Some computational aspects of metric fixed point theory

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Abstract

We show how bounds on asymptotic regularity for nonexpansive functions can effectively be converted into certain bounds on the convergence towards a fixed point.

1 Introduction

A substantial part of metric fixed point theory studies the fixed point property of selfmappings $f : C \to C$ of convex subsets of normed spaces $(X, \|\cdot\|)$ (or – more generally – hyperbolic spaces, see e.g. [7, 24, 21]) which are nonexpansive, i.e.

$$\|f(x) - f(y)\| \le \|x - y\| \text{ for all } x, y \in C.$$

An important aspect of the fixed point theory of such mappings f is that it has a computational flavor as one can define effective iteration schemata which under general conditions converge towards a fixed point.
The most common schema is the so-called Krasnoselski-Mann iteration which for a

given sequence (λ_n) in $[0,1]$ and starting point $x_0 \in C$ is defined as follows

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n f(x_n).$$

A central result is the following theorem of Ishikawa which holds in arbitrary Banach spaces and generalized many previous results for the more restricted case of uniformly convex spaces:

Theorem 1.1 ([9]) *Let C be a compact convex subset of a Banach space $(X, \|\cdot\|)$ and $f: C \to C$ nonexpansive. Let (λ_n) be a sequence in $[0,b]$ for some $b<1$ such that $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$. Then for any starting point $x_0 \in C$, the sequence (x_n) converges to a fixed point of f.*

Remark 1.2 *In contrast to the context of contractions or contractive mappings, the Picard iteration, which is the special case of the Krasnoselski-Mann iteration for $\lambda_n := 1$, does not work in the nonexpansive setting even for functions as simple as $f:[0,1] \to [0,1], f(x) := 1-x$: for $x \neq \frac{1}{2}$ the Picard iterates oscillate between x and $1-x$, whereas for $\lambda_n := \frac{1}{2}$ the Krasnoselski-Mann iteration yields the fixed point $\frac{1}{2}$ after one step.*[1]

Ishikawa obtains theorem 1.1 by proving the following theorem which only assumes the boundedness of C rather than its compactness:

Theorem 1.3 ([9]) *Let C be a closed bounded convex subset of a Banach space $(X, \|\cdot\|)$ and $f: C \to C$ nonexpansive. Let (λ_n) be as in theorem 1.1. Then for any starting point $x_0 \in C$, the sequence $(\|x_n - f(x_n)\|)_{n \in \mathbb{N}}$ converges to 0 (i.e. (x_n) is a so-called approximate fixed point sequence).*

Remark 1.4 *Ishikawa actually proved that instead of assuming C to be bounded it suffices that the sequence (x_n) is bounded. More general results in this direction can be found in [1]. Ishikawa's theorems were generalized to the setting of hyperbolic spaces (and even spaces of hyperbolic type) in [7].*

Theorem 1.1 follows from theorem 1.3 by the following simple argument: (x_n) has an accumulation point $\hat{x} \in C$ which – in view of theorem 1.3 – must be a fixed point of f. One easily verifies that

$$\|x_{n+1} - \hat{x}\| \le \|x_n - \hat{x}\|$$

[1] This special case where $\lambda_n := \frac{1}{2}$ was introduced by Krasnoselski [22] and is also called Krasnoselski iteration.

which implies that (x_n) itself converges to $\hat{x}$.

The proofs of both theorem 1.1 and 1.3 given in [9] are ineffective. So a natural question to ask is whether there is an effective operation which computes for given f, x_0 and (λ_n) a rate of convergence of (x_n) towards $\hat{x}$, i.e. a Cauchy modulus for (x_n).
Using a construction from [17] we show that even for $X := \mathbb{R}$, $C := [0,1]$ and $\lambda_n := \frac{1}{2}$ such a method does not exist. More specifically, we construct a computable sequence $(f_l)_{l\in\mathbb{N}}$ of nonexpansive functions $f_l : [0,1] \to [0,1]$ such that for the Krasnoselski-Mann iterations $(x^l_n)_{n\in\mathbb{N}}$ of the f_l's starting from $x^l_0 := 0$ there is no computable $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall l, k \forall m \geq \delta(l,k)(|x^l_m - x^l_{\delta(l,k)}| \leq 2^{-k}).$$

Actually, even for fixed $k := 1$ there is no computable function of l with this property.

Remark 1.5 *This result does not rule out that computability holds in a weak pointwise sense, i.e. that for any single fixed computable $f, x_0, (\lambda_n)$ there* **ineffectively exists** *a computable rate of convergence. Whether this holds or not is open.*

On the other hand, as shown in [16] there is an effective rate for the convergence $\|x_n - f(x_n)\| \to 0$ in theorem 1.3 despite the fact that the proof of [9] is ineffective as well. The effective rate of convergence, moreover, is independent from both x_0 and f and only depends on the error k, a bound d on the diameter of C, the upper bound $b < 1$ on λ_n and a rate of divergence of λ_n in sum. Furthermore, the proof does not need X to be complete or C to be closed.
This rate of convergence as well as several others (also for the more general context of hyperbolic spaces) established in [18, 21, 20] was obtained by analyzing proofs such as the one by Ishikawa using techniques from logic. Moreover, there are general logical metatheorems which guarantee this approach to work and predict such strong uniformity features of the bounds (see [19]). The tools from logic actually provide an algorithm for finding such bounds hidden in an ineffective proof of statements like the one in 1.3.

Remark 1.6 *Only for the more special cases of uniformly convex X and $\lambda_n := \frac{1}{2}$ resp. general normed spaces X and $\lambda_n := \lambda \in (0,1)$ effective rates of convergence of $(\|x_n - f(x_n)\|)$ had been obtained before in [13] resp. [2] (the latter actually proves an optimal quadratic bound in this case).*

From the logical point of view the main difference between the Cauchy property of (x_n)

$$(1)\ \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \geq n(\|x_n - x_m\| \leq 2^{-k})$$

and the so-called asympotic regularity[2] statement

$$(2)\ \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \geq n (\|x_m - f(x_m)\| \leq 2^{-k})$$

is that whereas the former has the form $\forall\exists\forall$ the latter can be written equivalently as

$$(3)\ \forall k \in \mathbb{N} \exists n \in \mathbb{N} (\|x_n - f(x_n)\| \leq 2^{-k})$$

which is only $\forall\exists$. This is due to the easy fact that $(\|x_n - f(x_n)\|)_{n\in\mathbb{N}}$ is non-increasing. Whereas general theorems from logic allow one to extract computable bounds from ineffective proofs of $\forall\exists$-theorems this in general is blocked for theorems of the form $\forall\exists\forall$ and – by the result just mentioned – the Cauchy property of (x_n) indeed is of that nature.

Given this state of affairs, the best effective information on (1) one can hope for is to find effective bounds for the so-called Herbrand normal form of (1)

$$(1)^H\ \forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists n \in \mathbb{N} \forall i, j \in [n; n + g(n)] (\|x_i - x_j\| \leq 2^{-k}),$$

where $[n; m]$ denotes the subset $\{n, n+1, \ldots, m-1, m\}$ of $\mathbb{N}$ for $m \geq n$.
(1) trivially implies $(1)^H$ (and a bound on (1) provides a bound on $(1)^H$ which is even independent from g). The converse implication also holds but only ineffectively: assume $(1)^H$ and suppose that (1) would be false. Then there would exist a $k \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \exists m \geq n (\|x_n - x_m\| > 2^{-k}).$$

Hence one can choose a sequence $g(n) := m_n$ such that

$$\forall n \in \mathbb{N} (g(n) \geq n \wedge \|x_n - x_{g(n)}\| > 2^{-k})$$

contradicting $(1)^H$ for $\tilde{g}(n) := g(n) - n$.

In this paper we present an effective procedure Ψ which transforms any given rate Φ of asymptotic regularity, i.e.

$$(4)\ \forall k \in \mathbb{N} \forall m \geq \Phi(k) (\|x_m - f(x_m)\| \leq 2^{-k})$$

into an effective bound $\Psi(k, g, \Phi, \alpha)$ for $(1)^H$, i.e.

$$(5)\ \forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists n \leq \Psi(k, g, \Phi, \alpha) \forall i, j \in [n; n + g(n)] (\|x_i - x_j\| \leq 2^{-k}).$$

[2] A selfmapping f of C is called asymptotical regular if $\|f^n(x) - f^{n+1}(x)\| \to 0$. For constant $\lambda_n := \lambda$ the statement $\|x_n - f(x_n)\| \to 0$ is equivalent to the asymptotic regularity of $f_\lambda := (1-\lambda)I + \lambda f$. We therefore call (following [1]) also the former statement (and for general λ_n) asymptotic regularity.

Here α is a quantitative form of the compactness of C, namely a so-called modulus of total boundedness (see below).
Since $(1)^H$ is equivalent to (1) but (1) is known to fail in general if C instead being compact is just bounded, it is clear that the compactness assumption somehow has to enter the construction. It turns out that it is sufficient to assume C to be totally bounded while it is not needed that X is complete or C is closed.
Since Ψ only depends on the error k, the function g and Φ, α it enjoys every uniformity property Φ has: if Φ does not depend on x_0 or f (as our bounds from [16]) then the same is true also for Ψ.
Suppose for the moment that (λ_n) is bounded away both from 1 **and** 0. Then (3) can be viewed as the special case of $(1)^H$ for $g(n) := 1$ since by

$$\|x_n - x_{n+1}\| = \lambda_n \|x_n - f(x_n)\|$$

any bound on asymptotic regularity provides a bound on the convergence of $(\|x_n - x_{n+1}\|)$ and vice versa. The same holds for any **constant** function $g(n) := k$. So whereas for constant g one essentially can take Ψ in (5) to be just Φ from (4) (without any requirement of total boundedness), the case of general g requires total boundedness as reflected by the dependence of Ψ on α.

Remark 1.7 *That the convergence in theorem 1.3 is uniform for x_0 and f was first shown ineffectively in [7] (for x_0 alone it was established already in [3]). For the significance of this result see e.g. [4].*

We conclude the treatment of nonexpansive functions by showing that the construction Ψ also applies to the context of hyperbolic spaces (and so a-fortiori to CAT(0) spaces, see [12]) in the sense of [10, 24] and also the slightly more general sense of [19] (though not to the still more general spaces of hyperbolic type from [7]).

In the final section of the paper we extend our results to the setting of asymptotically nonexpansive functions as introduced in [6]. In this context, the issue of asymptotic regularity was analyzed quantitatively first in [20]. Since here $(\|x_n - f(x_n)\|)$ no longer is nonincreasing, already a computable bound on asymptotic regularity can be obtained only in the weakened form of the Herbrand normal form. However, it turns out that this is still sufficient for the transformation $\Phi \mapsto \Psi(k, g, \Phi, \alpha)$ as all what is needed for Φ is

$$\forall k \in \mathbb{N} \exists n \leq \Phi(k)(\|x_n - f(x_n)\| \leq 2^{-k})$$

what we call an approximate fixed point bound for (x_n).
Such a Φ (even for asymptotically **quasi-**nonexpansive functions and for Krasnoselski-

Mann iterations with error terms in the sense of [25])[3] is constructed in [20]. Finally, we remark that also the results in the present paper are in fact instances of the general results from logic in [19, 5] (when applied to the sequential compactness argument used to prove theorem 1.1 from theorem 1.3) which (when combined with [15](prop.5.5)) even predict the correct complexity of Ψ.

2 Preliminaries

Throughout this paper $\mathbb{N}$ denotes the set of natural numbers including 0 and $\mathbb{Q}^*_+$ denotes the set of strictly positive rational numbers.

Definition 2.1 *Let (M,d) be a totally bounded metric space. We call $\alpha : \mathbb{N} \to \mathbb{N}$ a modulus of total boundedness for M if for any $k \in \mathbb{N}$ there exist elements $a_0, \ldots, a_{\alpha(k)} \in M$ such that*

$$\forall x \in M \exists i \leq \alpha(k)(d(x,a_i) \leq 2^{-k}).$$

Definition 2.2 *Let (M,d) be a metric space, $f : M \to M$ a selfmapping of M and (x_n) a sequence in M. A function $\Phi : \mathbb{Q}^*_+ \to \mathbb{N}$ is called an approximate fixed point bound for (x_n) if*

$$\forall q \in \mathbb{Q}^*_+ \exists m \leq \Phi(q)(d(x_m, f(x_m)) \leq q).$$

3 The nonexpansive case

We start this section with the negative result mentioned in the introduction:

Proposition 3.1 *The exists a computable sequence $(f_l)_{l\in\mathbb{N}}$ of nonexpansive functions $f_l : [0,1] \to [0,1]$ such that for $\lambda_n := \frac{1}{2}$ and $x^l_0 := 0$ and the corresponding Krasnoselski iterations (x^l_n) there is no computable function $\delta : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall m \geq \delta(l)(|x^l_m - x^l_{\delta(l)}| \leq \frac{1}{2}).$$

[3] In this still more general setting, f must be assumed to be uniformly Lipschitzian which is automatically the case for asymptotically nonexpansive functions.

Proof (based on [17]): Using the so-called Kleene T-predicate[4] from computability theory we define a computable function $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$\alpha(l,n) := \begin{cases} 1, & \text{if } \neg T(l,l,n) \\ 0, & \text{otherwise.} \end{cases}$$

Define $f_l : [0,1] \to [0,1]$ by

$$f_l(x) := a_l x + 1 - a_l, \text{ where } a_l := \sum_{i=0}^{\infty} \alpha(l,i) 2^{-i-1} \in [0,1].$$

(f_l) is a computable sequence (in the sense of computability theory, see e.g. [23] or [26]) of nonexpansive functions.
Assume that a computable function δ satisfying

$$\forall m \geq \delta(l)(|x^l_m - x^l_{\delta(l)}| \leq \frac{1}{2})$$

would exist. One easily verifies that for $m := \delta(l)$

$$a_l < 1 \Rightarrow \lim_{n\to\infty} x^l_n = 1 \Rightarrow x^l_m \in [\tfrac{1}{2}, 1] \text{ and}$$
$$a_l = 1 \Rightarrow \forall n(x^l_n = 0) \Rightarrow x^l_m = 0.$$

Because of this, we can use δ (by computing $x^l_{\delta(l)}$ up to say an error 1/3) to construct a computable function $\chi : \mathbb{N} \to \mathbb{N}$ such that

$$\forall l \in \mathbb{N}(\chi(l) = 0 \leftrightarrow a_l = 1)$$

and hence

$$\forall l \in \mathbb{N}(\chi(l) = 0 \leftrightarrow \forall n \in \mathbb{N} \neg T(l,l,n))$$

which contradicts the well-known undecidability of the so-called special halting problem. □

In the following, $(X, \|\cdot\|)$ is a normed linear space, $C \subset X$ a (nonempty) convex subset of X. Moreover, $f : C \to C$ is a nonexpansive selfmapping of C and (λ_n) a sequence in $[0,1]$. (x_n) denotes the corresponding Krasnoselski-Mann iteration starting from $x_0 \in C$.
The following lemma is easy and well-known (see the proof of lemma 2 in [9]):

[4] $T(l,l,n)$ expresses that the Turing machine with Gödel number l applied to the input l makes a terminating run which has code n.

Lemma 3.2 *$\|x_{n+1} - f(x_{n+1})\| \leq \|x_n - f(x_n)\|$ for all $n \in \mathbb{N}$.*

Remark 3.3 *By lemma 3.2, any approximate fixed point bound Φ for (x_n) is in fact a rate of convergence for $\|x_n - f(x_n)\| \stackrel{n\to\infty}{\to} 0$ which we also call 'rate of asymptotic regularity':*

$$\forall q \in \mathbb{Q}_+^* \forall m \geq \Phi(q)(\|x_m - f(x_m)\| \leq q).$$

Lemma 3.4 *Let $\varepsilon > 0$ and $u \in C$ be an ε-fixed point of f, i.e. $\|u - f(u)\| \leq \varepsilon$. Then for all $n, m \in \mathbb{N}$*

$$\|x_{n+m} - u\| \leq \|x_n - u\| + m \cdot \varepsilon.$$

Proof: Let $n \in \mathbb{N}$ be fixed. We proceed by induction on m :
For $m = 0$, the lemma trivially is true.
$m \mapsto m + 1$:

$$\begin{aligned}
&\|x_{n+m+1} - u\| = \|(1 - \lambda_{n+m})x_{n+m} + \lambda_{n+m} f(x_{n+m}) - u\| \\
&= \|(1 - \lambda_{n+m})x_{n+m} - (1 - \lambda_{n+m})u + \lambda_{n+m} f(x_{n+m}) - \lambda_{n+m} u\| \\
&\leq (1 - \lambda_{n+m})\|x_{n+m} - u\| + \lambda_{n+m}\|f(x_{n+m}) - f(u)\| + \lambda_{n+m}\|f(u) - u\| \\
&\leq \|x_{n+m} - u\| + \|f(u) - u\| \quad (f \text{ nonexpansive}) \\
&\leq \|x_{n+m} - u\| + \varepsilon \stackrel{\text{I.H.}}{\leq} \|x_n - u\| + (m + 1) \cdot \varepsilon.
\end{aligned}$$

$\square$

Notation 3.5 *For $n, m \in \mathbb{N}$ with $m \geq n$, we use $[n; m]$ to denote the set $\{n, n + 1, \ldots, m\} \subset \mathbb{N}$.*

We now assume that C is totally bounded.

Theorem 3.6 *Let $k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}, \Phi : \mathbb{Q}_+^* \to \mathbb{N}$ and $\alpha : \mathbb{N} \to \mathbb{N}$. We define a function $\Psi(k, g, \Phi, \alpha)$ (primitive) recursively as follows:*

$$\Psi(k, g, \Phi, \alpha) := \max_{i \leq \alpha(k+3)} \Psi_0(i, k, g, \Phi),$$

where

$$\begin{cases} \Psi_0(0, k, g, \Phi) := 0 \\ \Psi_0(n + 1, k, g, \Phi) := \Phi\left(2^{-k-2}/(\max_{l \leq n} g(\Psi_0(l, k, g, \Phi)) + 1)\right). \end{cases}$$

If Φ is an approximate fixed point bound for (x_n) and α a modulus of total boundedness[5] *for C then*

$$\forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists n \leq \Psi(k, g, \Phi, \alpha) \forall i, j \in [n; n + g(n)](\|x_i - x_j\| \leq 2^{-k}).$$

Proof: We first note that by lemma 3.2 $x_{\Phi(q)}$ is a q-fixed point of f for any $q \in \mathbb{Q}^*_+$. Define $n_i := \Psi_0(i, k, g, \Phi)$.
Claim: $\exists i, j \leq \alpha(k+3)+1, i \neq j(\|x_{n_i} - x_{n_j}\| \leq 2^{-k-2})$.
Proof of claim: By the assumption on α it follows that there exist points $a_0, \ldots, a_{\alpha(k+3)} \in C$ such that for at least two of the $(\alpha(k+3)+2)$-many indices $0 \leq i \leq \alpha(k+3)+1$ the corresponding x_{n_i}'s must be in a common 2^{-k-3}-ball around some a_l with $l \leq \alpha(k+3)$, i.e.

$$\begin{aligned} &\exists i, j \leq \alpha(k+3)+1, i \neq j, \exists l \leq \alpha(k+3) : \\ &\quad \|a_l - x_{n_i}\| \leq 2^{-k-3} \wedge \|a_l - x_{n_j}\| \leq 2^{-k-3} \end{aligned}$$

and hence $\|x_{n_i} - x_{n_j}\| \leq 2^{-k-2}$.
End of the proof of the claim.
By the claim, let $i < j \leq \alpha(k+3)+1$ be such that

$$\|x_{n_i} - x_{n_j}\| \leq 2^{-k-2}.$$

By construction and $j > 0$, x_{n_j} is a $(2^{-k-2}/(\max\limits_{l<j} g(\Psi_0(l, k, g, \Phi)) + 1))$-fixed point of f and hence a-fortiori a $(2^{-k-2}/(g(\Psi_0(i, k, g, \Phi)) + 1)) = (2^{-k-2}/(g(n_i) + 1))$-fixed point of f. By the lemma above we therefore obtain for all $l \leq g(n_i)$:

$$\begin{aligned} &\|x_{n_i+l} - x_{n_j}\| \leq \|x_{n_i} - x_{n_j}\| + l \cdot \tfrac{2^{-k-2}}{g(n_i)+1} \\ &\leq \|x_{n_i} - x_{n_j}\| + 2^{-k-2} \leq 2^{-k-1}. \end{aligned}$$

Thus

$$\forall j_1, j_2 \in [n_i; n_i + g(n_i)](\|x_{j_1} - x_{j_2}\| \leq 2^{-k}),$$

where $i \leq \alpha(k+3)$. Since $\Psi(k, g, \Phi, \alpha) = \max\{n_i : i \leq \alpha(k+3)\}$, the theorem follows. □

Remark 3.7 Ψ *is a primitive recursive functional in the sense of Kleene [14].*

[5] Here we consider C equipped with the metric induced by $\|\cdot\|$.

Except for the lemmas 3.2 and 3.4, the proof of theorem 3.6 has not really used the linear structure of $(X, \|\cdot\|)$ but only that $d(x,y) := \|x-y\|$ is a metric. Lemma 3.2 is well-known to hold even for spaces of hyperbolic type ([7]) and hence a-fortiori for hyperbolic spaces ([10, 24, 19]). Lemma 3.4 needs a bit more linear structure but still holds for hyperbolic spaces (even in the slightly more general sense of [19]) as we show now (using the notation from [24, 21] where, for $\lambda \in [0,1]$, $(1-\lambda)x \oplus \lambda y$ denotes the unique element z in the metric segment $[x,y]$ satisfying $d(x,z) := \lambda d(x,y)$) the inductive step proceeds as follows:

$$
\begin{aligned}
& d(x_{n+m+1}, u) = d((1-\lambda_{n+m})x_{n+m} \oplus \lambda_{n+m} f(x_{n+m}), u) \\
& \leq d((1-\lambda_{n+m})x_{n+m} \oplus \lambda_{n+m} f(x_{n+m}), (1-\lambda_{n+m})u \oplus \lambda_{n+m} f(u)) \\
& \qquad + d((1-\lambda_{n+m})u \oplus \lambda_{n+m} f(u), u) \\
& \overset{[21](2.8)}{\leq} (1-\lambda_{n+m}) d(x_{n+m}, u) + \lambda_{n+m} d(f(x_{n+m}), f(u)) \\
& \qquad + d((1-\lambda_{n+m})u \oplus \lambda_{n+m} f(u), u) \\
& \overset{f \text{ n.e.}}{\leq} (1-\lambda_{n+m}) d(x_{n+m}, u) + \lambda_{n+m} d(x_{n+m}, u) + d((1-\lambda_{n+m})u \oplus \lambda_{n+m} f(u), u) \\
& = d(x_{n+m}, u) + d((1-\lambda_{n+m})u \oplus \lambda_{n+m} f(u), u) \\
& \overset{[21](2.14)}{\leq} d(x_{n+m}, u) + (1-\lambda_{n+m}) d(u,u) + \lambda_{n+m} d(f(u), u) \\
& \leq d(x_{n+m}, u) + \varepsilon.
\end{aligned}
$$

Hence we get

Theorem 3.8 *Theorem 3.6 also holds for hyperbolic spaces (both in the sense of [10, 24] as well as in the slightly more general sense of [19]).*

Examples of effective asymptotic regularity bounds Φ:

1) In [16], the following rate of asymptotic regularity for Ishikawa's theorem is established: Let $(X, \|\cdot\|)$ be an arbitrary normed space and $C \subset X$ a bounded convex subset. Let $d \geq diam(C)$ and (λ_n) as in theorem 1.1 with $K \in \mathbb{N}$ such that $\lambda_n \leq 1 - \frac{1}{K}$ and $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n, i \in \mathbb{N}$

$$\beta(i,n) \leq \beta(i+1,n) \text{ and}$$

$$n \leq \sum_{s=i}^{i+\beta(i,n)-1} \lambda_s.$$

Let $(x_n)_{n\in\mathbb{N}}$ be the Krasnoselski-Mann iteration

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n f(x_n),\ x_0 := x$$

starting from $x \in C$. Then the following holds

$$\forall x \in C \forall q \in \mathbb{Q}^*_+ \forall n \geq \Phi(q,d,K,\beta)(\|x_n - f(x_n)\| \leq q),$$

where

$$\begin{array}{l} \Phi(q,d,K,\beta) := \widehat{\beta}(\lceil 2d \cdot \exp(K(M+1))\rceil - 1, M),\ \text{with} \\ M := \left\lceil \frac{1+2d}{q} \right\rceil \ \text{and} \\ \widehat{\beta}(0,M) := \tilde{\beta}(0,M),\ \widehat{\beta}(m+1,M) := \tilde{\beta}(\widehat{\beta}(m,M),M) \ \text{with} \\ \tilde{\beta}(m,M) := m + \beta(m,M) \ \ (m \in \mathbb{N}). \end{array}$$

In [21] it is shown that the same bound applies to hyperbolic spaces and a slight variant thereof also to so-called directionally nonexpansive functions (see [11]). For normed spaces and the special case $\lambda_n := \lambda \in (0,1)$ an in fact optimal quadratic bound is due to [2].

2) In the case of bounded convex subsets $C \subset X$ of **uniformly convex** spaces $(X, \|\cdot\|)$, asymptotic regularity is known to hold even under slightly weaker conditions on (λ_n) in $[0,1]$: as shown in [8] the only condition needed is that $\sum\limits_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty$ (note that (λ_n) is no longer required to be bounded away from 1). Let $\eta : (0,2] \to (0,1]$ be a modulus of uniform convexity for X, i.e.

$$\|x\|, \|y\| \leq 1, \|x-y\| \geq \varepsilon \Rightarrow \left\|\frac{1}{2}(x+y)\right\| \leq 1 - \eta(\varepsilon) \text{ for all } \varepsilon \in (0,2], x, y \in X,$$

and $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ \sum_{k=0}^{\gamma(n)} \lambda_k(1-\lambda_k) \geq n.$$

In [18](thm.3.4,cor.3.5) it is shown that

$$\Phi(q,d,\eta,\gamma) := \gamma\left(\left\lceil \frac{3(d+1)}{2q \cdot \eta(\frac{q}{d+1})} \right\rceil\right) \text{ for } q < 2d \text{ and } := 0 \text{ otherwise}$$

is a rate of asymptotic regularity (where, again, $d \geq diam(C)$).
Moreover, if $\eta(\varepsilon)$ can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with

$$\varepsilon_1 \geq \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \geq \tilde{\eta}(\varepsilon_2), \text{ for all } \varepsilon_1, \varepsilon_2 \in (0,2],$$

then the bound $\Phi(q,d,\eta,\gamma)$ can be replaced (for $q < 2d$) by

$$\tilde{\Phi}(q,d,\eta,\gamma) := \gamma\left(\left\lceil \frac{d+1}{2q \cdot \tilde{\eta}(\frac{q}{d+1})} \right\rceil\right).$$

The special case of $\lambda_n := \frac{1}{2}$ was already treated in [13].

4 The asymptotically nonexpansive case

Let $(X, \|\cdot\|)$ be a normed space, $\emptyset \neq C \subset X$ convex and (λ_n) a sequence in $[0,1]$. The class of asymptotically nonexpansive mappings $f : C \to C$ was introduced in [6]:

Definition 4.1 *$f : C \to C$ is said to be asymptotically nonexpansive with sequence $(k_n) \in [0,\infty)^{\mathbb{N}}$ if $\lim_{n\to\infty} k_n = 0$ and*

$$\|f^n(x) - f^n(y)\| \leq (1 + k_n)\|x - y\|, \ \forall n \in \mathbb{N}, \forall x, y \in C.$$

In the context of asymptotically nonexpansive mappings $f : C \to C$ the Krasnoselski-Mann iteration is defined in a slightly different way as follows

$$x_0 := x \in C, \ x_{n+1} := (1-\lambda_n)x_n + \lambda_n f^n(x_n).$$

In the following we assume that f is asymptotically nonexpansive with a sequence (k_n) which is bounded in sum by some $K \in \mathbb{N}$, i.e. $\sum\limits_{n=0}^{\infty} k_n \leq K$.

Lemma 4.2 *For all $\varepsilon > 0, n \geq 1$ and $u \in C$ we have*

$$\|u - f(u)\| \leq \varepsilon \to \|u - f^n(u)\| \leq (n + K) \cdot \varepsilon.$$

Proof: Let $\|u - f(u)\| \leq \varepsilon$. The lemma follows from

$$\|u - f^n(u)\| \leq \sum_{i=0}^{n-1} (1 + k_i) \cdot \varepsilon$$

which we prove by induction:
$n = 1$ is trivial. $n \mapsto n+1$:

$$\begin{aligned}
&\|f^{n+1}(u) - u\| = \|f^n(f(u)) - f^n(u) + f^n(u) - u\| \\
&\leq \|f^n(f(u)) - f^n(u)\| + \|f^n(u) - u\| \\
&\leq (1+k_n)\|f(u) - u\| + \|f^n(u) - u\| \\
&\overset{\text{I.H.}}{\leq} (1+k_n) \cdot \varepsilon + \sum_{i=0}^{n-1} (1+k_i) \cdot \varepsilon = \sum_{i=0}^{n} (1+k_i) \cdot \varepsilon.
\end{aligned}$$

□

Lemma 4.3 *Let* $\|u - f(u)\| \leq \varepsilon$. *Then for all* $n \in \mathbb{N}$

$$\|x_{n+1} - u\| \leq (1+k_n)\|x_n - u\| + (n+K) \cdot \varepsilon.$$

Proof:

$$\begin{aligned}
&\|x_{n+1} - u\| = \|(1-\lambda_n)x_n + \lambda_n f^n(x_n) - u\| \\
&= \|(1-\lambda_n)(x_n - u) + \lambda_n(f^n(x_n) - f^n(u)) + \lambda_n(f^n(u) - u)\| \\
&\overset{L.4.2}{\leq} (1-\lambda_n)\|x_n - u\| + \lambda_n(1+k_n)\|x_n - u\| + \lambda_n(n+K) \cdot \varepsilon \\
&\leq (1+k_n)\|x_n - u\| + (n+K) \cdot \varepsilon.
\end{aligned}$$

□

Lemma 4.4 *Let* $\|u - f(u)\| \leq \varepsilon$. *Then for all* $m, n \in \mathbb{N}$

$$\|x_{n+m} - u\| \leq e^K\|x_n - u\| + e^K m(n+m+K) \cdot \varepsilon.$$

Proof: Let $k(n,m) := \sum_{i=n}^{n+m-1} k_i$. The lemma follows from

$$\|x_{n+m} - u\| \leq e^{k(n,m)}\|x_n - u\| + e^{k(n,m)} \cdot \sum_{i=n}^{n+m-1} (i+K) \cdot \varepsilon$$

which we prove by induction on m : The case $m = 0$ is trivial. $m \mapsto m+1$:

$$\begin{aligned}
&\|x_{n+m+1} - u\| \overset{L.4.3}{\leq} (1 + k_{n+m})\|x_{n+m} - u\| + (n+m+K) \cdot \varepsilon \\
&\leq e^{k_{n+m}}\|x_{n+m} - u\| + (n+m+K) \cdot \varepsilon \quad (1 + x \leq e^x,\ x \geq 0) \\
&\overset{\text{I.H.}}{\leq} e^{k_{n+m}}\Big(e^{k(n,m)}\|x_n - u\| + e^{k(n,m)} \cdot \sum_{i=n}^{n+m-1}(i+K) \cdot \varepsilon\Big) + (n+m+K) \cdot \varepsilon \\
&= e^{k(n,m+1)}\|x_n - u\| + e^{k(n,m+1)} \cdot \sum_{i=n}^{n+m-1}(i+K) \cdot \varepsilon + (n+m+K) \cdot \varepsilon \\
&\leq e^{k(n,m+1)}\|x_n - u\| + e^{k(n,m+1)} \cdot \sum_{i=n}^{n+m}(i+K) \cdot \varepsilon.
\end{aligned}$$

□

Definition 4.5 *An approximate fixed point bound* $\Phi : \mathbb{Q}_+^* \to \mathbb{N}$ *is called monotone if*

$$q_1 \leq q_2 \to \Phi(q_1) \geq \Phi(q_2), \quad q_1, q_2 \in \mathbb{Q}_+^*.$$

Remark 4.6 *Any approximate fixed point bound* Φ *for a sequence* (x_n) *can effectively be converted into a monotone approximate fixed point bound for* (x_n) *by*

$$\Phi_M(q) := \Phi_m(\min k[2^{-k} \leq q]), \text{ where } \Phi_m(k) := \max_{i \leq k} \Phi(2^{-i}).$$

We now assume that C is totally bounded.

Theorem 4.7 *Let* $k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}, \Phi : \mathbb{Q}_+^* \to \mathbb{N}$ *and* $\alpha : \mathbb{N} \to \mathbb{N}$. *Let* $f : C \to C$ *be asymptotically nonexpansive with a sequence* (k_n) *such that* $\mathbb{N} \ni K \geq \sum_{n=0}^{\infty} k_n$ *and* $N \in \mathbb{N}$ *be such that* $N \geq e^K$. *We define a function* $\Psi(k, g, \Phi, \alpha)$ *(primitive) recursively as follows:*

$$\Psi(k, g, \Phi, \alpha) := \max_{i \leq \alpha(k + \lceil \log_2(N) \rceil + 3)} \Psi_0(i, k, g, \Phi),$$

where (writing $\Psi_0(l)$ *for* $\Psi_0(l, k, g, \Phi)$*)*

$$\begin{cases} \Psi_0(0) := 0 \\ \Psi_0(n+1) := \\ \quad \Phi\left(2^{-k-\lceil \log_2(N) \rceil - 2} / (\max_{l \leq n}[g^M(\Psi_0(l))(\Psi_0(l) + g^M(\Psi_0(l)) + \lceil \log_2(N) \rceil) + 1])\right) \end{cases}$$

with $g^M(n) := \max_{i \le n} g(i)$.
If Φ *is a monotone approximate fixed point bound for* (x_n) *and* α *a modulus of total boundedness for* C *then*

$$\forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists n \le \Psi(k, g, \Phi, \alpha) \forall i, j \in [n; n + g(n)](\|x_i - x_j\| \le 2^{-k}).$$

Proof: Since Φ is an approximate fixed point bound we can define a function $\varphi : \mathbb{Q}^*_+ \to \mathbb{N}$ by

$$\varphi(q) := \min\{n \le \Phi(q) \ : \ \|x_n - f(x_n)\| \le q\}.$$

Then $x_{\varphi(q)}$ is a q-fixed point of f.[6] Define $n_i := \Psi_0(i, k, g, \varphi)$.
Analogously to the claim in the proof of theorem 3.6 one shows that there exist points $a_0, \ldots, a_{\alpha(k+\lceil \log_2(N) \rceil+3)} \in C$ such that

$$\exists i < j \le \alpha(k + \lceil \log_2(N) \rceil + 3) + 1, \exists l \le \alpha(k + \lceil \log_2(N) \rceil + 3) :$$
$$\|a_l - x_{n_i}\| \le 2^{-k-\lceil \log_2(N) \rceil - 3} \wedge \|a_l - x_{n_j}\| \le 2^{-k-\lceil \log_2(N) \rceil - 3}$$

and hence $\|x_{n_i} - x_{n_j}\| \le 2^{-k-\lceil \log_2(N) \rceil - 2}$. By construction x_{n_j} is a $(2^{-k-\lceil \log_2(N) \rceil - 2}/q)$-fixed point of f where
$q = \max_{l<j} (g^M(\Psi_0(l, k, g, \varphi))(\Psi_0(l, k, g, \varphi) + g^M(\Psi_0(l, k, g, \varphi)) + \lceil \log_2(N) \rceil) + 1)$.
Hence x_{n_j} is a-fortiori a $(2^{-k-\lceil \log_2(N) \rceil - 2}/(g(n_i)(n_i + g(n_i) + \lceil \log_2(N) \rceil) + 1))$-fixed point of f. By lemma 4.4 we therefore obtain for all $l \le g(n_i)$:

$$\|x_{n_i+l} - x_{n_j}\| \le N\|x_{n_i} - x_{n_j}\| + N \cdot l(n_i + l + \lceil \log_2(N) \rceil) \cdot \frac{2^{-k-\lceil \log_2(N) \rceil - 2}}{g(n_i)(n_i+g(n_i)+\lceil \log_2(N) \rceil)+1}$$
$$\le N\|x_{n_i} - x_{n_j}\| + 2^{-k-2} \le 2^{-k-1}.$$

So

$$\forall j_1, j_2 \in [n_i; n_i + g(n_i)](\|x_{j_1} - x_{j_2}\| \le 2^{-k}).$$

The theorem now follows from the fact that (using the monotonicity of Φ)

$$\Psi_0(n, k, g, \Phi) \ge \Psi_0(n, k, g, \varphi) \quad (n \in \mathbb{N})$$

which is easily verified by induction on n. $\square$

Remark 4.8 Ψ *is a primitive recursive functional in the sense of Kleene [14].*

[6] Note that in general $(\|x_n - f(x_n)\|)_{n \in \mathbb{N}}$ will not (as in the nonexpansive case) be nonincreasing anymore so that we cannot just take $\Phi(q)$ instead of $\varphi(q)$.

Let $(X, \|\cdot\|)$ be a uniformly convex space and η a modulus of uniform convexity for X. Let $C \subset X$ be a nonempty, convex and bounded subset and $d \geq diam(C)$. In [20](cor. 28 specialized to $E = 0$ and $g(n) := 0$) the following approximate fixed point bound Φ is established for functions $f : C \to C$ which are asymptotically nonexpansive with a sequence (k_n) which is bounded in sum by $K \geq \sum\limits_{n=0}^{\infty} k_n$. Let (λ_n) be a sequence in $[\frac{1}{M}, 1 - \frac{1}{M}]$ for some $M \in \mathbb{N}$. Then

$$\Phi(K, M, d, \eta, q) := \left\lceil \frac{3(5Kd+d)M^2}{\tilde{q} \cdot \eta(\tilde{q}/(d(1+K)))} \right\rceil, \text{ where}$$
$$\tilde{q} := q/(2(1 + (K+1)(K+2)(K+3)))$$

is an approximate fixed point bound for the Krasnoselski-Mann iteration of f starting from any point $x_0 \in C$ and with the sequence (λ_n), i.e.

$$\exists n \leq \Phi(K, M, d, \eta, q)(\|x_n - f(x_n)\| \leq q).$$

For special η, an improvement analogous to the one in the 2nd example at the end of the previous section is possible.

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