims 3-5 and the fact that $\mathrm{PRA}^2_- +$ $\mathrm{Limsup}^- \vdash \mathrm{PCM}^-_{ar}$ (which in turn yields Σ^0_1-IA^- by proposition 5.2.2, so that PRA $+\Sigma^0_1$-IA is a subsystem of $\mathrm{PRA}^2_- + \mathrm{Limsup}^-$)[12], we obtain in $\mathrm{PRA}^2_- + \mathrm{Limsup}^-$ the least number principle instance

$$\exists k\forall n\exists m\,R(l,k,n,m) \rightarrow \exists k(\ k \text{ minimal such that } \forall n\exists m\,R(l,k,n,m)).$$

Hence $\mathrm{PRA}^2_- + \mathrm{Limsup}^-$ proves every function parameter-free Π^0_2-instance of the least number principle, i.e. Π^0_2-LNP^-. It is an easy exercise to show that this in turn implies Σ^0_2-IA^- which concludes the proof of the theorem since PRA$+\Sigma^0_2$-IA is a pure first-order theory. $\square$

As an immediate corollary of the theorems 3.4 and 6.1 we get theorem 1.4 from the introduction. Corollary 1.5 follows from theorem 1.4 using the fact that PRA$+\Sigma^0_2$-IA has via negative translation a Gödel functional interpretation in T_1 (see [20]) and that the functions definable in T_1 are exactly the $\alpha(<\omega^{(\omega^\omega)})$-recursive ones (see [19]).

[12] Here and below, the superscript '$-$' again indicates that function parameters are not allowed whereas number parameters are allowed.

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