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Elimination of Skolem functions for monotone formulas in analysis

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1 Introduction

In [14] we have introduced a hierarchy $(\mathrm{G}_n\mathrm{A}^\omega)_{n\in\mathbb{N}}$ of subsystems of arithmetic in all finite types where the growth of the definable functions of $\mathrm{G}_n\mathrm{A}^\omega$ corresponds to the well-known Grzegorczyk hierarchy. For certain (in general) non-constructive analytical axioms Δ and the schema of quantifier–free choice AC-qf the following rule is shown:

¿From a proof

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta\text{+AC-qf} \vdash \forall u^1, k^0 \forall v \leq_\tau tuk \exists w^\gamma A_0(u,k,v,w),$$

(where t is a closed term, A_0 is quantifier-free and contains only u,k,v,w free, $\gamma \leq 2, \rho$ is an arbitrary type and $\leq_\tau$ is defined pointwise) one can extract (by monotone functional interpretation) a uniform bound Φ on $\exists w^\gamma$ which is given by a closed term of $\mathrm{G}_n\mathrm{A}^\omega$ and does not depend on v, i.e.

$$\forall u, k \forall v \leq tuk \exists w \leq \Phi uk\, A_0(u,k,v,w)$$

holds in the full set-theoretic model.

In particular Φuk is a polynomial (an elementary recursive function) in $u^M := \lambda x^0. \max_{i\leq x} u(i)$ and k^0 in case $n=2$ (resp. $n=3$).

In a paper under preparation we will show that substantial parts of classical analysis can be carried out in $\mathrm{G}_3\mathrm{A}^\omega + \Delta$+AC-qf and even in $\mathrm{G}_2\mathrm{A}^\omega + \Delta$+AC-qf for suitable Δ (see [14] for more information on this).

On the other hand there are central theorems in analysis whose proofs use arithmetical instances of AC, i.e. instances of

$$\mathrm{AC}_{ar}: \ \forall x^0 \exists y^0 A(x,y) \rightarrow \exists f^1 \forall x^0 A(x,fx),$$

where $A \in \Pi^0_\infty$ (A may contain parameters of arbitrary type), and which are not covered by the results mentioned above.

Examples are the following theorems

1. The principle of convergence for bounded monotone sequences of real numbers (or equivalently: every bounded monotone sequence of reals has a Cauchy modulus (PCM)).
2. For every sequence of real numbers which is bounded from above there exists a least upper bound.
3. The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$ (for every fixed d).
4. The Arzelà–Ascoli lemma.
5. The existence of the limit superior for bounded sequences of real numbers.

We will investigate these theorems (w.r.t. to their contribution to the rate of growth of uniform bounds extractable from proofs which use them) in a subsequent paper using the method developed in this paper and discuss now only (PCM) in order to motivate the results of the present paper which is the second one in a sequence of papers resulting from the authors Habilitationsschrift [12]. All undefined notions are used in the sense of [14] on which this paper relies. $A_0, B_0, C_0, \ldots$ always denote quantifier-free formulas.

Using a convenient representation of real numbers, (PCM) can be formalized as follows:

$$(\mathrm{PCM}) \; : \; \begin{cases} \forall a^{1(0)}_{(\cdot)}, c^1 (\forall n^0 (c \leq_{\mathbb{R}} a_{n+1} \leq_{\mathbb{R}} a_n) \\ \qquad \to \exists h^1 \forall k^0 \forall m, \tilde{m} \geq_0 hk (|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1})). \end{cases}$$

(PCM) immediately follows from its arithmetical weakening

$$(\mathrm{PCM}^-) \; : \; \begin{cases} \forall a^{1(0)}_{(\cdot)}, c^1 (\forall n^0 (c \leq_{\mathbb{R}} a_{n+1} \leq_{\mathbb{R}} a_n) \\ \qquad \to \forall k^0 \exists n^0 \forall m, \tilde{m} \geq_0 n (|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1})) \end{cases}$$

by an application of AC_{ar} to

$$A :\equiv \forall m, \tilde{m} \geq n (|a_m -_{\mathbb{R}} a_{\tilde{m}}| \leq_{\mathbb{R}} \frac{1}{k+1}) \in \Pi^0_1$$

($\leq_{\mathbb{R}} \in \Pi^0_1$ follows from the fact that real numbers are given as Cauchy sequences of rationals with fixed rate of convergence in our theories).

It is well–known that a constructive functional interpretation of the negative translation of AC_{ar} requires so–called bar-recursion and cannot be caried out e.g. in Gödel's term calculus T (see [21] and [15]). AC_{ar} is (using classical logic) equivalent to CA_{ar}+$\mathrm{AC}^{0,0}$–qf, where

$$\mathrm{CA}_{ar} : \; \exists g^1 \forall x^0 (g(x) =_0 0 \leftrightarrow A(x)) \text{ with } A \in \Pi^0_\infty,$$

(and $\mathrm{AC}^{0,0}$–qf is the restriction of AC_{ar} to quantifier-free formulas) and therefore causes an immense rate of growth (when added to e.g. $\mathrm{G}_2\mathrm{A}^\omega$). ¿From the work in the context of 'reverse mathematics' (see e.g. [3],[20]) it is known that 1)–5) imply CA_{ar} relatively to (a second-order version of) $\widehat{\mathrm{PA}}^\omega\!\upharpoonright$+$\mathrm{AC}^{0,0}$–qf (see [1] for the definition of $\widehat{\mathrm{PA}}^\omega\!\upharpoonright$). In [12] it is shown that this holds even relatively to $\mathrm{G}_2\mathrm{A}^\omega$.

In contrast to these general facts we prove in this paper a meta–theorem which in particular implies that if (PCM) is applied in a proof only to sequences (a_n) which are given explicitely in the parameters of the proposition (which is proved) then this proof can be (effectively) transformed (without causing new growth) into a proof of the same conclusion which uses only (PCM^-) for these sequences. By this transformation the use of AC_{ar} is eliminated and the determination of the growth caused (potentially by (PCM)) reduces to the determination of the growth caused by (PCM^-).

More precisely our meta–theorem has the following consequence:

Let $\mathcal{T}^\omega :=\mathrm{G}_n\mathrm{A}^\omega$ and χ denote a closed term of $\mathrm{G}_n\mathrm{A}^\omega$ (having an appropriate type). Then the following rule holds

$$(1)\quad \begin{cases} \mathcal{T}^\omega + \text{AC–qf } \vdash \forall u^1 \forall v \leq_\rho tu \\ \;(\exists h^1 \forall k^0 \forall m, \tilde{m} \geq_0 hk(|(\widetilde{\chi uv})_m -_{\mathbb{R}} (\widetilde{\chi uv})_{\tilde{m}}| \leq \frac{1}{k+1}) \to \exists w^\tau A_0(u,v,w)) \\ \Rightarrow \text{ there exists a } \Phi \in \mathrm{G}_n\mathrm{A}^\omega \text{ such that} \\ \mathcal{T}^\omega \ \vdash \Big((\forall u^1 \forall v \leq_\rho tu (\forall k^0 \exists n^0 \forall m, \tilde{m} \geq_0 n(|(\widetilde{\chi uv})_m -_{\mathbb{R}} (\widetilde{\chi uv})_{\tilde{m}}| \leq \frac{1}{k+1}) \\ \qquad\qquad \to \exists w^\tau A_0(u,v,w))) \\ \qquad \wedge \Phi \text{ fulfils the mon. funct. interpr. of the negative trans. of } (\ldots) \Big). \end{cases}$$

(Here $\tilde{a}(n) := \max_{\mathbb{R}}(0, \min\limits_{i \leq n}(a(i)))$. By this construction every sequence $a^{1(0)}$ represents a decreasing sequence of positive real numbers. The restriction to the special lower bound $c :=_{\mathbb{R}} 0$ is convenient but of course not essential.)

In contrast to (PCM) the (negative translation of the) principle (PCM^-) has a simple constructive monotone functional interpretation which is fulfilled by a functional Ψ which is primitive recursive in the sense of [6]. Because of the nice behaviour of the monotone functional interpretation with respect to the modus ponens one obtains (by applying Φ to Ψ) a monotone functional interpretation of

$$\forall u^1 \forall v \leq_\rho tu \exists w^\tau A_0(u,v,w)$$

and so (if $\tau \leq 2$) using tools from [11],[14] a uniform bound ξ for $\exists w$, i.e.

$$\forall u^1 \forall v \leq_\rho tu \exists w \leq_\tau \xi u A_0(u,v,w),$$

where ξ is primitive recursive in the sense of Kleene [6] (and not only in the generalized sense of Gödel's calculus T).

This conclusion also holds for sequences of instances $\forall n^0$ PCM(χuvn) of PCM(a) instead of PCM(χuv).

Let us consider the following general situation:

For

$$F(\underline{a}) :\equiv \forall x_1^0 \exists y_1^0 \ldots \forall x_k^0 \exists y_k^0 F_0(x_1, y_1, \ldots, x_k, y_k, \underline{a}),$$

where $\underline{x}, \underline{y}, \underline{a}$ are all the free variables of F_0, we define the **Skolem normal form** F^S of F by

$$F^S(\underline{a}) :\equiv \exists f_1, \ldots, f_k \forall x_1^0, \ldots, x_k^0 \; F_0(x_1, f_1 x_1, \ldots, x_k, f_k x_1 \ldots x_k, \underline{a}).$$

If we could prove that

$$(2)\ \begin{cases} \mathcal{T}^\omega(+\text{AC–qf}) \vdash \forall u^1 \forall v \leq_\rho tu(F^S(u,v) \to \exists w^\tau A_0(u,v,w)) \Rightarrow \\ \mathcal{T}^\omega \vdash \forall u^1 \forall v \leq_\rho tu(F(u,v) \to \exists w^\tau A_0(u,v,w)), \end{cases}$$

then (1) would follow as a special case.

(2) in turn is implied by

$$(3)\ \mathcal{T}^\omega(+\text{AC–qf}) \vdash G^H \Rightarrow \mathcal{T}^\omega \vdash G,$$

where

$$G^H :\equiv \begin{cases} \forall u^1 \forall v \leq_\rho tu \forall h_1, \ldots, h_k \exists y_1^0, \ldots, y_k^0, w^\tau \\ \qquad G_0(u, v, y_1, h_1 y_1, y_2, h_2 y_1 y_2, \ldots, y_k, h_k y_1 \ldots y_k, w) \end{cases}$$

is the (generalized)[1] **Herbrand normal form** of

$$G :\equiv \forall u^1 \forall v \leq_\rho tu \exists y_1^0 \forall x_1^0 \ldots \exists y_k^0 \forall x_k^0 \exists w^\tau G_0(u, v, y_1, x_1, \ldots, y_k, x_k, w).$$

Since $\forall u^1 \forall v \leq_\rho tu(F(u,v) \to \exists w^\tau A_0)$ can be transformed into a prenex normal form G whose Herbrand normal form is logically equivalent to

$\forall u \forall v \leq tu(F^S(u,v) \to \exists w A_0)$, (2) is a special case of (3).

Unfortunately (3) is wrong (even without AC–qf) for $\mathcal{T}^\omega = \mathrm{G}_n\mathrm{A}^\omega$, PRA^ω and much weaker theories. In fact it is false already for the first-order fragments of these theories augmented by function variables. For (a second-order fragment of) $\mathrm{PRA}^\omega + \Sigma^0_1$–IA this was proved firstly in [10] (thereby detecting a false argument in the literature). In §2 below we will prove a result which implies this as a special case and refutes (3) also for $\mathrm{G}_n\mathrm{A}^\omega$ (and their second-order fragments even when the universal axioms 9) from the definition of $\mathrm{G}_n\mathrm{A}^\omega$ are replaced by the schema of quantifier–free induction).

On the other hand we will show that (3) is true for $\mathcal{T}^\omega = \mathrm{G}_n\mathrm{A}^\omega$ (but remains false for $\mathcal{T}^\omega = \mathrm{PRA}^\omega$) if G satisfies a certain monotonicity condition (see def.26 below) which is fulfilled e.g. in (1). We may add also axioms Δ to $\mathrm{G}_n\mathrm{A}^\omega$ having the form $\forall x^\delta \exists y \leq_\tau sx \forall z^\gamma G_0(x,y,z)$, where G_0 is quantifier-free and s a closed term. As mentioned above such axioms cover substantial parts of classical analysis relatively to $\mathrm{G}_2\mathrm{A}^\omega$ (see [12] and [14] for details).

This result will be used in §3 to determine the growth caused by (sequences of) instances of the restriction of AC_{ar} and CA_{ar} to Π^0_1 formulas: Π^0_1–AC, Π^0_1–CA.

In a subsequent paper we will treat the analytical principles mentioned above. It will turn out that 1)-4) have the same contribution to the growth of uniform bounds as Π^0_1-CA, whereas 5) may produce a growth of the Ackermann type.

[1] The Herbrand normal form is usually defined only for arithmetical formulas, i.e. if u, v, w are not present. In this case it coincides with our definition. In $\mathrm{G}_2\mathrm{A}^+$ in §2 below, u, v, v do not occur and the h_i are free function variables.

2 Elimination of Skolem functions of type $0(0)\ldots(0)$ in higher type theories for monotone formulas: no additional growth

We first prove a result which in particular refutes (3) from the introduction (even without AC–qf) for G_nA^ω (with $n \geq 2$), $G_\infty A^\omega$ and PRA^ω:

Let G_2A be the first-order part of G_2A^ω (without the universal axioms 9) from [14] but only with the schema of quantifier-free induction instead of them) and G_2A^+ be G_2A augmented by function variables and a substitution rule

$$SUB: \ \frac{A(f)}{A(g)}.$$

G_2A^+ contains the schema of quantifier–free induction **with function parameters** .

Proposition 21 *Let $A \in \Pi^0_\infty$ be a theorem of (first-order) Peano arithmetic PA. Then one can construct a sentence $\tilde{A} \in \Pi^0_\infty$ such that*

$$G_2A^+ \vdash \tilde{A}^H \quad \text{and} \quad G_2A \ \vdash A \leftrightarrow \tilde{A}.$$

Proof: If PA $\vdash A$, then there are arithmetical instances (without function parameters) of the induction schema such that for their universal closure $\tilde{F}_1, \ldots, \tilde{F}_k$

$$\mathrm{G_2A} \ \vdash \bigwedge_{i=1}^{k} \tilde{F}_i \to A,$$

since PA $\subset$ $G_2A + \Pi^0_\infty$–IA$^-$, where Π^0_∞–IA$^-$ is the induction schema for all arithmetical formulas without function variables.

Let B be any prenex normal form of $(\bigwedge_{i=1}^{k}(y_i =_0 0 \leftrightarrow F_i(x_i)) \to A)$, where F_i denotes the induction formula of $\tilde{F}_i$, then

$$\tilde{A} :\equiv \exists \underline{a}, x_1, \ldots, x_k \forall y_1, \ldots, y_k \, B(x_1, \ldots, x_k, y_1, \ldots, y_k, \underline{a})$$

is a prenex normal form of

$$\forall \underline{a}, x_1, \ldots, x_k \exists y_1, \ldots, y_k \bigwedge_{i=1}^{k} (y_i = 0 \leftrightarrow F_i(x_i)) \to A,$$

where $\underline{a}$ are the (number) parameters of the induction formulas F_i. Because of

$$\mathrm{G_2A} \vdash \forall \underline{a}, x_1, \ldots, x_k \exists y_1, \ldots, y_k \bigwedge_{i=1}^{k} (y_i = 0 \leftrightarrow F_i(x_i)),$$

we obtain

$$\mathrm{G_2A} \vdash A \leftrightarrow \tilde{A}.$$

Since $\tilde{A}^H$ is logically implied by

$$C :\equiv \exists \underline{a}, x_1, \ldots, x_k\, B(x_1, \ldots, x_k, f_1\underline{a}x_1 \ldots x_k, \ldots, f_k\underline{a}x_1 \ldots x_k, \underline{a}),$$

it remains to show that $G_2A^+ \vdash C$:

Assume $\forall \underline{a}, x_1, \ldots, x_k \bigwedge_{i=1}^{k} (f_i\underline{a}x_1 \ldots x_k = 0 \leftrightarrow F_i(x_i))$. Quantifier–free induction applied to $A_0(x_i) :\equiv (f_i(\underline{a}, 0, \ldots, 0, x_i, 0, \ldots, 0) = 0)$ yields $\tilde{F}_i$. Hence

$$G_2A^+ \vdash \forall \underline{a}, x_1, \ldots, x_k \bigwedge_{i=1}^{k} (f_i\underline{a}x_1 \ldots x_k = 0 \leftrightarrow F_i(x_i)) \to A,$$

i.e. $G_2A^+ \vdash C$.

Corollary 22 (to the proof) *Let $G_2A[f_1, \ldots, f_k]$ denote the extension of G_2A which is obtained by adding new function symbols $f_1, \ldots, f_k$ which may occur in instances of QF–IA. Then $G_2A[f_1, \ldots, f_k] \vdash \tilde{A}^H$ and $G_2A \vdash A \leftrightarrow \tilde{A}$ (with $A, \tilde{A}$ as in the proof above), where $f_1, \ldots, f_k$ are the function symbols used in the definition of $\tilde{A}^H$.*

Corollary 23 *1. For each $n \in \mathbb{N}$ one can construct a sentence $A \in \Pi^0_\infty$ such that*

$$G_2A^\omega \vdash A^H, \text{ but } G_\infty A^\omega + \Sigma^0_n\text{–}IA \subset PRA^\omega + \Sigma^0_n\text{–}IA \nvdash A.$$

2. For each $n \in \mathbb{N}$ one can construct sentences $A \in \Pi^0_\infty$ and a sentence $\forall x^0 \exists y^0 B_0(x, y) \in \Pi^0_2$ such that

$$G_2A^\omega \vdash A^H, \text{ but } G_2A^\omega + A \vdash \forall x^0 \exists y^0 B_0(x, y),$$

where $fx := \min y[B_0(x, y)]$ is not $< \omega_n(\omega)$–recursive.

Proof: 1) Let $n \geq 1$ and $A \in \mathcal{L}(\text{PA})$ be an instance of Σ^0_{n+1}–IA which is not provable in $\text{PRA}^\omega + \Sigma^0_n$–IA (such an instance exists since every $< \omega_{n+1}(\omega)$–recursive function is provably recursive in $\text{PRA}^\omega + \Sigma^0_{n+1}$–IA, but in $\text{PRA}^\omega + \Sigma^0_n$–IA only $< \omega_n(\omega)$–recursive functions are provably recursive (This follows from [18](thm.5) using the fact that $\text{PRA}^\omega + \Sigma^0_n$–IA has a functional interpretation by functionals in Parsons calculus T_{n-1}) and there are $< \omega_{n+1}(\omega)$–recursive functions which are not $< \omega_n(\omega)$–recursive). Construct now $\tilde{A}$ as in prop.21 . It follows that $G_2A^\omega \vdash \tilde{A}^H$, but $\text{PRA}^\omega + \Sigma^0_n$–IA $\nvdash \tilde{A}$.

2) follows from prop.21 and the fact that every $\alpha(< \varepsilon_0)$–recursive function is provably recursive in PA.

The reason for the provability of $\tilde{A}^H$ in prop.21 is that the schema of quantifier–free induction is applicable to the index functions used in defining $\tilde{A}^H$. This always is the case in the presence of the substitution rule SUB or $\forall^1$–elimination in theories like G_2A^ω where quantification over functions is possible.

In the following we show that the same phenomenon occurs if QF–IA in G_2A^+ is restricted to formulas **without function variables** but instead of this new functional symbols $\Phi_{\max,n}$ are added (for each number $n \in \mathbb{N}$) together with the axioms

$$(\max, n) : \bigwedge_{i=1}^{n} (y_i \le_0 x_i) \to f\underline{y} \le_0 \Phi_{\max,n} f \underline{x},$$

where f is an n–ary function variable.

$$(\max) := \cup_n (\max, n).$$

We call the resulting system $G_2A+(\max)$.

Remark 24 $(\max, 1)$ *is fulfilled by the functional* $\Phi_1 fx = \max(f0, \ldots, fx)$ *from* G_nA^ω. *By* λ*–abstraction and finite iteration of* Φ_1 *one can easily define a functional satisfying* $(\max, n)$ *(Hence* $G_2A+(\max)$ *is a subsystem of* G_2A^ω*). This is the reason for calling this axiom* $(\max)$*. Of course instead of* Φ_1 *one could also use e.g.* $\Phi_2 fx = \sum_{i=0}^{x} fi$.

Proposition 25 *Let* $A \in \Pi^0_\infty$ *be a theorem of PA. Then one can construct a sentence* $\tilde{A} \in \Pi^0_\infty$ *such that*

$$G_2A + (\max) \vdash \tilde{A}^H \text{ and } G_2A \vdash A \leftrightarrow \tilde{A}.$$

Proof: Since PA $\vdash A$, there are arithmetical instances (without function parameters) of the induction schema such that for their universal closure $\tilde{F}_1, \ldots, \tilde{F}_k$

$$G_2A \vdash \bigwedge_{i=1}^{k} \tilde{F}_i \to A.$$

Lets consider now the so-called collection principle

$$\mathbf{CP} : \forall \tilde{x}^0 (\forall x <_0 \tilde{x} \exists y^0 F(x, y, \underline{a}) \to \exists z \forall x <_0 \tilde{x} \exists y <_0 z\, F(x, y, \underline{a})),$$

where $x, y, \underline{a}$ are all free variables of F. This principle has been studied proof–theoretically in [17] and also in [19]. By [19] (prop.4.1 (iv)) one can construct for every instance $\tilde{F}$ of Σ^0_n–IA instances F_i of Σ^0_{n+1}–CP (i.e. CP restricted to Σ^0_{n+1}–formulas) such that $\bigwedge_i F_i \to \tilde{F}$. From the proof in [19] (which uses only QF–IA and the function $\dot{-}$) it follows that $G_2A \vdash \bigwedge_i F_i \to \tilde{F}$. Let $F_1, \ldots, F_l$ denote such instances of CP whose universal closures imply $\tilde{F}_1, \ldots, \tilde{F}_k$. F_i has the form

$$F_i :\equiv (\forall x <_0 \tilde{x} \exists y^0 G_i(x, y, \underline{a}) \to \exists z \forall x < \tilde{x} \exists y < z\, G_i(x, y, \underline{a})).$$

Thus G_2A proves

$$(1)\ \forall \underline{a}, \tilde{x} \bigwedge_{i=1}^{l} (\forall x_i <_0 \tilde{x} \exists y_i^0 G_i(x_i, y_i, \underline{a}) \to \exists z_i \forall x_i < \tilde{x} \exists y_i < z_i\, G_i(x_i, y_i, \underline{a})) \to A.$$

Consider now

$$B :\equiv \begin{cases} \forall \underline{a}, \tilde{x}, x_1, \ldots, x_l \exists y_1, \ldots, y_l \\ \bigwedge\limits_{i=1}^{l} (\forall u_i < \tilde{x} \exists w_i G_i(u_i, w_i, \underline{a}) \to (x_i < \tilde{x} \to G_i(x_i, y_i, \underline{a}))) \to A \end{cases}$$

and

$$C :\equiv \Big(\bigwedge_{i=1}^{l} (\forall u_i < \tilde{x} \exists w_i G_i(u_i, w_i, \underline{a}) \to (x_i < \tilde{x} \to G_i(x_i, y_i, \underline{a}))) \to A \Big).$$

Let C^{pr} be an (arbitrary) prenex normal form of C. Then

$$B^{pr} :\equiv \exists \underline{a}, \tilde{x}, x_1, \ldots, x_l \forall y_1, \ldots, y_l C^{pr}(\tilde{x}, x_1, \ldots, x_l, y_1, \ldots, y_l, \underline{a})$$

is a prenex normal form of B.

We now show i) $\mathrm{G_2A}$+(max) $\vdash (B^{pr})^H$ and ii) $\mathrm{G_2A} \vdash B^{pr} \leftrightarrow A$.

i) Define

$$\widehat{B} :\equiv \exists \underline{a}, \tilde{x}, x_1, \ldots, x_l C^{pr}(\tilde{x}, x_1, \ldots, x_l, f_1 \underline{a} \tilde{x} x_1 \ldots x_l, \ldots, f_l \underline{a} \tilde{x} x_1 \ldots x_l, \underline{a}).$$

The implication $\widehat{B} \to (B^{pr})^H$ holds logically. Hence we have to show that $\mathrm{G_2A}$+(max) $\vdash \widehat{B}$:

$\widehat{B}$ is logically equivalent to

$$(2)\ \forall \underline{a}, \tilde{x} \bigwedge_{i=1}^{l} (\underbrace{\forall u_i < \tilde{x} \exists w_i G_i \to \forall \underline{x}(x_i < \tilde{x} \to G_i(x_i, f_i \underline{a} \tilde{x} x_1 \ldots x_l, \underline{a}))}_{H_i :\equiv}) \to A.$$

By (max) applied to f_i, $\forall x_i (x_i < \tilde{x} \to G_i(x_i, f_i \underline{a} \tilde{x} x_1 \ldots x_l, \underline{a}))$ implies

$\exists z_i \forall x_i < \tilde{x} \exists y_i < z_i G_i(x_i, y_i, \underline{a})$. Thus

$$\mathrm{G_2A} + (\max) \vdash H_i \to F_i \text{ for } i = 1, \ldots l.$$

By (1),(2) this yields $\mathrm{G_2A}$+(max) $\vdash \widehat{B}$.

ii) We have to show that $\mathrm{G_2A} \vdash B \leftrightarrow A$. This follows immediately from the fact that

$$\forall \underline{a}, \tilde{x}, x_1, \ldots, x_l \exists y_1, \ldots, y_l \bigwedge_{i=1}^{l} (\forall u_i < \tilde{x} \exists w_i G_i(u_i, w_i, \underline{a}) \to (x_i < \tilde{x} \to G_i(x_i, y_i, \underline{a})))$$

holds logically.

Prop.21 and prop.25 show that for theories like $\mathrm{G}_n\mathrm{A}^\omega$ the Herbrand normal form A^H of an arithmetical formula A in general is much weaker than A with respect to provability in $\mathrm{G}_n\mathrm{A}^\omega$ (compare cor.23). This phenomenon does not occur if A satisfies the following monotonicity condition:

Definition 26 *Let $A \in \mathcal{L}(G_nA^\omega)$ be a formula having the form*

$$A \equiv \forall u^1 \forall v \leq_\tau tu \exists y_1^0 \forall x_1^0 \ldots \exists y_k^0 \forall x_k^0 \exists w^\gamma A_0(u,v,y_1,x_1,\ldots,y_k,x_k,w),$$

where A_0 is quantifier–free and contains only $u,v,\underline{y},\underline{x},w$ free. Furthermore let t be $\in G_nR^\omega$ and τ,γ are arbitrary finite types.

1. A is called (arithmetically) **monotone** *if*

$$Mon(A) :\equiv \begin{cases} \forall u^1 \forall v \leq_\tau tu \forall x_1,\tilde{x}_1,\ldots,x_k,\tilde{x}_k,y_1,\tilde{y}_1,\ldots y_k,\tilde{y}_k \\ \Big(\bigwedge_{i=1}^{k}(\tilde{x}_i \leq_0 x_i \wedge \tilde{y}_i \geq_0 y_i) \wedge \exists w^\gamma A_0(u,v,y_1,x_1,\ldots,y_k,x_k,w) \\ \qquad\qquad \rightarrow \exists w^\gamma A_0(u,v,\tilde{y}_1,\tilde{x}_1,\ldots,\tilde{y}_k,\tilde{x}_k,w)\Big). \end{cases}$$

2. The **Herbrand normal form** *A^H of A is defined to be*

$$A^H :\equiv \forall u^1 \forall v \leq_\tau tu \forall h_1^{\rho_1},\ldots,h_k^{\rho_k} \exists y_1^0,\ldots,y_k^0,w^\gamma \underbrace{A_0(u,v,y_1,h_1y_1,\ldots,y_k,h_ky_1\ldots y_k,w)}_{A_0^H :\equiv}, \text{ where } \rho_i = 0\underbrace{(0)\ldots(0)}_{i}.$$

Theorem 27 *Let $n \geq 1$ and $\Psi_1,\ldots,\Psi_k \in G_nR^\omega$. Then*

$$G_nA^\omega + Mon(A) \vdash \forall u^1 \forall v \leq_\tau tu \forall h_1,\ldots,h_k\Big(\bigwedge_{i=1}^{k}(h_i \text{ monotone}) \rightarrow \exists y_1 \leq_0 \Psi_1 u\underline{h} \ldots \exists y_k \leq_0 \Psi_k u\underline{h} \exists w^\gamma A_0^H\Big) \rightarrow A,$$

where $(h_i \text{ monotone}) :\equiv \forall x_1,\ldots,x_i,y_1,\ldots,y_i(\bigwedge_{j=1}^{i}(x_j \geq_0 y_j) \rightarrow h_i\underline{x} \geq_0 h_i\underline{y})$.

Theorem 28 *Let A be as in thm.27 and Δ be as in [14](thm.3.2.2), i.e. a set of sentences $\forall x^\delta \exists y \leq_\rho sx \forall z^\eta G_0(x,y,z)$ where s is a closed term of G_nA^ω and G_0 a quantifier-free formula, and let A' denote the negative translation[2] of A. Then the following rule holds:*

$$\begin{cases} G_nA^\omega + AC\text{–}qf + \Delta \vdash A^H \wedge Mon(A) \Rightarrow \\ G_nA^\omega + \tilde{\Delta} \vdash A \text{ and by monotone functional interpretation} \\ \text{one can extract a tuple } \underline{\Psi} \in G_nR^\omega \text{ such that} \\ G_nA_i^\omega + \tilde{\Delta} \vdash \underline{\Psi} \text{ satisfies the monotone functional interpretation of } A', \end{cases}$$

where $\tilde{\Delta} := \{\exists Y \leq_{\rho\delta} s \forall x^\delta, z^\eta G_0(x,Yx,z) : \forall x^\delta \exists y \leq_\rho sx \forall z^\eta G_0(x,y,z) \in \Delta\}$. (In particular the second conclusion can be proved in $G_nA_i^\omega + \Delta +$ b-AC)[3].

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[2] Here we can use Gödel's [5] translation or any of the various negative translations. For a systematical treatment of negative translations see [15].

[3] Here b–AC:= $\bigcup_{\delta,\rho \in \mathbf{T}} \{(\text{b–AC}^{\delta,\rho})\}$ denotes the schema

$$(\text{b–AC}^{\delta,\rho}) \;:\; \forall Z^{\rho\delta}\big(\forall x^\delta \exists y \leq_\rho Zx\; A(x,y,Z) \rightarrow \exists Y \leq_{\rho\delta} Z \forall x A(x,Yx,Z)\big).$$

Remark 29 *In theorems 27,28 one may also have tuples '$\exists \underline{w}$' instead of '$\exists w^\gamma$' in A.*

Proof of theorem 27: We assume that

$$(0)\forall u^1 \forall v \leq_\tau tu \forall h_1,\ldots,h_k(\bigwedge_{i=1}^{k}(h_i \text{ monotone}) \to \exists y_1,\ldots,y_k \leq_0 \Psi u\underline{h} \exists w^\gamma A_0^H)$$

(This assumption follows from the implicative premise in the theorem by taking $\Psi u\underline{h} := \max_0(\Psi_1 u\underline{h},\ldots,\Psi_k u\underline{h})$). By [14](cor.2.2.24 and rem.2.2.25) one can construct a term $\Psi^*[u,\underline{h}]^0$ such that

1. $\Psi^*[u,\underline{h}]$ is built up from $u,\underline{h},A_0,\ldots,A_n,S^1,0^0,\max_0$ only (by application).
2. $\lambda u,\underline{h}.\Psi^*[u,\underline{h}]$ s–maj Ψ (see [14] for the definition of s–maj).

1) in particular implies:

1*) Every occurrence of an $h_j \in \{h_1,\ldots,h_k\}$ in $\Psi^*[u,\underline{h}]$ has the form $h_j(r_{n_1},\ldots,r_{n_j})$, i.e. h_j occurs only with a full stock of arguments but not as a function argument in the form $s(h_j r_{n_1}\ldots r_{n_l})$ for some $l < j$.

By 2), $\forall u^1(u^M$ s–maj $u)$ (where $u^M x := \max_{i\leq x} ui$) and ($h_i$ monotone $\to h_i$ s–maj h_i) we have

2*) $\mathrm{G}_n\mathrm{A}^\omega \vdash \forall u \forall h_1,\ldots,h_k(\bigwedge_{i=1}^{k}(h_i \text{ monotone}) \to \Psi^*[u^M,\underline{h}] \geq_0 \Psi u\underline{h})$.

(Note the the replacement of h_i by $h_i^M := \lambda x_1,\ldots,x_i.\ \max_{\substack{\tilde{x}_1\leq x_1\\ \vdots \\ \tilde{x}_i\leq x_i}} h(\tilde{x}_1,\ldots,\tilde{x}_i)$, which would make the monotonicity assumption on h_i superfluous, would destroy property 1*) on which the proof below is based. This is the reason why we have to assume h_i to be monotone. In order to overcome this assumption we will use essentially the monotonicity of A).

Let $r_1,\ldots,r_l$ be all subterms of $\Psi^*[u^M,\underline{h}]$ which occur as an argument of a function $\in \{h_1,\ldots,h_k\}$ in $\Psi^*[u^M,\underline{h}]$ plus the term $\Psi^*[u^M,\underline{h}]$ itself.

Let $\widehat{r}_j[a_1,\ldots,a_{q_j}]$ be the term which results from r_j if every occurrence of a maximal $h_1,\ldots,h_k$–subterm (i.e. a maximal subterm which has the form $h_i(s_1,\ldots,s_i)$ for an $i = 1,\ldots,k$) is replaced by a new variable and let $a_1,\ldots,a_{q_j}$ denote these variables. We now define

$$\tilde{r}_j a_1\ldots a_{q_j} := \max\Big(\max_{\substack{\tilde{a}_1\leq a_1\\ \vdots \\ \tilde{a}_{q_j}\leq a_{q_j}}} \widehat{r}_j[\tilde{a}_1,\ldots,\tilde{a}_{q_j}],a_1,\ldots,a_{q_j}\Big).$$

($\tilde{r}_j$ can be defined in $\mathrm{G}_n\mathrm{R}^\omega$ from $\widehat{r}_j$ by successive use of Φ_1).

By the construction of $\tilde{r}_j$ we get

$$\mathrm{G}_n\mathrm{A}^\omega \vdash (\lambda\underline{a}.\tilde{r}_j\underline{a} \text{ s–maj } \lambda\underline{a}.\widehat{r}_j[a_1,\ldots,a_{q_j}]) \wedge \forall\underline{a}(\tilde{r}_j\underline{a} \geq_0 a_1,\ldots,a_{q_j}).$$

Since $\Psi^*[u^M,\underline{h}]$ is built up from $\widehat{r}_j,\underline{h}$ only (by substitution) and (h_i monotone $\to h_i$ s–maj h_i), u^M s–maj u, this implies

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \forall u, h_1,\ldots,h_k(\bigwedge_{i=1}^{k}(h_i \text{ monotone}) \to \overline{\Psi}[u^M,\underline{h}] \geq_0 \Psi^*[u^M,\underline{h}] \geq_0 \Psi u\underline{h}),$$

where $\overline{\Psi}[u^M,\underline{h}]$ is built up as $\Psi^*[u^M,\underline{h}]$ but with $\tilde{r}_j(a_1,\ldots,a_{q_j})$ instead of $\widehat{r}_j[a_1,\ldots,a_{q_j}]$.

Summarizing the situation achieved so far we have obtained a term $\overline{\Psi}[u^M,\underline{h}]$ such that

(α) $\forall u^1\forall v \leq_\tau tu\forall\underline{h}(\underline{h}$ monotone $\to \exists y_1,\ldots,y_k \leq_0 \overline{\Psi}[u^M,\underline{h}]\exists w^\gamma A_0^H)$.
(β) $h_1,\ldots,h_k$ occur in $\overline{\Psi}[u^M,\underline{h}]$ only as in 1*), i.e. with all places for arguments filled and not as function arguments themselves.
(γ) For $\overline{\Psi}[u^M,\underline{h}]$ and all subterms s which occur as an argument of a function $h_1,\ldots,h_k$ in $\overline{\Psi}[u^M,\underline{h}]$ we have $\mathrm{G}_n\mathrm{A}^\omega \vdash \widehat{s}[a_1,\ldots,a_q] \geq_0 a_1,\ldots,a_q$, where $\widehat{s}$ results by replacing every occurrence of a maximal $h_1,\ldots,h_k$–subterm in s by a new variable a_l.

(β), (γ) do not depend on any assumption and (α) follows from (0):
$\mathrm{G}_n\mathrm{A}^\omega \vdash (0) \to (\alpha)$.

In the following we only use (α)–(γ) and $Mon(A)$.

¿From now on let $r_1,\ldots,r_l$ denote all subterms of $\overline{\Psi}[u^M,\underline{h}]$ which occur as an argument of a function $\in \{h_1,\ldots,h_k\}$ in $\overline{\Psi}[u^M,\underline{h}]$ plus $\overline{\Psi}[u^M,\underline{h}]$ itself. $M := \{r_1,\ldots,r_l\}$ (This set formation is meant w.r.t. identity $\equiv$ of terms and not $=_0$, i.e. '$s \in M$' means '$s \equiv r_1 \vee \ldots \vee s \equiv r_l$').

We now show that we can reduce '$\exists y_1,\ldots,y_k \leq \overline{\Psi}[u^M,\underline{h}]$' in ($\alpha$) to a disjunction with fixed length, namely to the disjunction over M:

$$(1)\ \begin{cases} \forall u^1\forall v \leq_\tau tu\forall\underline{h}(\underline{h} \text{ monotone on } M \to \exists s_1,\ldots,s_k \in M\exists w^\gamma \\ \qquad\qquad A_0(u,v,s_1,h_1s_1,\ldots,s_k,h_ks_1\ldots s_k,w)). \end{cases}$$

Proof of (1): Let $h_1,\ldots,h_k$ be monotone on M. We order the terms r_i w.r.t. $\leq_0$. The resulting ordered tuple depends of course on $u,h_1,\ldots,h_k$. For notational simplicity we assume that $r_1 \leq_0 \ldots \leq_0 r_l$. We now define (again depending on $u,\underline{h}$) functions $\tilde{h}_1,\ldots,\tilde{h}_k$ by

$$\tilde{h}_i y_1^0 \ldots y_i^0 := h_i(r_{j_{y_1}},\ldots,r_{j_{y_i}}), \text{where}$$

$$j_{y_q} := \begin{cases} 1, \text{ if } y_q \leq_0 r_1 \\ j+1, \text{ if } r_j <_0 y_q \leq_0 r_{j+1} \\ l, \text{ if } r_l <_0 y_q. \end{cases}$$

Since l (and therefore the number of cases in this definition of $\tilde{h}_i$) is a (from outside) fixed number depending only on the term structure of $\overline{\Psi}[u^M,\underline{h}]$ but not on $u,\underline{h}$, the functions $\tilde{h}_i$ can be defined uniformly in $u,\underline{h}$ within $\mathrm{G}_n\mathrm{A}^\omega$. On M, $\tilde{h}_i$ equals h_i.

By the definition of $\tilde{h}_i$ and the assumption that $h_1,\ldots,h_k$ are monotone on M we conclude

$$(a)\ \tilde{h}_1,\ldots,\tilde{h}_k \text{ are monotone everywhere.}$$

By (β) we know that $h_1,\dots,h_k$ occur in $\overline{\Psi}[u^M,\underline{h}]$ only in the form $h_i(s_1,\dots,s_i)$ for certain terms $s_1,\dots,s_i \in M$. Hence we can define the $\underline{h}$**–depth** of a term $s \in M$ as the maximal number of nested occurrences of $h_1,\dots,h_k$ in s and show by induction on this rank (on the meta–level):

$$(b)\ \begin{cases} \bigwedge\limits_{i=1}^{l}(r_i =_0 \tilde{r}_i), \text{ where } \tilde{r}_i \text{ results if in } r_i \in M \text{ the functions } h_1,\dots,h_k \\ \text{are replaced by } \tilde{h}_1,\dots,\tilde{h}_k \text{ everywhere.} \\ \text{In particular } \overline{\Psi}[u^M,\underline{\tilde{h}}] =_0 \overline{\Psi}[u^M,\underline{h}]. \end{cases}$$

By (α), (a) and (b) it follows (for all $u^1, v \le tu$ and all $\underline{h}$ which are monotone on M) that

$$(c)\ \exists y_1,\dots,y_k \le_0 \overline{\Psi}[u^M,\underline{h}]\exists w^\gamma A_0(u,v,y_1,\tilde{h}_1 y_1,\dots,y_k,\tilde{h}_k y_1\dots y_k,w).$$

Let $y_1,\dots,y_k \le_0 \overline{\Psi}[u^M,\underline{h}]$ be such that (c) is fulfilled. Because of $\tilde{h}_i y_1\dots y_i = h_i(r_{j_{y_1}},\dots,r_{j_{y_i}})$ this implies

$$(d)\ \exists w^\gamma A_0(u,v,y_1,h_1 r_{j_{y_1}},\dots,y_k,h_k r_{j_{y_1}}\dots r_{j_{y_k}},w).$$

With $y_q \le r_{j_{y_q}}$ for $q = 1,\dots.k$ (since $y_q \le \overline{\Psi}[u^M,\underline{h}] \le r_l$ –because of $\overline{\Psi}[u^M,\underline{h}] \in M$ and the y_q–assumption– the case '$y_q > r_l$' does not occur) and $Mon(A)$ we conclude

$$\exists w^\gamma A_0(u,v,r_{j_{y_1}},h_1 r_{j_{y_1}},\dots,r_{j_{y_k}},h_k r_{j_{y_1}}\dots r_{j_{y_k}},w)$$

and therefore

$$(e)\ \exists s_1,\dots,s_k \in M\exists w^\gamma A_0(u,v,s_1,h_1 s_1,\dots,s_k,h_k s_1\dots s_k,w).$$

This concludes the proof of (1) which can easily be carried out in $\mathrm{G}_n\mathrm{A}^\omega$ (assuming $Mon(A)$, (α) and using (β)), i.e.

$$\mathrm{G}_n\mathrm{A}^\omega \vdash Mon(A) \wedge (\alpha) \to (1).$$

We now define $N := \bigcup\limits_{i=1}^{k} N_i$, where $N_i := \{h_i(s_1,\dots,s_i) : s_1,\dots,s_i \in M\}$ (Again this set is meant w.r.t. identity $\equiv$ between terms). With the terms in N we associate new number variables according to their $\underline{h}$–depth as follows: Let p the maximal $\underline{h}$–depth of all terms $\in N$.

1. Let $t \in N$ be a term with $\underline{h}$–depth$(t) = p$. Then $t \mapsto y_i^1$, if $t \in N_i$.
2. Let $t \in N$ be a term with $\underline{h}$–depth$(t) = p-1$. Then $t \mapsto y_i^2$, if $t \in N_i$.

$\vdots$

p. Let $t \in N$ be a term with $\underline{h}$–depth$(t) = 1$. Then $t \mapsto y_i^p$, if $t \in N_i$.

This association of variables to the terms in N has the following properties:

(i) Terms $s_1, s_2 \in N$ with different $\underline{h}$–depth have different variables associated with.
(ii) If $s_1, s_2 \in N$ have the same $\underline{h}$–depth, then the variables associated with s_1 and s_2 are equal iff $s_1, s_2 \in N_i$ for an $i = 1,\dots,k$.

For $r \in M \cup N$ we define $\widehat{r}$ as the term which results if every maximal $\underline{h}$–subterm occurring in r is replaced by its associated variable. Thus $\widehat{r}$ does not contain $h_1, \ldots, h_k$. For $r \in N$, $\widehat{r}$ is just the variable associated with r. $\widehat{M} := \{\widehat{r} : r \in M\}$.

We now show that (1) implies a certain index function–free (i.e. $h_1, \ldots, h_k$–free) disjunction ((2) below):

For q with $2 \leq q \leq p$ let $\widehat{r}^q_1, \ldots, \widehat{r}^q_{n_q}$ be all terms $\in \widehat{M}$ whose smallest upper index i of a variable y^i_j occurring in them equals q (i.e. there occurs a variable y^q_j in the term and for all variables y^i_m occurring in the term, $i \geq q$ holds). Since for $r \in M$ the $\underline{h}$–depth of $h_1(r) \in N$ is strictly greater than those of subterms of r, there are no terms $\widehat{r} \in \widehat{M}$ containing a variable y^1_j. $\widehat{r}^{p+1}_1, \ldots, \widehat{r}^{p+1}_{n_{p+1}}$ denote those terms $\in \widehat{M}$ which do not contain any variable y^i_j at all.

We now show that (1) implies (for all u and for all $v \leq tu$)

$$(2)\ \begin{cases} \forall y^1_1, \ldots, y^1_k; \ldots; y^p_1, \ldots, y^p_k \\ \qquad \Big((+) \rightarrow \bigvee\limits_{\widehat{s}_1, \ldots, \widehat{s}_k \in \widehat{M}} \exists w^\gamma A_0(u, v, \widehat{s}_1, \widehat{h_1 s_1}, \ldots, \widehat{s}_k, \widehat{h_k s_1 \ldots s_k}, w)\Big), \end{cases}$$

where[4]

$$(+) :\equiv \begin{cases} \bigwedge\limits_{\substack{q=1,\ldots,p-1 \\ l=1,\ldots,p-q}} (y^q_1, \ldots, y^q_k > \widehat{r}^{q+l}_1, \ldots, \widehat{r}^{q+l}_{n_{q+l}}, y^{q+l}_1, \ldots, y^{q+l}_k) \wedge \\ \bigwedge\limits_{q=1,\ldots,p} (y^q_1, \ldots, y^q_k > \widehat{r}^{p+1}_1, \ldots, \widehat{r}^{p+1}_{n_{p+1}}). \end{cases}$$

Assume that there are values $y^1_1, \ldots, y^1_k; \ldots; y^p_1, \ldots, y^p_k$ such that (+) holds and

$$\bigwedge_{\widehat{s}_1, \ldots, \widehat{s}_k \in \widehat{M}} \neg \exists w^\gamma A_0(u, v, \widehat{s}_1, \widehat{h_1 s_1}, \ldots, \widehat{s}_k, \widehat{h_k s_1 \ldots s_k}, w).$$

We construct (working in $\mathrm{G}_n\mathrm{A}^\omega$) functions $h_1, \ldots, h_k$ which are monotone on M and satisfy

$$\forall s_1, \ldots, s_k \in M \neg \exists w\, A_0(u, v, s_1, h_1 s_1, \ldots, s_k, h_k s_1 \ldots s_k, w)$$

yielding a contradiction to (1): Define for $i = 1, \ldots, k$

$$h_i(x_1, \ldots, x_i) := \begin{cases} y_i^{\min_{1 \leq l \leq i}(q_l) - 1}, \text{ if } \exists \widehat{r}^{q_1}_{j_1}, \ldots, \widehat{r}^{q_i}_{j_i} \in \widehat{M}((x_1, \ldots, x_i) =_0 (\widehat{r}^{q_1}_{j_1}, \ldots, \widehat{r}^{q_i}_{j_i})) \\ 0^0, \text{ otherwise.}^5 \end{cases}$$

---

[4] Here $a_1, \ldots, a_k > b_1, \ldots, b_l$ means $\bigwedge\limits_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} (a_i > b_j)$.

In (2) above we actually show the disjunction '$\bigvee\limits_{s_1, \ldots, s_k \in M} \exists w^\gamma A_0(u, v, \widehat{s}_1, \widehat{h_1 s_1}, \ldots)$' instead of '$\bigvee\limits_{\widehat{s}_1, \ldots, \widehat{s}_k \in \widehat{M}} \exists w^\gamma A_0(u, v, \widehat{s}_1, \widehat{h_1 s_1}, \ldots)$'. However the later follows from the former disjunction by contraction since $\widehat{s}_1 \equiv \widehat{s}'_1 \wedge \ldots \wedge \widehat{s}_i \equiv \widehat{s}'_i$ implies $h_i \widehat{s_1 \ldots} s_i \equiv h_i \widehat{s'_1 \ldots} s'_i$ for $s_1, s'_1, \ldots, s'_i, s'_i \in M$. Alternatively we could also use the non-contracted disjunction in the following proof.

We have to show:

(i) The h_i are well–defined functions : $\underbrace{\mathbb{N}\times\ldots\times\mathbb{N}}_{i}\to\mathbb{N}$ and the definition above can be carried out in G_nA^ω.

(ii) $\widehat{r}=_0 r$ for all $r\in M\cup N$ (for these $h_1,\ldots,h_k$).

(iii) $h_1,\ldots,h_k$ are monotone on $\widehat{M}$ (and hence –by (ii)– on M).

Ad (i): Consider $(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i})$ and $(\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i})$. We show that $y_i^{\min_{1\le l\le i}(q_l)-1}\neq y_i^{\min_{1\le l\le i}(\tilde{q}_l)-1}$ implies $(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i})\neq(\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i})$:

We may assume $\min\limits_{1\le l\le i}(q_l)<\min\limits_{1\le l\le i}(\tilde{q}_l)$. Let l_0 be such that $q_{l_0}=\min\limits_{1\le l\le i}(q_l)\wedge 1\le l_0\le i$. $\widehat{r}^{q_{l_0}}_{j_{l_0}}$ contains a variable $y_d^{q_{l_0}}$ for some $d=1,\ldots,k$. By the property (γ) of $\overline{\Psi}[u^M,\underline{h}]$ this implies

$$\widehat{r}^{q_{l_0}}_{j_{l_0}}\ge y_d^{q_{l_0}}\overset{(+),q_{l_0}<\tilde{q}_{l_0}}{>}\widehat{r}^{\tilde{q}_{l_0}}_{\tilde{j}_{l_0}}\text{ and thus }(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i})\neq(\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i}).$$

Hence h_i can be defined in G_nA^ω by a definition by cases which are pairwise exclusive.

Ad (ii): (ii) follows from the definition of $h_1,\ldots,h_k$ by induction on the $\underline{h}$–depth of r.

Ad (iii): Assume $\bigwedge\limits_{l=1}^{i}(\widehat{r}^{q_l}_{j_l}\le_0\widehat{r}^{\tilde{q}_l}_{\tilde{j}_l})$. Let l_0 $(1\le l_0\le i)$ be such that $q_{l_0}=\min\limits_{1\le l\le i}(q_l)$. By contraposition of the implication established in the proof of (i) one has: $\min\limits_{1\le l\le i}(q_l)\ge\min\limits_{1\le l\le i}(\tilde{q}_l)$.

Case 1: $\min\limits_{1\le l\le i}(q_l)=\min\limits_{1\le l\le i}(\tilde{q}_l)$. Then (by h_i–definition)

$$h_i(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i})=y_i^{\min(q_l)-1}=y_i^{\min(\tilde{q}_l)-1}=h_i(\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i}).$$

Case 2: $q_{l_0}=\min\limits_{1\le l\le i}(q_l)>\min\limits_{1\le l\le i}(\tilde{q}_l)=\tilde{q}_{\tilde{l}_0}$ (where $1\le l_0,\tilde{l}_0\le i$). Then

$$h_i(\widehat{r}^{q_1}_{j_1},\ldots,\widehat{r}^{q_i}_{j_i})=y_i^{q_{l_0}-1}\overset{(+)}{<}y_i^{\tilde{q}_{\tilde{l}_0}-1}=h_i(\widehat{r}^{\tilde{q}_1}_{\tilde{j}_1},\ldots,\widehat{r}^{\tilde{q}_i}_{\tilde{j}_i}).$$

Hence $h_1,\ldots,h_k$ are monotone on $\widehat{M}$ and therefore (by (ii)) on M, which concludes the proof of (2) from (1) in G_nA^ω (using $(\beta),(\gamma)$). Since (1) follows (in G_nA^ω) from $Mon(A)\wedge(\alpha)$ (using (β)), and

$$F:\equiv\forall u^1\forall v\le_\tau tu\forall\underline{h}(\underline{h}\text{ monotone }\to\exists y_1,\ldots,y_k\le_0\Psi u\underline{h}\exists w^\gamma A_0^H)$$

implies (in G_nA^ω) (α), we have shown altogether

$$(3)\begin{cases}G_nA^\omega+Mon(A)\vdash\\ F\to\Big[v\le tu\wedge(+)\to\bigvee\limits_{\widehat{s}_1,\ldots,\widehat{s}_k\in\widehat{M}}\exists w^\gamma A_0(u,v,\widehat{s}_1,\widehat{h_1s_1},\ldots,\widehat{s}_k,\widehat{h_ks_1\ldots s_k},w)\Big].\end{cases}$$

[5] For $\widehat{r}^{q_i}_{j_i}\in\widehat{M}$ we have $q_i\ge 2$ since e.g. $h_1r_{j_i}(\in N)$ has an $\underline{h}$–depth which is strictly greater than those of subterms in r_{j_i}.

It remains to show that (3) implies

$$(4)\ \mathrm{G}_n\mathrm{A}^\omega + Mon(A) \vdash F \to A.$$

We prove this by a suitable application of quantifier introduction rules: We start with the variables with smallest upper index, i.e. $y_1^1, \ldots, y_k^1$. Under these variables we first take those of maximal lower index, i.e. with y_k^1: We split the assumption

$$(+) \equiv \begin{cases} \bigwedge\limits_{\substack{q=1,\ldots,p-1\\ l=1,\ldots,p-q}} (y_1^q, \ldots, y_k^q > \widehat{r}_1^{q+l}, \ldots, \widehat{r}_{n_{q+l}}^{q+l}, y_1^{q+l}, \ldots, y_k^{q+l}) \wedge \\ \bigwedge\limits_{q=1,\ldots,p} (y_1^q, \ldots, y_k^q > \widehat{r}_1^{p+1}, \ldots, \widehat{r}_{n_{p+1}}^{p+1}) \end{cases}$$

as well as the disjunction

$$A^d :\equiv \bigvee_{\widehat{s}_1,\ldots,\widehat{s}_k \in \widehat{M}} \exists w^\gamma A_0(u, v, \widehat{s}_1, \widehat{h_1 s_1}, \ldots, \widehat{s}_k, \widehat{h_k s_1 \ldots s_k}, w)$$

into the part in which y_k^1 occurs and into its y_k^1–free part:

$$(5) \begin{cases} F \to \Big[v \le tu \wedge \bigwedge\limits_{l=1,\ldots,p-1} (y_k^1 > \widehat{r}_1^{1+l}, \ldots, \widehat{r}_{n_{1+l}}^{1+l}, \widehat{r}_1^{p+1}, \ldots, \widehat{r}_{n_{p+1}}^{p+1}, y_1^{1+l}, \ldots, y_k^{1+l}) \\ \quad \wedge \underbrace{\bigwedge{}'(\ldots)}_{y_k^1\text{–free part of } (+)} \to \bigvee\limits_j \exists w^\gamma A_0(u, v, \widehat{s}_1^j, \widehat{h_1 s_1^j}, \ldots, \widehat{s}_k^j, y_k^1, w) \vee \underbrace{\bigvee\limits_{j'}(\ldots)}_{y_k^1\text{–free part of } A^d} \Big]. \end{cases}$$

y_k^1 does not occur at any place other than indicated. Hence ∀–introduction applied to y_k^1 yields:

$$(6)\ F \to \forall y_k^1 [v \le tu \wedge \bigwedge_l (y_k^1 > \ldots) \wedge \bigwedge{}'(\ldots) \to \bigvee_j \exists w^\gamma A_0(\ldots, y_k^1, w) \vee \bigvee_{j'}(\ldots)],$$

where y_k^1 does not occur at any place other than indicated.

Using $Mon(A)$ this implies

$$(7)\ F \to [v \le tu \wedge \bigwedge{}'(\ldots) \to \forall y_k^1 \bigvee_j \exists w^\gamma A_0(\ldots, y_k^1, w) \vee \bigvee_{j'}(\ldots)].$$

(Proof: In (6) put
$\tilde{y}_k^1 := \max\limits_{1 \le l \le p-1} \max(y_k^1, \widehat{r}_1^{1+l}, \ldots \widehat{r}_{n_{1+l}}^{1+l}, \widehat{r}_1^{p+1}, \ldots, \widehat{r}_{n_{p+1}}^{p+1}, y_1^{1+l}, \ldots, y_k^{1+l}) + 1$ for y_k^1.
(6) then gives

$$F \to [v \le tu \wedge \bigwedge{}'(\ldots) \to \bigvee_j \exists w^\gamma A_0(\ldots, \tilde{y}_k^1, w) \vee \bigvee_{j'}(\ldots)].$$

$Mon(A)$ and $\bigvee_j \exists w^\gamma A_0(\ldots,\tilde{y}^1_k,w)$ imply $\bigvee_j \exists w^\gamma A_0(\ldots,y^1_k,w)$, since $\tilde{y}^1_k \geq y^1_k$. Now $\forall$–introduction applied to y^1_k and shifting $\forall y^1_k$ in front of $\bigvee_j$, which is possible since y^1_k occurs only in this disjunction, proves (7)).

Again by $Mon(A)$ we obtain

$$\bigvee_j \forall y^1_k \exists w^\gamma A_0(\ldots,y^1_k,w)$$

from $\forall y^1_k \bigvee_j \exists w^\gamma A_0(\ldots,y^1_k,w)$:

Assume $\bigwedge_j \exists y^1_k \forall w^\gamma \neg A_0(\ldots,y^1_k,w)$. Then $\exists y \bigwedge_j \exists y^1_k \leq_0 y \forall w^\gamma \neg A_0(\ldots,y^1_k,w)$. Using $Mon(A)$ this implies $\exists y \bigwedge_j \forall w^\gamma \neg A_0(\ldots,y,w)$.

Hence (7) implies (since y^1_k does not occur in $\widehat{s}^j_k$)

$$(8)\ \begin{cases} F \to [v \leq tu \wedge \bigwedge'(\ldots) \to \\ \quad \bigvee_j \exists x \forall y \exists w A_0(u,v,\widehat{s}^j_1,\widehat{h_1 s^j_1},\ldots,h_{k-1}\widehat{s^j_1 \ldots s^j_{k-1}},x,y,w) \vee \bigvee_{j'}(\ldots)]. \end{cases}$$

Next we apply the same procedure to the variable y^1_{k-1} and then to y^1_{k-2} and so on until all $y^1_1,\ldots,y^1_k$ are bounded. We then continue with y^2_k, y^2_{k-1} and so on. This corresponds to the sequence of quantifications used in the usual proofs of Herbrand's theorem in order to show that there is a direct proof from the Herbrand disjunction of a first-order formula to this formula itself: By taking always variables of minimal upper index it is ensured that any variable to which the $\forall$–introduction rule is applied occurs in the disjunction $\bigvee A_0$ only at places where it is universal quantified in the original formula A. By quantifying under these variables firstly the one with maximal lower index one ensures that a universal quantifier is introduced only if the quantifiers which stand behind this one in A have already been introduced. In addition to these two reasons for the special sequence of quantifications there is in our situation another (essentially used) property which is fulfilled only if variables which have minimal upper index are quantified first: If y^i_j has minimal index i (under all variables which still have to be quantified), then y^i_j occurs in the still remaining part of the implicative assumption (+) only in the form '$y^i_j >$ (...y^i_j–free...)'. So we are in the situation at the begining for y^1_k and are able to eliminate this part of (+) which is connected with y^i_j altogether using $Mon(A)$ (as we have shown for y^1_k).

Finally we have derived

$$(9)\ F \to [v \leq tu \to \bigvee \exists x^0_1 \forall y^0_1 \ldots \exists x^0_k \forall y^0_k \exists w^\gamma A_0(u,v,x_1,y_1,\ldots,x_k,y_k,w)$$

and therefore (by contraction of $\bigvee$)

$$(10)\ F \to [v \leq tu \to \exists x^0_1 \forall y^0_1 \ldots \exists x^0_k \forall y^0_k \exists w^\gamma A_0(u,v,x_1,y_1,\ldots,x_k,y_k,w)$$

which (by ∀–introduction applied to u, v) yields

$$(11)\ F \to A.$$

Remark 210 The proof of thm.27 also works for various other systems $\mathcal{T}$ and domains of terms S than $\mathrm{G}_n\mathrm{A}^\omega$ and $\mathrm{G}_n\mathrm{R}^\omega$. What actually is used in the proof is:

1. Every term $\Psi^\rho \in S$ with $\deg(\rho) \le 2$ has a majorant $\Psi^*[\underline{h}^1]$ such that
   (i) $\mathcal{T} \vdash \lambda\underline{h}.\Psi^*[\underline{h}]$ s–maj Ψ,
   (ii) $\Psi^*[\underline{h}]$ is built up only from $\underline{h}$ and terms $\in S$ of type level ≤ 1 (by substitution).
2. S is (provably in $\mathcal{T}$) closed under the successor, definition by cases, λ –abstraction and contains the variable maximum–functional Φ_1.

Condition 1) is a sort of an upper bound for the complexity of $\mathcal{T}, S$. E.g. 1) is not satisfied if S contains the iteration functional Φ_{it} defined by $\Phi_{it}0yf =_0 y$, $\Phi_{it}x'yf =_0 f(\Phi_{it}xyf)$ (Note that Φ_{it} is primitive recursive in the usual sense of [6] and not only in the extended sense of [5]). In the next paragraph we will show that thm.27 becomes false if $\mathrm{G}_n\mathrm{R}^\omega$ is replaced by $\widehat{PR}^\omega$ (see also remark 214). Since Φ_{it} is on some sense the simplest functional for which 1) fails, this shows that the upper bound provided by 1) is quite sharp. 1) essentially says that Ψ^{001} can be majorized by a term $\Psi^*[x^0, h^1]$ which uses h only at a fixed number of arguments, i.e. there exists a fixed number n (which depends only on the structure of Ψ^* but not on x, h) such that for all h, x the value of of $\Psi^*[x, h]$ only depends on (at most) n–many h–values. Let us illustrate this by an example: Φ defined by $\Phi hx = \max(h0, \ldots, hx)$ depends on $x+1$–many h–values but is majorized by Φ^* defined by $\Phi^*hx := hx$ which for every x depends only on one h–value, namely on hx. If a term Ψ has a majorant which satisfies 1) we say that Ψ **is majorizable with finite support**. One easily convinces oneself that Φ_{it} is not majorizable with finite support.

2) is a lower bound on the complexity of $\mathcal{T}, S$, which also is essential. E.g. take $\mathcal{T} := \mathcal{L}^2$ and $S := \{0^0\}$, where $\mathcal{L}^2$ is first–order logic with $=_0, \le_0$ extended by quantification over functions and two constants $0^0, 1^0$. Consider now

$$G :\equiv \exists x^0 \forall y^0 \exists z^0, f^1 (F_0(f, z) \to A_0(x, y)),$$

where $F_0(f, z) :\equiv (fz = 0 \land 0 \ne 1)$ and $A_0(x, y) :\equiv (y \ne 0 \land x = x \to \bot)$. Then

$$\mathcal{L}^2 \vdash \forall g^1 \exists x, z \le_0 0 \exists f (F_0(f, z) \to A_0(x, gx)) \land Mon(G), \text{ but } \mathcal{L}^2 \nvdash G,$$

i.e. thm.27 fails for $\mathcal{L}^2, S$. If however $\mathcal{L}^2$ is extended by λ–abstraction, then G becomes derivable since we can form $f := \lambda x^0.1^0$ now.

Let F^- denote the 'non-standard' axiom introduced in [14] (def.4.16) and WKL^2_{seq} be the generalization of the binary König's lemmas WKL as defined in [14](def.4.25). Theorem 27 combined with the elimination procedure for F^- from [14] yields the following new conservation result for WKL^2_{seq}:

Corollary 211 *Let A be as in def.26 and thm.27, $n \geq 3$. Then*[6]

1. $G_nA^\omega \oplus F^- \oplus AC\text{–}qf \vdash A^H \Rightarrow G_nA^\omega + Mon(A) \vdash A$. *In particular:* $G_nA^\omega \oplus F^- \oplus AC\text{–}qf \vdash A \Rightarrow G_nA^\omega + Mon(A) \vdash A$.
2. $G_nA^\omega \oplus WKL^2_{seq} \oplus AC\text{–}qf \vdash A^H \Rightarrow G_nA^\omega + Mon(A) \vdash A$. *In particular:* $G_nA^\omega \oplus WKL^2_{seq} \oplus AC\text{–}qf \vdash A \Rightarrow G_nA^\omega + Mon(A) \vdash A$.

If $\tau \leq 1$ (in A) then $G_nA^\omega \oplus F^- \oplus AC\text{–}qf$ can be replaced by $E\text{–}G_nA^\omega + F^- + AC^{\alpha,\beta}\text{–}qf$ (with $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$).

An analogous result holds for the corresponding variant of G_nA^ω where the universal axioms 9) are replaced by the schema of quantifier-free induction.

Proof: 1) By [14](thm.4.21 and remark 3.2.4) $\mathrm{G}_n\mathrm{A}^\omega \oplus F^- \oplus \mathrm{AC\text{–}qf} \vdash A^H$ implies the extractability of a $\Psi \in \mathrm{G}_n\mathrm{R}^\omega$ such that

$$\mathrm{G}_n\mathrm{A}^\omega \vdash \forall u^1 \forall v \leq_\tau tu \forall \underline{h} \exists y_1, \ldots, y_k \leq_0 \Psi u \underline{h} \exists w^\gamma A_0^H.$$

Theorem 27 now yields $\mathrm{G}_n\mathrm{A}^\omega + Mon(A) \vdash A$.

2) follows from 1) by [14](cor.4.28).

Remark 212 *Cor.211 is optimal in the following sense: For simplicity let us consider only the variant of G_3A^ω with the universal axioms replaced by the schema of quantifier-free induction and let us denote this system for the moment also by G_3A^ω. Then $G_3A^\omega \oplus WKL^2_{seq} \oplus AC^{0,0}$-qf is neither conservative over G_3A^ω w.r.t. sentences $\forall u^2 \exists w^1 A_0(u,w)$ nor w.r.t. Π^0_3-sentences, which do not satisfy $Mon(A)$. The first assertion follows analogously to the proof of 4.11 (ii) in [8]. The second claim follows from the fact that even $G_3A^\omega \oplus AC^{0,0}$-qf proves the Σ^0_1-collection principle, whereas G_3A^ω – which is conservative over its first-order fragment, i.e. over the Kalmar–elementary arithmetic – does not prove this principle (see the proof of thm.1 in [17]). Since every instance of the Σ^0_1–collection principle (having only arithmetical parameters) can be prenexed into a Π^0_3–sentence, already $G_3A^\omega \oplus AC^{0,0}$–qf is not Π^0_3–conservative over G_3A^ω. The condition $Mon(A)$ in cor.211 just rules out any non-trivial instances of collection.*

Proof of theorem 28:

$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC\text{–}qf} \vdash A^H$ implies the extractability by monotone functional interpretation (see [14](thm.3.2.2, rem.3.2.4 and the remarks after 3.2.6)) of uniform bounds $\Psi_1, \ldots, \Psi_k \in \mathrm{G}_n\mathrm{R}^\omega$ on $\exists y_i$ (provably in $\mathrm{G}_n\mathrm{A}^\omega_i + \tilde{\Delta}$, where

[6] Here $\oplus$ means that F^- and AC–qf must not be used in the proof of the premise of an application of the quantifier–free rule of extensionality QF–ER. $\mathrm{G}_n\mathrm{A}^\omega$ satisfies the deduction theorem w.r.t $\oplus$ but not w.r.t $+$. In fact the theorem also holds for $(\mathrm{G}_n\mathrm{A}^\omega + \mathrm{AC\text{–}qf}) \oplus F^-$ since the deduction property is used in the proof only for F^-.

$\tilde{\Delta} := \{\exists Y \leq_{\rho\delta} s \forall x^\delta, z^\eta G_0(x, Yx, z) : \forall x^\delta \exists y \leq_\rho sx \forall z^\eta G_0(x,y,z) \in \Delta\}$) which depend only on u and $\underline{h}$:

$$(1)\ \mathrm{G}_n\mathrm{A}^\omega_i + \tilde{\Delta} \vdash \forall u \forall v \leq tu \forall \underline{h} \exists y_1 \leq_0 \Psi_1 u\underline{h} \ldots \exists y_k \leq_0 \Psi_k u\underline{h} \exists w\, A_0^H.$$

The assumption $\mathrm{G}_n\mathrm{A}^\omega + \Delta$+AC–qf $\vdash Mon(A)$ implies (by monotone functional interpretation, since $Mon(A)$ is implied by the monotone functional interpretation of its negative translation) that

$$(2)\ \mathrm{G}_n\mathrm{A}^\omega_i + \tilde{\Delta} \vdash Mon(A).$$

Theorem 27 combined with (1) and (2) yields

$$\mathrm{G}_n\mathrm{A}^\omega + \tilde{\Delta} \vdash A.$$

The second part of the theorem now follows by monotone functional interpretation, since $\tilde{\Delta}$ also is a set of allowed axioms Δ in [14](thm.3.2.2) and $\mathrm{G}_n\mathrm{A}^\omega_i + \Delta$+b-AC $\vdash \tilde{\Delta}$.

For our applications in the next paragraph we need the following corollary of theorem 28:

Corollary 213 *Let* $\forall x^0 \exists y^0 \forall z^0 A_0(u^1, v^\tau, x, y, z) \in \mathcal{L}(G_n A^\omega)$ *be a formula which contains only* u, v *as free variables and satisfies provably in* $G_n A^\omega + \Delta$*+AC–qf the following monotonicity property:*

$$(*)\ \forall u, v, x, \tilde{x}, y, \tilde{y} (\tilde{x} \leq_0 x \wedge \tilde{y} \geq_0 y \wedge \forall z^0 A_0(u,v,x,y,z) \rightarrow \forall z^0 A_0(u,v,\tilde{x},\tilde{y},z)),$$

(i.e. $Mon(\exists x \forall y \exists z \neg A_0)$*). Furthermore let* $B_0(u,v,w^\gamma) \in \mathcal{L}(G_n A^\omega)$ *be a (quantifier-free) formula which contains only* u, v, w *as free variables and* $\gamma \leq 2$*, then the following rule holds:*

$$\begin{cases} \textit{From a proof} \\ G_n A^\omega + \Delta + AC\text{–}qf \vdash \\ \qquad \forall u^1 \forall v \leq_\tau tu (\exists f^1 \forall x, z\, A_0(u,v,x,fx,z) \rightarrow \exists w^\gamma B_0(u,v,w)) \wedge (*) \\ \textit{one can extract a term } \chi \in G_n R^\omega \textit{ such that} \\ G_n A^\omega_i + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \forall \Psi^* ((\Psi^* \textit{ satisfies the mon.funct.interpr.} \\ \qquad \textit{of } \forall x^0, g^1 \exists y^0 A_0(u,v,x,y,gy)) \rightarrow \exists w \leq_\gamma \chi u \Psi^*\, B_0(u,v,w))^7. \end{cases}$$

Proof: We may assume that $\gamma = 2$. The property $Mon(G)$ for

$$G :\equiv \forall u^1 \forall v \leq_\tau tu \exists x^0 \forall y^0 \exists z^0, w^2 (A_0(u,v,x,y,z) \rightarrow B_0(u,v,w))$$

follows logically from the monotonicity assumption $(*)$. By the assumption of the rule to be proved we have

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \mathrm{AC–qf} \vdash G^H + Mon(G).$$

[7] 'Ψ^* satisfies the mon. funct.interpr. of $\forall x, g \exists y A_0(u,v,x,y,gy)$' is meant here for fixed u, v (and not uniformly as a functional in u, v), i.e. $\exists \Psi \big(\Psi^*$ s–maj $\Psi \wedge \forall x, g\, A_0(u,v,x,\Psi xg, g(\Psi xg))\big)$.

¿From this we conclude by thm.28 that

$$\mathrm{G}_n\mathrm{A}^\omega_i + \Delta + \text{b-AC} \vdash \underset{\sim}{\chi} \text{ satisfies the monotone functional interpretation of } G',$$

for a suitable tuple $\underset{\sim}{\chi}$ of terms $\in \mathrm{G}_n\mathrm{R}^\omega$ which can be extracted from the proof.

G' is intuitionistically equivalent to (using the fact that $\mathrm{G}_n\mathrm{A}^\omega_i \vdash \neg\neg A_0 \leftrightarrow A_0$ for quantifier–free formulas A_0)

$$\forall u \forall v \leq tu \neg\neg\exists x^0 \forall y^0 \neg\neg\exists z, w(A_0 \to B_0)$$

of G (This follows immediately if one uses the negative translation which is denoted by $*$ in [15]). By intuitionistic logic the following implication holds

$$G' \to \forall u \forall v \leq_\tau tu(\forall x \neg\neg\exists y \forall z\, A_0(u,v,x,y,z) \to \neg\neg\exists w\, B_0(u,v,w)).$$

Hence from $\underset{\sim}{\chi}$ we obtain a term which satisfies the monotone functional interpretation of the right side of this implication. In particular we obtain a term $\widehat{\chi} \in \mathrm{G}_n\mathrm{R}^\omega$ such that

$$\begin{aligned}\mathrm{G}_n\mathrm{A}^\omega_i + \Delta + \text{b-AC} \vdash \exists W(&\widehat{\chi} \text{ s–maj } W \wedge \forall u \forall v \leq tu \forall \Psi \\ &(\forall x, g\, A_0(u,v,x,\Psi xg, g(\Psi xg)) \to B_0(u,v,Wuv\Psi))).\end{aligned}$$

Define $\chi \in \mathrm{G}_n\mathrm{R}^\omega$ by $\chi := \lambda u^1, \Psi, y^1.\widehat{\chi} u^M(t^* u^M)\Psi y^M$, where $t^* \in \mathrm{G}_n\mathrm{R}^\omega$ is such that $\mathrm{G}_n\mathrm{A}^\omega_i \vdash t^*$ s–maj t and $u^M := \lambda x^0.\max\limits_{i \leq x}(ui)$. Then

$$\begin{aligned}\forall u \forall v \leq tu \forall \Psi^*(\exists \Psi(\Psi^* \text{ s–maj } \Psi \wedge \forall x, g\, A_0(u,v,x,\Psi xg, g(\Psi xg))) \\ \to \exists w \leq_2 \chi u \Psi^*\, B_0(u,v,w)),\end{aligned}$$

since $\widehat{\chi}$ s–maj W and Ψ^* s–maj Ψ imply $\forall u \forall v \leq tu(\chi u \Psi^* \geq_2 Wuv\Psi)$.

Remark 214 *At the end of the next paragraph we will show that cor.213 does not hold for* $PRA^\omega, \widehat{PR}^\omega, PRA^\omega_i$ *(or* $G_nA^\omega + \Sigma^0_1$*–IA,* $\widehat{PR}^\omega$*,* $G_nA^\omega_i + \Sigma^0_1$*–IA) instead of* G_nA^ω *,*G_nR^ω*,* $G_nA^\omega_i$ *(even for* $\Delta = \emptyset$*). Since the proof of cor.213 from thm.28 as well as the proof of thm.28 from thm.27 extends to these theories it follows that also the theorems 27 and 28 do not hold for them. The proof of thm.27 fails for* $\Psi_i \in \widehat{PR}^\omega$ *since* $\widehat{PR}^\omega$ *contains functionals like* Φ_{it} *which are not majorizable with finite support (see also remark 210). The proof of thm.28 fails for* $G_nA^\omega + \Sigma^0_1$*–IA since the (monotone) functional interpretation of* Σ^0_1*–IA requires* Φ_{it} *and thus thm. 27 is not applicable.*

The mathematical significance of corollary 213 for the growth of bounds extractable from given proofs rests on the following fact: Direct monotone functional interpretation of

$$\mathrm{G}_n\mathrm{A}^\omega + \Delta + \text{AC–qf} \vdash \forall u^1 \forall v \leq_\tau tu(\exists f^1 \forall x, z\, A_0(u,v,x,fx,z) \to \exists w^\gamma B_0(u,v,w))$$

yields only a bound on $\exists w$ which depends on a functional which satisfies the monotone functional interpretation of (1) $\exists f \forall x, z\, A_0$ or if we let remain the double negation in front of $\exists$ (which comes from the negative translation) (2) $\neg\neg\exists f \forall x, z\, A_0$. However in our applications the monotone functional interpretation of (1) would require non–computable functionals (since f is not recursive) and the monotone functional interpretation of (2) can be carried out only using bar-recursive functionals. In contrast to this the bound χ only depends on a functional which satisfies the monotone functional interpretation of the negative translation of $\forall x \exists y \forall z\, A_0(x,y,z)$: In our applications such a functional can be constructed in $\widehat{PR}^{\omega}$. In particular the use of the **analytical** premise

$$\exists f^1 \forall x, z A_0$$

has been reduced to the **arithmetical** premise

$$\forall x^0 \exists y^0 \forall z^0 A_0.$$

3 The rate of growth caused by sequences of instances of arithmetical comprehension and choice for Π^0_1-formulas

Using the results from the previous paragraph combined with the methods from [14] one can determine the impact on the rate of growth of uniform bounds for provable $\forall u^1 \forall v \leq_\tau tu \exists w^\gamma A_0$–sentences which may result from the use of sequences of instances (which may depend on the parameters of the proposition to be proved) of:

1. Π^0_1–CA and Π^0_1–AC.
2. The convergence of bounded monotone sequences of real numbers (PCM).
3. The existence of a greatest lower bound for every sequence of real numbers which is bounded from below.
4. The Bolzano–Weierstraß property for bounded sequences in $\mathbb{R}^d$ (for every fixed d).
5. The Arzelà–Ascoli lemma.
6. The existence of lim sup and lim inf for bounded sequences in $\mathbb{R}$.

In this paper we only consider Π^0_1–CA and Π^0_1–AC as well as certain arithmetical consequences of these principles. The treatment of the other analytical principles listed above needs a careful representation of analytical objects like continuous functions in $\mathrm{G}_n\mathrm{A}^\omega$ as well as -in the case of 4),5)- the 'non-standard' axiom F^- introduced in [14] and will be carried out in a subsequent paper.

Definition 31 Π^0_1*–CA*$(f^{1(0)}) :\equiv \exists g^1 \forall x^0 (gx =_0 0 \leftrightarrow \forall y^0 (fxy =_0 0))$.

(Note that iteration of $\forall f^{1(0)}(\Pi^0_1\text{–CA}(f))$ yields CA_{ar}).

Definition 32
Define $A^C_0(f^{1(0)}, x^0, y^0, z^0) :\equiv \forall \tilde{x} \leq_0 x \exists \tilde{y} \leq_0 y \forall \tilde{z} \leq_0 z (f\tilde{x}\tilde{y} \neq_0 0 \vee f\tilde{x}\tilde{z} =_0 0)$.
A^C_0 *can be expressed as a quantifier–free formula in* G_nA^ω *(see [14]).*

Proposition 33 *For* $n \geq 1$ *one has:*

1. $G_nA^\omega_i$ *proves*

$$\forall f, x, \tilde{x}, y, \tilde{y} (\tilde{x} \leq_0 x \wedge \tilde{y} \geq_0 y \wedge \forall z^0 A^C_0(f,x,y,z) \to \forall z^0 A^C_0(f,\tilde{x},\tilde{y},z)).$$

2. *$G_nA_i^\omega \vdash \forall f^{1(0)}(\exists g^1 \forall x^0, z^0 A_0^C(f,x,gx,z) \rightarrow \Pi_1^0$-$CA(f))$.*
3. *For the functional $\Phi \in \widehat{PR}^\omega$ defined by $\Phi x^0 h^1 := \max_{i \leq x+1} h^i(0)$ we have*

$$PRA_i^\omega \vdash \Phi \text{ s-maj} \, \Phi \wedge \forall f^{1(0)}, x^0, h^1 \exists y \leq_0 \Phi xh \, A_0^C(f,x,y,hy).$$

Hence Φ satisfies (provably in PRA_i^ω) the monotone functional interpretation of $\forall x, h \exists y \, A_0^C(f,x,y,hy)$ for all $f^{1(0)}$.

Proof: 1) is obvious.

2) Let g be such that $\forall x, z \forall \tilde{x} \leq x \exists \tilde{y} \leq gx \forall \tilde{z} \leq z (f\tilde{x}\tilde{y} \neq 0 \vee f\tilde{x}\tilde{z} = 0)$. By taking $\tilde{x} := x$ and $\tilde{z} := z$ we obtain $\forall x, z \exists \tilde{y} \leq gx (fx\tilde{y} \neq 0 \vee fxz = 0)$ and thus

$$\forall x (\forall \tilde{y} \leq gx (fx\tilde{y} = 0) \leftrightarrow \forall z (fxz = 0)).$$

Hence $hx := \begin{cases} 0, & \text{if } \forall \tilde{y} \leq gx (fx\tilde{y} = 0) \\ 1, & \text{otherwise} \end{cases}$ satisfies Π_1^0-CA(f).

3) Assume that

$$(*) \; \forall y \leq \Phi xh \exists \tilde{x} \leq x \forall \tilde{y} \leq y \exists \tilde{z} \leq hy (f\tilde{x}\tilde{y} = 0 \wedge f\tilde{x}\tilde{z} \neq 0).$$

Case 1: $\exists i < x+1 (h(h^i 0) \leq h^i 0)$:

$(*)$ applied to $y := h^i 0 \leq \Phi xh$ yields an $\tilde{x} \leq x$ such that

$$(**) \; \forall \tilde{y} \leq h^i 0 \exists \tilde{z} \leq h(h^i 0)(f\tilde{x}\tilde{y} = 0 \wedge f\tilde{x}\tilde{z} \neq 0)$$

and thus for $\tilde{y} := 0$ one has a $\tilde{z} \leq h(h^i 0)$ such that $f\tilde{x}\tilde{z} \neq 0$. But on the other hand –again by $(**)$– one has $f\tilde{x}\tilde{z} = 0$ (since $\tilde{z} \leq h(h^i 0) \leq h^i 0$) which is a contradiction.

Case 2: $\forall i < x+1 (h(h^i 0) > h^i 0)$:

By the pigeon-hole principle, $(*)$ implies that there exists $i < j \leq x+1$ and $\tilde{x} \leq x$ such that

$$(1) \; \forall \tilde{y} \leq h^i 0 \exists \tilde{z} \leq h(h^i 0)(f\tilde{x}\tilde{y} = 0 \wedge f\tilde{x}\tilde{z} \neq 0)$$

and

$$(2) \; \forall \tilde{y} \leq h^j 0 \exists \tilde{z} \leq h(h^j 0)(f\tilde{x}\tilde{y} = 0 \wedge f\tilde{x}\tilde{z} \neq 0).$$

Hence $\exists \tilde{z} \leq h(h^i 0)(f\tilde{x}\tilde{z} \neq 0)$ by (1) (take $\tilde{y} := 0$) and $\forall \tilde{y} \leq h^j 0 (f\tilde{x}\tilde{y} = 0)$ by (2) which is a contradiction since by the case (and $i < j \leq x+1$) $h(h^i 0) = h^{i+1} 0 \leq h^j 0$.

Put together we have proved that $\forall f, x, h \exists y \leq_0 \Phi xh \, A_0^C(f,x,y,hy)$ which is equivalent to a purely universal sentence and hence an axiom of $\mathrm{G}_n\mathrm{A}_i^\omega$ (In fact one easily verifies that this assertion would also be provable in $\mathrm{G}_n\mathrm{A}_i^\omega$ if we would have instead of the universal axioms only the schema of quantifier–free induction included as axioms of $\mathrm{G}_n\mathrm{A}_i^\omega$).

It remains to show that Φ s–maj Φ: Assume that $\tilde{h}$ s–maj$_1 h$. By quantifier-free induction on x one shows that $\forall x (\tilde{h}^x 0 \geq h^x 0)$. Hence (by quantifier–free induction on $\tilde{x}$): $\forall \tilde{x}, x (\tilde{x} \geq x \rightarrow \Phi \tilde{x} \tilde{h} \geq \Phi xh)$.

Cor.213 combined with prop.33 yields

Proposition 34 *Let $n \geq 1$ and $B_0(u^1, v^\tau, w^\gamma) \in \mathcal{L}(G_nA^\omega)$ be a quantifier-free formula which contains only u^1, v^τ, w^γ free, where $\gamma \leq 2$. Furthermore let $\xi, t \in G_nR^\omega$ and Δ be as in thm.28. Then the following rule holds*

$$\begin{cases} G_nA^\omega + \Delta + AC\text{–}qf \vdash \forall u^1 \forall v \leq_\tau tu(\Pi^0_1\text{–}CA(\xi uv) \to \exists w^\gamma B_0(u,v,w)) \\ \Rightarrow \exists (eff.) \chi \in G_nR^\omega \text{ such that} \\ G_nA^\omega_i + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \forall \Psi^* ((\Psi^* \text{ satifies the mon. funct.interpr.} \\ \qquad\qquad \text{of } \forall x^0, h^1 \exists y^0 A^C_0(\xi uv, x, y, hy)) \to \exists w \leq_\gamma \chi u \Psi^* \, B_0(u,v,w)) \\ \text{and in particular} \\ PRA^\omega_i + \Delta + b\text{-}AC \vdash \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \chi u \Psi \, B_0(u,v,w), \end{cases}$$

where $\Psi := \lambda x^0, h^1. \max\limits_{i<x+1} (\Phi_{it} i0h) (= \lambda x^0, h^1. \max\limits_{i<x+1} (h^i 0))$.

In the conclusion, Δ+b-AC can be replaced by $\tilde{\Delta}$, where $\tilde{\Delta}$ is defined as in thm.28. If $\Delta = \emptyset$, then b–AC can be omitted from the proof of the conclusion. If $\tau \leq 1$ and the types of the $\exists$–quantifiers in Δ are ≤ 1, then $G_nA^\omega + \Delta + AC$–qf may be replaced by E–$G_nA^\omega + \Delta + AC^{\alpha,\beta}$–qf, where α, β are as in cor.211.

Remark 35 *In general prop.34 only guarantees a primitive recursive (in the sense of Kleene [6],[7] and not only in the generalized sense of Gödel's T) bound $\Phi := \lambda u.\chi u \Psi \in \widehat{PR}^\omega$ on $\exists w$. This is not avoidable since Π^0_1-CA$(\xi(f))$ proves Σ^0_1-IA(f) relative to G_nA^ω for suitable ξ. If however the proof applies Ψ only to special functions like e.g. $h := S$ then much better bounds will result.*

We now consider Π^0_1–instances of AC$_{ar}$:

$$\Pi^0_1\text{–AC}(f^{1(0)(0)(0)}) :\equiv \forall l^0 (\forall x^0 \exists y^0 \forall z^0 (flxyz =_0 0) \to \exists g^1 \forall x^0, z^0 (flx(gx)z =_0 0)).$$

Π^0_1–AC(f) can be reduced to Π^0_1–CA(g) uniformly by

Proposition 36

$$G_2A^\omega + AC^{0,0}\text{–}qf \vdash \forall f^{1(0)(0)(0)} (\Pi^0_1\text{–}CA(f') \to \Pi^0_1\text{–}AC(f)),$$

where $f' := \lambda v^0, z^0. f(\nu^3_1(v), \nu^3_2(v), \nu^3_3(v), z)$[8].

Proof: By Π^0_1–CA(f') there exists a function h^1 such that

$$\forall v^0 (hv = 0 \leftrightarrow \forall z (f'vz = 0)).$$

$\tilde{h}lxy := h(\nu^3(l,x,y))$. Then

$$\forall l, x, y (\tilde{h}lxy = 0 \leftrightarrow \forall z (flxyz = 0)).$$

AC^{0,0}–qf applied to $\forall x \exists y (\tilde{h}lxy = 0)$ yields $\exists g \forall x, z (flx(gx)z = 0)$.

As a corollary of prop.34 and prop.36 we obtain

[8] Here ν^k_i are the coding functions for tuples from [14].

Corollary 37 *For $n \geq 2$ proposition 34 also holds with Π^0_1–$AC(\xi uv)$ (but now with $A^C_0((\xi uv)', x, y, hy)$ in the conclusion).*

Remark 38 *Suppose that $n \geq 2$.*

1. *We may also have finite conjunctions $\bigwedge\limits_{i=1}^{l} \Pi^0_1$-$CA(\xi_i uv) \wedge \bigwedge\limits_{i=1}^{j} \Pi^0_1$-$AC(\tilde{\xi}_i uv)$ of instances of Π^0_1-CA and Π^0_1-AC in prop. 34 (with a suitable $\tilde{\xi} \in G_nR^\omega$ instead of ξ in the conclusion): Since instances of Π^0_1-AC reduce to instances of Π^0_1-CA by the proposition above we only have to verify (in G_2A^ω) that Π^0_1-$CA(f) \wedge \Pi^0_1$-$CA(g) \rightarrow \Pi^0_1$-$CA(\varphi fg)$, where $\varphi \in G_2R^\omega$ is defined by*
   $\varphi fgxy := \begin{cases} f(j_2x, y), & \text{if } j_1x = 0 \\ g(j_2x, y), & \text{otherwise.} \end{cases}$
   This however is clear.
2. *In prop.34 even sequences $\forall l^0 \Pi^0_1$-$CA(\xi uvl), \forall l^0 \Pi^0_1$-$AC(\xi uvl)$ of instances of Π^0_1-CA,Π^0_1-AC are allowed (instead of Π^0_1-$CA(\xi uv)$,Π^0_1-$AC(\xi uv)$ only) since such sequences of instances can be reduced to single instances in G_2A^ω: $\forall l^0 \Pi^0_1$-$CA(fl)$ follows from Π^0_1-$CA(f')$, where $f'xy := f(j_1x, j_2x, y)$. Similarly for Π^0_1-AC (note that the universal closure under arithmetical parameters has already been incorporated within the definition of Π^0_1-$AC(f)$).*

By Π^0_1–CA$^-$, Π^0_1–AC$^-$and Σ^0_2–AC$^-$ we denote the schemas of Π^0_1-comprehension and Π^0_1,Σ^0_2–choice for formulas **without parameters of type $\geq$ 1**, i.e.

$$\Pi^0_1\text{–CA}^- \;:\; \exists f \forall x^0 (fx =_0 0 \leftrightarrow \forall y^0 A_0(x, y, \underline{a}^0)),$$

$$\Pi^0_1\text{–AC}^- \;:\; \forall x^0 \exists y^0 \forall z^0 A_0(x, y, z, \underline{a}^0) \rightarrow \exists f \forall x \forall z\, A_0(x, fx, z, \underline{a}),$$

$$\Sigma^0_2\text{–AC}^- \;:\; \forall x^0 \exists y^0 \exists z^0 \forall v^0 A_0(x, y, z, v, \underline{a}^0) \rightarrow \exists f \forall x \exists z \forall v\, A_0(x, fx, z, v, \underline{a}),$$

where only x,y,$\underline{a}$ ($x, y, z, \underline{a}$ resp. $x, y, z, v, \underline{a}$) occur free in $A_0(x, y, \underline{a})$ ($A_0(x, y, z, \underline{a})$ resp. $A_0(x, y, z, v, \underline{a})$).

As a special case of prop.34 and cor.37 we have

Proposition 39 *Let $n \geq 2$ and $\gamma \leq 2$ and $B_0(u^1, v^\tau, w^\gamma)$ contains only u, v, w as free variables; $t \in G_nR^\omega$. Then the following rule holds*

$$\begin{cases} G_nA^\omega \oplus AC\text{–}qf \oplus \Pi^0_1\text{–}CA^- \oplus \Sigma^0_2\text{–}AC^- \vdash \forall u^1 \forall v \leq_\tau tu \exists w^\gamma B_0(u, v, w) \\ \Rightarrow \exists \Psi \in \widehat{PR}^\omega \text{ such that} \\ PRA^\omega_i \vdash \forall u^1 \forall v \leq_\tau tu \exists w \leq_\gamma \Psi u\, B_0(u, v, w). \end{cases}$$

If $\tau \leq 1$, we may replace $G_nA^\omega \oplus AC$–$qf \oplus \Pi^0_1$–$CA^- \oplus \Sigma^0_2$–AC^- by
E–G_nA^ω+$AC^{\alpha,\beta}$–qf +Π^0_1–CA^- + Σ^0_2–AC^-, where $(\alpha = 0 \wedge \beta \leq 1)$ or $(\alpha = 1 \wedge \beta = 0)$.

In particular

$$\begin{cases} E\text{–}G_nA^\omega + AC^{\alpha,\beta}\text{–}qf + \Pi^0_1\text{–}CA^- + \Sigma^0_2\text{–}AC^- \ \vdash \forall u^0 \exists v^0 R(u,v) \\ \Rightarrow \exists \text{ primitive recursive function } \varphi : \\ \forall u\, R(u, \varphi u) \text{ is true,} \end{cases}$$

where R is a primitive recursive relation. If in the definition of G_nA^ω the universal axioms 9) are replaced by the schema of quantifier–free induction one has $PRA \vdash R(u, \varphi u)$

(Note that this proposition also holds for $n = \infty$. Since all primitive recursive functions (but not all primitive recursive functionals of type 2!) can be defined in $G_\infty A^\omega$ (see §2 of [14]) we may assume that $G_\infty A^\omega \supset PRA$).

Proof: Since Π^0_1-CA$^-$ follows from Σ^0_2-AC$^-$ which in turn is implied by Π^0_1-AC$^-$ (using pairing) it suffices to consider an instance A of the later (for simplicity we may assume that we have only one arithmetical parameter):

$$A(a^0) :\equiv \forall x^0 \exists y^0 \forall z^0 A_0(x,y,z,a) \to \exists f^1 \forall x,z\, A_0(x, fx, z, a),$$

where $A_0(x,y,z,a)$ is quantifier-free and contains only x,y,z,a as free variables. Let $\xi \in \mathrm{G}_n\mathrm{R}^\omega$ be such that $\xi axyz =_0 0 \leftrightarrow A_0(x,y,z,a)$. Then Π^0_1-AC(ξ) implies $\forall a^0 A(a)$. The corollary now follows from prop.34, cor.37 and rem.38.1) by the deduction theorem for $\oplus$.

Remark 310 Even for the second-order fragment $\mathrm{G}_2\mathrm{A}^2$ of $\mathrm{G}_2\mathrm{A}^\omega$ (and without the universal axioms 9) but only the schema of quantifier–free induction instead of them) the theory $\mathrm{G}_2\mathrm{A}^2 + \Pi^0_1$-CA$^-$ + Π^0_2-IR$^-$ proves the totality of the Ackermann function (see [13]). This refutes a result stated in Mints [16] and a fortiori various generalizations of this result stated in [19] (thm.5.8,cor.5.9,thm.5.13,cor.5.14(ii)). For details see [12](chapter 12) and also [13].

Proposotion 39 also becomes false if the primitive recursive functional $\tilde{\Phi}0yf =_0 y, \tilde{\Phi}x'yf =_0 f(x,y,\tilde{\Phi}xyf)$ is added to $\mathrm{G}_n\mathrm{A}^\omega$ (see [12](chapter 12)). Therefore any proof of a result like this proposition has to exploit carefully the structure of the type-2-functionals of $\mathrm{G}_n\mathrm{A}^\omega$.

Arithmetical consequences of Π^0_1–CA(f) and Π^0_1–AC(f)

Using Π^0_1–CA(f) we can prove (relatively to $\mathrm{G}_2\mathrm{A}^\omega$) every instance of Δ^0_2–IA with fixed function parameters:

$$\Delta^0_2\text{–IA}(f,g) :\equiv \begin{cases} \forall l^0 \Big(\forall x^0 (\exists u^0 \forall v^0 (flxuv =_0 0) \leftrightarrow \forall \tilde{u}^0 \exists \tilde{v}^0 (glx\tilde{u}\tilde{v} =_0 0)) \to \\ [\exists u \forall v (fl0uv = 0) \wedge \forall x (\exists u \forall v (flxuv = 0) \to \exists u \forall v (flx'uv = 0)) \\ \qquad \to \forall x \exists u \forall v (flxuv = 0)] \Big). \end{cases}$$

Define $f' := \lambda i^0, v^0 . f(\nu^3_1(i), \nu^3_2(i), \nu^3_3(i), v)$ and
$g' := \lambda i^0, v^0 . \overline{sg}(g(\nu^3_1(i), \nu^3_2(i), \nu^3_3(i), v))$. We now show

Proposition 311

$$G_2A^\omega + AC^{0,0}\text{–}qf \vdash \forall f, g(\ \Pi^0_1\text{–}CA(f') \wedge \Pi^0_1\text{–}CA(g') \to \Delta^0_2\text{–}IA(f,g)).$$

Proof: Π^0_1–CA(f') and Π^0_1–CA(g') imply the existence of functions h_1, h_2 such that for all l, x, u

$$h_1 lxu =_0 0 \leftrightarrow \forall v(flxuv =_0 0) \text{ and } h_2 lxu =_0 0 \leftrightarrow \exists v(glxuv =_0 0).$$

Assume now that

$$\forall x^0(\exists u^0 \forall v^0(flxuv =_0 0) \leftrightarrow \forall \tilde{u}^0 \exists \tilde{v}^0(glx\tilde{u}\tilde{v} =_0 0)).$$

Then

$$\forall x(\exists u(h_1 lxu = 0) \leftrightarrow \forall \tilde{u}(h_2 lx\tilde{u} = 0)).$$

With classical logic this yields

$$\forall x^0 \exists z^0(\underbrace{[\forall \tilde{u}(h_2 lx\tilde{u} = 0) \to z = 0] \wedge [z = 0 \to \exists u(h_1 lxu = 0)]}_{\in \Sigma^0_1}).$$

By AC^{0,0}–qf we obtain a function α such that

$$\forall x(\alpha x = 0 \leftrightarrow \exists u(h_1 lxu = 0)).$$

Δ^0_2–IA(f, g) now follows by applying QF–IA to $A_0(x) :\equiv (\alpha x = 0)$.

Next we show that Π^0_1–instances (with fixed function parameters) of the collection principle

$$\text{CP} \ : \ \forall \tilde{x} <_0 x \exists y^0 A(x, \tilde{x}, y) \to \exists y_0 \forall \tilde{x} <_0 x \exists y <_0 y_0 A(x, \tilde{x}, y).$$

are derivable from Π^0_1–AC–instances.

Π^0_1–CP$(f) :\equiv$
$\forall l^0, x^0(\forall \tilde{x} < x \exists y^0 \forall z^0(flx\tilde{x}yz =_0 0) \to \exists y_0 \forall \tilde{x} < x \exists y <_0 y_0 \forall z(flx\tilde{x}yz = 0)).$

Proposition 312

$$G_2A^\omega \vdash \forall f(\Pi^0_1\text{–}AC(f') \to \Pi^0_1\text{–}CP(f)),$$

where f' *such that* $f' i\tilde{x}yz =_0 0 \leftrightarrow (\tilde{x} < \nu^2_2(i) \to f(\nu^2_1(i), \nu^2_2(i), \tilde{x}, y, z) =_0 0).$

Proof: Π^0_1–AC(f') yields

$$\forall l^0, x^0(\forall \tilde{x} < x \exists y \forall z(flx\tilde{x}yz = 0) \to \exists h^1 \forall \tilde{x} < x \forall z(flx\tilde{x}(h\tilde{x})z = 0)).$$

Define $y_0 := 1 + \Phi_1 hx$ (Recall that $\Phi_1 hx := \max\limits_{i \le x}(hi)$).

We conclude this paper by showing that cor.213 is false (even for $\Delta = \emptyset$) when G$_n$A$^\omega$, G$_n$R$^\omega$, G$_n$A$^\omega_i$ are replaced by G$_n$A$^\omega + \Sigma^0_1$–IA, $\widehat{PR}^\omega$, G$_n$A$^\omega_i + \Sigma^0_1$–IA or PRA$^\omega$, $\widehat{PR}^\omega$, PRA$^\omega_i$: It is well–known that there is an (function parameter–free) instance G of Π^0_2–IA such that

$$\text{G}_3\text{A}^\omega + \Sigma^0_1\text{–IA } + G \vdash \forall x^0 \exists y^0 A_0(x, y),$$

where $\forall x \exists y \leq fx\, A_0(x,y)$ implies that f has the growth of the Ackermann function.

Let $B(x^0) :\equiv \forall u^0 \exists v^0 B_0(a^0,u,v,x)$ be the induction formula of G, where $B_0(a,u,v,x)$ contains only a,u,v,x as free variables. By applying Π^0_1–CA(f) to $f(i,v) := t_{B_0}(\nu^3_1(i),\nu^3_2(i),v,\nu^3_3(i))$, where t_{B_0} is the characteristic function of B_0, G reduces to Π^0_1–IA (with function parameters) and hence to Σ^0_1–IA (Π^0_n–IA and Σ^0_n–IA are equivalent, see [19]). Hence

$$\mathrm{G_3A}^\omega + \Sigma^0_1\text{–IA} \vdash \Pi^0_1\text{–CA}(f) \rightarrow \forall x \exists y\, A_0(x,y).$$

If cor.213 would apply to $\mathrm{G_3A}^\omega + \Sigma^0_1$–IA and $\widehat{PR}^\omega$ we would obtain (by the proof of prop. 34) a term $s^1 \in \widehat{PR}^\omega$ such that $\forall x \exists y \leq sx\, A_0(x,y)$. This however would contradict the well–known fact that every $s^1 \in \widehat{PR}^\omega$ is primitive recursive.

The same argument applies to PRA^ω since $\mathrm{PRA}^\omega + \mathrm{AC}^{0,0}$ –qf $\vdash \Sigma^0_1$–IA.

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