# Relative constructivity*

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## 1 Introduction

In a previous paper [10] we introduced a hierarchy $(\mathrm{G}_n\mathrm{A}^\omega)_{n\in\mathbb{N}}$ of subsystems of classical arithmetic in all finite types where the growth of definable functions of $\mathrm{G}_n\mathrm{A}^\omega$ corresponds to the well–known Grzegorczyk hierarchy. Let AC–qf denote the schema of quantifier–free choice.
[8],[10] and subsequent papers (under preparation) study various analytical principles $\Gamma$ in the context of the theories $\mathrm{G}_n\mathrm{A}^\omega$+AC–qf (mainly for $n = 2$) and use proof–theoretic tools like e.g. monotone functional interpretation (which was introduced in [9]) to determine their impact on the growth of uniform bounds $\Phi$ such that

$$\forall u^1, k^0 \forall v \leq_\rho tuk \exists w \leq_0 \Phi uk\, A_0(u,k,v,w)$$

which are extractable from given proofs (based on these principles $\Gamma$) of sentences

$$\forall u^1, k^0 \forall v \leq_\rho tuk \exists w^0\, A_0(u,k,v,w).$$

Here $A_0(u,k,v,w)$ is **quantifier–free** and contains only $u,k,v,w$ as free variables; $t$ is a closed term and $\leq_\rho$ is defined pointwise. The term '**uniform** bound' refers to the fact that $\Phi$ does not depend on $v \leq_\rho tuk$ (see [9] for the relevance of such uniform bounds in numerical analysis and for concrete applications to approximation theory).
It turns out that many principles (e.g. the attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$, the mean value theorems for differentiation and integrals, the Cauchy–Peano existence theorem, Brouwer's fixed point theorem for continuous functions $f : [0,1]^d \to [0,1]^d$, the existence of a modulus of uniform continuity for every pointwise continuous function $f : [0,1]^d \to \mathbb{R}$, the (sequential form of the) Heine–Borel covering property for $[0,1]^d$, Dini's theorem and others) do not contribute significantly to the growth at all and for proofs using these principles relative to $\mathrm{G}_2\mathrm{A}^\omega$+AC–qf the extractability of bounds $\Phi uk$ which are polynomials in $u^M n := \max(u0,\ldots,un), k$ is guaranteed (or if the proof relies on certain functions of exponential growth which are not iterated in the proof, then the bound will be of polynomial growth relative to these functions, see [8],[10], [12]).

*This paper essentially contains material from chapter 8 of the author's Habilitationsschrift. Some of the results were presented at the Logic Colloquium 94 at Clermont–Ferrand (see [7]).

As is well-known (cf. the discussion at the end of §3 of [10]), the use of **classical** logic (on which the systems $\mathrm{G}_n\mathrm{A}^\omega$ are based) has the consequence that the extractability of an effective (and for $n = 2$ polynomial) bound from a proof of an $\forall\exists A$–sentence is (in general) guaranteed only if $A$ is quantifier–free (or purely existential). In the present paper we study proofs which may use mathematically strong **non–constructive analytical principles** as e.g.

1) Attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$
2) Mean value theorem for integrals
3) Cauchy–Peano existence theorem
4) Brouwer's fixed point theorem for continuous functions $f : [0,1]^d \to [0,1]^d$
5) A generalization $\mathrm{WKL}^2_{seq}$ of the binary König's lemma WKL
6) Comprehension for negated formulas:

$$CA^\rho_\neg : \ \exists \Phi \leq_{0\rho} \lambda x^\rho.1^0 \forall y^\rho \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big), \text{ where } A \text{ is arbitrary.}$$

as well as the non-intuitionistic logical principles

7) The 'double negation shift' DNS : $\forall x^\rho \neg\neg A \to \neg\neg\forall x^\rho\, A$ for arbitrary types $\rho$ and formulas $A$
8) The 'lesser limited principle of omniscience'

$$\mathrm{LLPO} \ : \ \forall x^1, y^1 \exists k \leq_0 1([k = 0 \to x \leq_{\mathbb{R}} y] \wedge [k = 1 \to y \leq_{\mathbb{R}} x])$$

9) The independence of premise principle for negated formulas

$$\mathrm{IP}_\neg \ : \ (\neg A \to \exists y^\rho B) \to \exists y^\rho(\neg A \to B),$$

where $y$ is not free in $A$,

plus the schema AC of full choice but apply these principles only **in the context of the intuitionistic versions (E)–$\mathbf{G}_n\mathbf{A}^\omega_i$** of the theories (E)–$\mathrm{G}_n\mathrm{A}^\omega$. The restriction to intuitionistic logic guarantees the extractability of (uniform) effective bounds for **arbitrary** $\forall\exists A$–sentences (see theorem 4.1 below). Indeed we are able to extract uniform bounds $\Phi$ (given by closed terms of $\mathrm{G}_n\mathrm{A}^\omega_i$) such that

$$\forall u^1, k^0 \forall v \leq_\rho tuk \exists w \leq_0 \Phi uk(\neg G \to H(w))$$

from such proofs of sentences

$$(+)\ \forall u^1, k^0 \forall v \leq_\rho tuk(\neg G \to \exists w^0 H),$$

where $G, H$ are **arbitrary** formulas (such that (+) is closed).
The phenomenon that we may use even strong positive existence principles as the comprehension schema $CA^\rho_\neg$ for all types $\rho$ (which both classical and intuitionistically produces the strength of classical simple type theory) without any impact on the growth of $\Phi$ is a consequence of the fact that instead of analytical axioms $\Delta$ only, having the form $\forall x^\delta \exists y \leq_\rho sx \forall z^\tau A_0(x,y,z)$ with quantifier–free

$A_0$ (which we have treated in the classical context of [10]), we now may use more general sentences as axioms, e.g. arbitrary sentences having the form

$(*)\ \forall x^\delta (A \to \exists y \leq_\rho sx \neg B)$,

where $A, B$ are arbitrary formulas (such that $(*)$ is closed).

For a somewhat restricted class of formulas (+), DNS dropped and $CA^\rho_\neg$ replaced by the comprehension schema for $\exists$–free formulas one may add also

10) Every pointwise continuous function $F : [0,1]^d \to \mathbb{R}$ is uniformly continuous (together with a modulus of uniform continuity)

11) Every sequence of functions $F_n : [0,1]^d \to \mathbb{R}$ which converges pointwise to a function $F : [0,1]^d \to \mathbb{R}$ converges uniformly on $[0,1]^d$ (together with a modulus of convergence)

12) Every sequence of balls (not necessarily open ones) which cover $[0,1]^d$ contains a finite subcovering

to the list of allowed principles above. Although 11) and 12) are classically refutable strengthened versions of Dini's theorem resp. the Heine–Borel theorem we may use them (combined with the non–constructive principles listed above) and the extractable bounds $\Phi$ are nevertheless classically valid (i.e. the conclusion holds in the full set–theoretic type structure $\mathcal{S}^\omega$). For this result essential use of the 'non–standard' axiom $F$ introduced in [10] is made.

These results also apply to the theory $\mathrm{PRA}^\omega_i$, which contains all primitive recursive functionals $\Phi \in \widehat{\mathrm{PR}}^\omega$ in the sense of Kleene, as well as to $\mathrm{PA}^\omega_i$ which has the schema of full induction and is based on Gödel's primitive recursive functionals $T$. Then the extractable bounds are $\in \widehat{\mathrm{PR}}^\omega$ resp. $\in T$.

The methods by which these extractions of bounds are achieved are new monotone versions of the 'modified realizability' and 'modified realizability with truth' interpretations.

# 2 Majorization and monotone realizability

The set $\mathbf{T}$ of all finite types is defined inductively by

$(i)\ 0 \in \mathbf{T}$ and $(ii)\ \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}$.

Terms which denote a natural number have type 0. Elements of type $\tau(\rho)$ are functions which map objects of type $\rho$ to objects of type $\tau$.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

$(i)\ 0 \in \mathbf{P}$ and $(ii)\ \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}$.

Brackets whose occurrences are uniquely determined are often omitted, e.g. we write $0(00)$ instead of $0(0(0))$. Furthermore we write for short $\tau\rho_k \ldots \rho_1$ instead of $\tau(\rho_k)\ldots(\rho_1)$. Pure types can be represented by natural numbers: $0(n) := n+1$. The types $0, 00, 0(00), 0(0(00)) \ldots$ are so represented by $0, 1, 2, 3 \ldots$. For arbitrary types $\rho \in \mathbf{T}$ the degree of $\rho$ (for short $\deg(\rho)$ ) is defined by $\deg(0) := 0$

and deg$(\tau(\rho))$ := max(deg$(\tau)$,deg$(\rho)$ + 1). For pure types the degree is just the number which represents this type.

**Description of the theories (E)–$\mathbf{G}_n\mathbf{A}^\omega_{(i)}$, (E)–$\mathbf{PRA}^\omega_{(i)}$ and (E)–$\mathbf{PA}^\omega_{(i)}$**

Our theories $\mathcal{T}^\omega_i, \mathcal{T}^\omega$ used in this paper are based on many–sorted intuitionistic (indicated by the subscript $i$) or classical logic formulated in the language of all finite types plus the combinators $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$ which allow the definition of $\lambda$–abstraction.
The systems $\mathrm{G}_n\mathrm{A}^\omega_{(i)}$ (for all $n \geq 1$) are introduced in [10] to which we refer for details. $\mathrm{G}_n\mathrm{A}^\omega_i$ has as primitive relations $=_0, \leq_0$ for type–0–objects, the constant $0^0$, functions $\min_0, \max_0, S$ (successor), $A_0, \ldots, A_n$, where $A_i$ is the $i$–th branch of the Ackermann function (more precisely $A_0(x,y) = y', A_1(x,y) = x + y, A_2(x,y) = x \cdot y, A_3(x,y) = x^y, \ldots$), functionals of type level 2: $\Phi_1, \ldots, \Phi_n$, where $\Phi_1 fx = \max_0(f0, \ldots, fx)$ and for $i \geq 2$, $\Phi_i$ is the iteration of $A_{i-1}$ on the $f$–values, i.e. $\Phi_2 fx = \sum\limits_{i=0}^{x} fi, \Phi_3 fx = \prod\limits_{i=0}^{x} fi, \ldots$. Moreover we have a bounded search functional $\mu_b$ and bounded predicative recursion given by recursor constants $\tilde{R}_\rho$ (where 'predicative' means that recursion is possible only at the type–0–level as in the case of the (unbounded) Kleene-Feferman recursors $\widehat{R}_\rho$). Furthermore we have a quantifier-free rule of extensionality QF–ER.
In addition to the defining axioms for the constants of our theories we add all true sentences having the form $\forall x^\rho A_0(x)$, where $A_0$ is quantifier–free and $deg(\rho) \leq 2$, as axioms. Here 'true' refers to the full set–theoretic model $\mathcal{S}^\omega$. Of course in concrete proofs only very special universal axioms will be used which can be proved in suitable extensions of our theories. However in order to stress that (proofs of) universal sentences do not contribute to the growth of extractable bounds we include them all as axioms. In particular this covers all instances of the schema of quantifier-free induction (The main results in section 3 are also valid for the variant of $\mathrm{G}_n\mathrm{A}^\omega_i$ where the universal axioms are replaced by the schema of quantifier–free induction). The restriction $deg(\rho) \leq 2$ has the reason that at some places we make use of the type structure $\mathcal{M}^\omega$ of all so–called strongly majorizable functionals (which was introduced in [2]) and the fact that $\mathcal{S}^\omega \models \forall x^\rho A_0(x)$ implies $\mathcal{M}^\omega \models \forall x^\rho A_0(x)$ if $deg(\rho) \leq 2$.
The systems $\mathrm{PRA}^\omega_i$, $\mathrm{PRA}^\omega$ result if unbounded predicative recursion (i.e. the Kleene–Feferman recursors $\widehat{R}_\rho$) are added to $\mathrm{G}_n\mathrm{A}^\omega_i$, $\mathrm{G}_n\mathrm{A}^\omega$.
$\mathrm{PA}^\omega_i$, $\mathrm{PA}^\omega$ are the extensions of $\mathrm{G}_n\mathrm{A}^\omega_i$, $\mathrm{G}_n\mathrm{A}^\omega$ by the addition of the schema of full induction and all (impredicative) primitive recursive functionals in the sense of [4].
E–$\mathcal{T}^\omega_{(i)}$ denotes the theory which results from $\mathcal{T}^\omega_{(i)}$ when the quantifier–free rule of extensionality is replaced by the axioms of extensionality (E)

$$\forall x^\rho, y^\rho, z^{\tau\rho}(x =_\rho y \to zx =_\tau zy)$$

for all finite types ($x =_\rho y$ is defined as $\forall z_1^{\rho_1}, \ldots, z_k^{\rho_k}(xz_1 \ldots z_k =_0 yz_1 \ldots z_k)$ where $\rho = 0\rho_k \ldots \rho_1$).
$\mathrm{G}_n\mathrm{R}^\omega$, $\widehat{\mathrm{PR}}^\omega$, $T$ denote the sets of all closed terms of (E)–$\mathrm{G}_n\mathrm{A}^\omega_{(i)}$, (E)–$\mathrm{PRA}^\omega_{(i)}$, (E)–$\mathrm{PA}^\omega_{(i)}$.

**Definition 2.1** *Between functionals of type $\rho$ we define relations $\leq_\rho$ ('less or equal') and s–maj$_\rho$ ('strongly majorizes') by induction on the type:*

$$\begin{cases} x_1 \leq_0 x_2 :\equiv (x_1 \leq_0 x_2), \\ x_1 \leq_{\tau\rho} x_2 :\equiv \forall y^\rho(x_1 y \leq_\tau x_2 y); \end{cases}$$

$$\begin{cases} x^* \ s\text{–}maj_0 \ x :\equiv x^* \geq_0 x, \\ x^* \ s\text{–}maj_{\tau\rho} \ x :\equiv \forall y^{*\rho}, y^\rho (y^* \ s\text{–}maj_\rho \ y \to x^* y^* \ s\text{–}maj_\tau \ x^* y, \ xy). \end{cases}$$

**Remark 2.2** *'s–maj' is a variant of W.A. Howard's relation 'maj' from [5] which is due to [2]. For more details see [6].*

**Notation 2.3** *For $x^1$ we define $x^M := \Phi_1 x$, i.e. $x^M y^0 = \max_{i \leq y}(xi)$.*

**Lemma 2.4** ([10]) *$G_1A_i^\omega$ proves the following facts:*

1) $\tilde{x}^* =_\rho x^* \wedge \tilde{x} =_\rho x \wedge x^* \ s\text{–}maj_\rho \ x \to \tilde{x}^* \ s\text{–}maj_\rho \ \tilde{x}$.

2) $x^* \ s\text{–}maj_\rho \ x \to x^* \ s\text{–}maj_\rho \ x^*$.

3) $x_1 \ s\text{–}maj_\rho \ x_2 \wedge x_2 \ s\text{–}maj_\rho \ x_3 \to x_1 \ s\text{–}maj_\rho \ x_3$.

4) $x^* \ s\text{–}maj_\rho \ x \wedge x \geq_\rho y \to x^* \ s\text{–}maj_\rho \ y$.

5) *For $\rho = \tau\rho_k \ldots \rho_1$ we have*

$$x^* \ s\text{–}maj_\rho \ x \leftrightarrow \forall y_1^*, y_1, \ldots, y_k^*, y_k \Big( \bigwedge_{i=1}^{k} (y_i^* \ s\text{–}maj_{\rho_i} \ y_i) \to x^* y_1^* \ldots y_k^* \ s\text{–}maj_\tau \ x^* y_1 \ldots y_k, x y_1 \ldots y_k \Big).$$

6) $x^* \ s\text{–}maj_1 \ x \leftrightarrow x^* \ \text{monotone} \ \wedge x^* \geq_1 x$, *where $x^*$ is monotone iff* $\forall u, v(u \leq_0 v \to x^* u \leq_0 x^* v)$. *In particular:* $\forall x^1 (x^M s\text{–}maj_1 \ x)$.

7) $x^* \ s\text{–}maj_2 \ x \to \lambda y^1 . x^*(y^M) \geq_2 x$.

**Definition 2.5** 1) *The subset $G_nR_-^\omega \subset G_nR^\omega$ denotes the set of all terms which are built up from $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}, 0^0, A_0, \ldots, A_n$ only (i.e. in particular without $\Phi_1, \ldots, \Phi_n, \tilde{R}_\rho$ or $\mu_b$).*

2) *$G_nR_-^\omega[\Phi_1]$ is the set of all term built up from $G_nR_-^\omega$ plus $\Phi_1$.*

**Proposition 2.6** *For all $n \geq 1$ the following holds: For each term $t^\rho \in G_nR^\omega$ one can construct by induction on the structure of $t$ (without normalization) a term $t^{*\rho} \in G_nR_-^\omega$ such that $G_nA_i^\omega \vdash t^* \ s\text{–}maj_\rho \ t$.*

*An analogous result holds for $G_nR^\omega$, $G_nR_-^\omega$, $G_nA_i^\omega$ replaced by $\widehat{\mathrm{PR}}^\omega$, $\widehat{\mathrm{PR}}^\omega$, $PRA_i^\omega$ resp. $T$, $T$, $PA_i^\omega$.*

***Proof:*** For $\mathrm{G}_n\mathrm{R}^\omega$ the result is proved in [10]. For $T$ it is essentially due to Howard [5] and follows from [2]. An analogous proof applies to $\widehat{\mathrm{PR}}^\omega$ observing that quantifier–free induction is sufficient for the proof the majorizability of the Kleene-recursors.

**Corollary 2.7** *Assume $n \geq 1$, $deg(\rho) \leq 2$ (i.e. $\rho = 0\rho_k \ldots \rho_1$ where $deg(\rho_i) \leq 1$ for $i = 1, \ldots, k$) and $t^\rho \in G_nR^\omega$. Then one can construct (by majorization and subsequent 'logical' normalization) a term $t^*[x_1^{\rho_1}, \ldots, x_k^{\rho_k}]$ such that*

1) *$t^*[x_1, \ldots, x_k]$ contains at most $x_1 \ldots, x_k$ as free variables,*

*2)* $t^*[x_1,\ldots,x_k]$ *is built up only from* $0^0, x_1,\ldots,x_k, A_0,\ldots,A_n$*,*

*3)* $G_nA_i^\omega \vdash \lambda x_1,\ldots,x_k.t^*[x_1,\ldots,x_k]$ *s–maj* $t$*. In particular:*

$$\forall x_1^*, x_1,\ldots,x_k^*, x_k\big(\bigwedge_{i=1}^{k}(x_i^* \text{ s–maj}_{\rho_i} x_i \to t^*[x_1^*,\ldots,x_k^*] \geq_0 tx_1\ldots x_k\big).$$

***Proof:*** See [10] (cor.2.2.24 and remark 2.2.25).

We call $\Phi^{0(0)(1)}uk$ a **polynomial (resp. a finitely iterated exponential function) in** $u^1, k^0$ if $\Phi uk$ can be written as a term $t[u,k]^0$ which is built up from $0^0, k^0, u^1, S, +, \cdot$ (resp. $0^0, k^0, u^1, S, +, \cdot, x^y$) only (see [10] for a detailed discussion of these notions).
¿From the corollary above and the fact that $u^M$ s–maj$_1$ $u$ it follows that for every $\Phi^{0(0)(1)} \in \mathrm{G}_2\mathrm{R}^\omega$ (resp. $\mathrm{G}_3\mathrm{R}^\omega$) one can construct a polynomial (resp. a finitely iterated exponential function) $t[u,k]$ in $u^1, k^0$ such that

$$\forall u^1, k^0(t[u^M,k] \geq_0 \Phi uk),$$

i.e. $\Phi uk$ is bounded by a polynomial (resp. a finitely iterated exponential function) in $u^M$ and $k$.

The methods by which our extraction of bounds is achieved are monotone versions of the so–called 'modified realizability' interpretations $mr$ and $mrt$. Modified realizability was introduced in [13] and is studied in great detail in [14] and [16] (to which we refer).[1] In [14],[16] these interpretations are developed for theories like E-HA$^\omega$ (and immediately apply also to E–PA$_i^\omega$ and E–PRA$_i^\omega$). Furthermore both interpretations apply to our theories E–$\mathrm{G}_n\mathrm{A}_i^\omega$:
The interpretation of the logical part can be carried out using only $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}, \overline{sg}, 0^0$ and definition by cases which is available in E–$\mathrm{G}_n\mathrm{A}_i^\omega$. The non–logical axioms can be expressed (using $\mu_b$ and $\min(x,y) = 0 \leftrightarrow x = 0 \vee y = 0$) as purely universal sentences (without $\vee$) which are trivially interpreted (with the empty tuple of realizing terms).

Whereas the usual modified realizability interpretation extracts tuples of closed terms $\underline{t} = t_1,\ldots,t_k$ such that $\underline{t}$ $mr$ $A$ (where $A$ is a closed formula, the types of $t_i$ and the length $k$ of the tuple depends only on the logical form of $A$, and '$\underline{x}$ $mr$ $A$' (in words '$\underline{x}$ (modified) realizes $A$') is a formula defined by induction on $A$) we are interested in majorants of such realizing terms, i.e. $t_1^*,\ldots,t_k^*$ such that

$$(+)\ \exists x_1,\ldots,x_k \bigwedge_{i=1}^{k}\big(t_i^* \text{ s–maj } x_i \wedge \underline{x}\ mr\ A\big).$$

By saying that '$\underline{t}^*$ fulfils the monotone $mr$–interpretation of $A$' we simply mean that '$\underline{t}^*$ fulfils (+)' (analogously for the 'modified realizability with truth' variant $mrt$ of $mr$).[2] For E–$\mathrm{G}_n\mathrm{A}_i^\omega$ (resp. E–PRA$_i^\omega$, E–PA$_i^\omega$) such terms $\underline{t}^*$ can be obtained by applying at first the usual $mr$–interpretation and subsequent construction of majorants for the resulting terms by proposition 2.6. As in the case of functional interpretation (see our development of the 'monotone functional interpretation' for PA$_i^\omega$ in [9] and its application to $\mathrm{G}_n\mathrm{A}_i^\omega$ in [10]) it is also possible to extract such majorizing terms directly from a given proof, i.e. without extracting $\underline{t}$ at first. However the simplification achieved in this way is not as significant as for the functional interpretation since no decision of prime formulas is needed

[1] In [17] '$mrt$' is denoted by '$mq$'. But note that in [14] '$mq$' denotes a slightly different interpretation.
[2] This variant has the property that $\underline{x}$ $mrt$ $A$ implies $A$; see [17], [16] for information on this.

for the $mr$–interpretation of intuitionistic logic (in contrast to usual functional interpretation, where this is avoided only by our monotone variant) and it will be therefore not studied further.
The monotone $mr$–interpretation has the same nice behaviour w.r.t. to the modus ponens as the usual $mr$–interpretation. Hence in order to treat the extension of E–$\mathrm{G}_n\mathrm{A}^\omega_i$ by new axioms, we only have to consider what terms are needed to fulfil their monotone $mr$–interpretation (and what principles are necessary to verify them). We will show that for a closed axiom

$(*)\ \forall x^\delta(A \to \exists y \leq_\rho sx\neg B)$

any majorant $s^*$ for $s$ satisfies its monotone $mr$–interpretation (provably in E–$\mathrm{G}_n\mathrm{A}^\omega_i + (*)$+b-AC), whereas such axioms in general do not have a usual $mr$-interpretation by computable functionals at all. So sentences $(*)$ contribute to extractable bounds only by majorants for the terms occuring in their formulation but not by their proofs. That is why we can treat them as axioms (if they are true in the full set–theoretical type structure $\mathcal{S}^\omega$ or –as the non-standard axiom $F$ from [10] – in the type structure of all strongly majorizable functionals $\mathcal{M}^\omega$, see below).

**Definition 2.8** *1) The schema of choice is defined as* $AC := \bigcup\limits_{\delta,\rho\in\mathbf{T}} \Big\{(AC^{\delta,\rho})\Big\}$, *where*

$$(AC^{\delta,\rho})\ :\ \forall x^\delta\exists y^\rho\ A(x,y) \to \exists Y^{\rho\delta}\forall x A(x,Yx)\big),$$

*2) The schema of 'bounded' choice is defined as* $b\text{–}AC := \bigcup\limits_{\delta,\rho\in\mathbf{T}} \Big\{(b\text{–}AC^{\delta,\rho})\Big\}$, *where*

$$(b\text{–}AC^{\delta,\rho})\ :\ \forall Z^{\rho\delta}\big(\forall x^\delta\exists y \leq_\rho Zx\ A(x,y,Z) \to \exists Y \leq_{\rho\delta} Z\forall x A(x,Yx,Z)\big),$$

*(a discussion of this principle can be found in [6] ).*

# 3 Extraction of uniform bounds from partially constructive proofs by monotone realizability

**Definition 3.1 ([14])** *The independence–of–premise schema* $IP_\neg$ *for negated formulas is defined as*[3]

$IP_\neg\ :\ (\neg A \to \exists y^\rho B) \to \exists y^\rho(\neg A \to B),$

*where $y$ is not free in $A$.*

**Notational convention 3.2** *In the theorems of this paper we consider* **always closed formulas***, i.e. e.g. in the theorem below $A, B, C$ resp. $D$ contain (at most) $x$, $(x,y)$, $(u,v)$ resp. $(u,v,w)$ as free variables.*

**Theorem 3.3** *Let $s,t$ be $\in G_nR^\omega$ $(n \geq 1)$, $A,B,C,D \in \mathcal{L}(E\text{–}G_nA^\omega_i)$. Then the following holds:*

$$\begin{cases} E\text{–}G_nA^\omega_i + \forall x^\delta(A \to \exists y \leq_\rho sx\neg B)(+AC\text{+}IP_\neg) \vdash \forall u^1\forall v \leq_\gamma tu(\neg C \to \exists w^2 D) \\ \Rightarrow \exists\ \textit{(eff.)}\ \Psi \in G_nR^\omega_-[\Phi_1]\ \textit{such that} \\ E\text{–}G_nA^\omega_i + \exists Y \leq_{\rho\delta} s\forall x(A\to \neg B(x,Yx))(+AC\text{+}IP_\neg) \vdash \forall u^1\forall v \leq_\gamma tu\exists w \leq_2 \Psi u(\neg C \to D) \\ \textit{and therefore} \\ E\text{–}G_nA^\omega + b\text{-}AC + \forall x^\delta(A \to \exists y \leq_\rho sx\neg B)(+AC) \vdash \forall u^1\forall v \leq_\gamma tu\exists w \leq_2 \Psi u(\neg C \to D). \end{cases}$$

[3]In [14] $\mathrm{IP}_\neg$ is denoted by $\mathrm{IP}^\omega$.

*(If the type of $w$ is 0 and $n=2$ (resp. $n=3$) $\Psi u$ is a polynomial (resp. a finitely iterated exponential function) in $u^M$).*
*An analogous result holds for $E\text{–}PRA^\omega_i, \widehat{PR}^\omega$, $E\text{–}PRA^\omega$ and $E\text{–}PA^\omega_i$, $T$, $E\text{–}PA^\omega$ instead of $E\text{–}G_nA^\omega_i$, $G_nR^\omega_{\_}[\Phi_1]$, $E\text{–}G_nA^\omega$.*

***Proof:*** By intuitionistic logic (and the decidability of prime formulas) one shows

$$\exists Y\neg\neg\big(Y \le s \wedge \forall x(A \to \neg B(x,Yx))\big) \leftrightarrow \exists Y\big(Y \le s \wedge \forall x(A \to \neg B(x,Yx))\big)$$

and

$$\exists Y\big(Y \le s \wedge \forall x(A \to \neg B(x,Yx))\big) \to \forall x(A \to \exists y \le sx\neg B(x,y)).$$

Hence the assumption gives

$$\text{E–G}_n\text{A}^\omega_i + \exists Y\neg\neg(Y \le s \wedge \forall x(A \to \neg B(x,Yx)))(+\text{AC}+\text{IP}_\neg) \vdash \forall u^1\forall v \le_\gamma tu(\neg C \to \exists wD).$$

By prop.2.6 we can construct a term $s^* \in \text{G}_n\text{R}^\omega_{\_}$ such that $\text{E–G}_n\text{A}^\omega_i \vdash s^*$ s–maj $s$.
$\mathcal{T} :=\text{E–G}_n\text{A}^\omega_i + \exists Y \le s\forall x(A \to \neg B(x,Yx))$ proves

$$(+)\ \exists u\big(s^* \text{ s–maj } u \wedge u\ mrt\big(\exists\tilde{Y}\neg\neg(\tilde{Y} \le s \wedge \forall x(A \to \neg B(x,\tilde{Y}x)))\big)\big) :$$

By the definition of $mrt$ and the easy fact that $(\underline{x}\ mrt\ \neg F) \leftrightarrow \neg F$ (and $\underline{x}$ is the empty sequence) for negated formulas one shows

$$u\ mrt\ \big(\exists\tilde{Y}\neg\neg(\tilde{Y} \le s \wedge \forall x(A \to \neg B(x,\tilde{Y}x)))\big) \leftrightarrow \neg\neg\big(u \le s \wedge \forall x(A \to \neg B(x,ux))\big).$$

(+) now follows by taking $u := Y$ since $s^*$ s–maj $s \wedge s \ge Y$ implies $s^*$ s–maj $Y$ (see lemma 2.4 ).
Thus $\mathcal{T}(+\text{AC}+\text{IP}_\neg)$ has a monotone $mrt$–interpretation in itself by terms $\in\text{G}_n\text{R}^\omega_{\_}$.
In particular (by the assumption) one can extract $\underline{\Psi} = \Psi_1,\ldots,\Psi_k \in \text{G}_n\text{R}^\omega_{\_}$ such that[4]

$$\mathcal{T}(+\text{AC}+\text{IP}_\neg) \vdash \exists\underline{\chi}\big(\underline{\Psi} \text{ s–maj } \underline{\chi} \wedge \underline{\chi}\ mrt\ \big(\forall u\forall v \le tu(\neg C \to \exists w^2 D(w))\big)\big).$$

Let $t^* \in\text{G}_n\text{R}^\omega_{\_}$ be such that $\text{E–G}_n\text{A}^\omega_i \vdash t^*$ s–maj $t$ (prop.2.6).
The following implications hold in $\text{E–G}_n\text{A}^\omega_i$:

$$\begin{aligned}
&\underline{\chi}\ mrt\ \big(\forall u\forall v \le tu(\neg C \to \exists w^2 D(w))\big) \ \to \\
&\forall u\forall v(v \le tu \wedge \neg C \to \chi_2 uv\ldots\chi_k uv\ mrt\ D(\chi_1 uv)) \ \to \ \text{(because } \underline{x}\ mrt\ D \to D) \\
&\forall u,v(v \le tu \wedge \neg C \to D(\chi_1 uv)) \overset{\Psi_1 \text{ s–maj } \chi_1}{\to} \ \text{(by lemma 2.4)} \\
&\forall u\forall v \le tu\big(\underbrace{\lambda y^1.\Psi_1 u^M(t^* u^M)y^M}_{\Psi u:=} \ge_2 \chi_1 uv \wedge (\neg C \to D(\chi_1 uv))\big) \ \to \\
&\forall u\forall v \le tu\exists w \le_2 \Psi u(\neg C \to D(w)).
\end{aligned}$$

It remains to show that

$$\text{E–G}_n\text{A}^\omega + \text{b-AC} \vdash \forall x(A \to \exists y \le sx\neg B) \to \exists Y \le s\forall x(A \to \neg B(x,Yx)) :$$

[4]Here $\underline{\Psi}$ s-maj $\underline{\chi}$ means $\bigwedge_{i=1}^{k}(\Psi_i \text{ s–maj } \chi_i)$.

$$\begin{aligned}\forall x(A\to\exists y\le sx\neg B) &\overset{(E)}{\to} \forall x(A\to\exists y\neg B(x,\min_\rho(y,sx)))\\ &\overset{\text{class.logic}}{\to} \forall x\exists y(A\to\neg B(\min_\rho(y,sx)))\\ &\to \forall x\exists y\le sx(A\to\neg B(x,y))\\ &\overset{\text{(b–AC)}}{\to} \exists Y\le s\forall x(A\to\neg B(x,Yx)).\end{aligned}$$

**Remark 3.4** *Instead of single variables $u^1, w^2$ we may have also tuples $u_1^{\rho_1},\ldots,u_k^{\rho_k},\ w_1^{\gamma_1},\ldots,w_l^{\gamma_l}$ where $deg(\rho_i)\le 1$ and $deg(\gamma_j)\le 2$ for $1\le i\le k$ and $1\le j\le l$. Similarly we may have tuples $x_1^{\delta_1},\ldots,x_p^{\delta_p}$ instead of $x^\delta$.*
*In case $u_1^1, u_2^0$ and $w^0$ the bound $\Psi u_1u_2$ is a polynomial (resp. a finitely iterated exponential function) in $u_1^M$ and $u_2$ if $n=2$ (resp. $n=3$).*
*An analogous remark applies to theorems 3.10 and 3.18 below.*

**Corollary 3.5 (to the proof)** *1) If $A\equiv\neg\tilde{A}$ is a negated formula, then the conclusion can be proved in E–$G_nA_i^\omega$+b-AC+$\forall x(A\to\exists y\le sx\neg B)$+IP$_\neg$(+AC).*

*2) If the variable $x$ is not present (i.e. if we only have closed axioms $A\to\exists y\le s\neg B(y)$, then the conclusion can be proved without b-AC.*

*3) Instead of a single axiom $\forall x(A\to\exists y\le sx\neg B)$ we may also use a finite set of such axioms.*

**Definition 3.6** ([14]) *A formula $A\in\mathcal{L}(E\text{–}G_nA_i^\omega)$ is called $\exists$–free (or 'negative') if $A$ is built up from quantifier–free formulas by means of $\wedge,\to,\neg,\forall$ (i.e. $A$ does not contain $\exists$ and contains $\vee$ only within quantifier–free subformulas[5]).*

**Definition 3.7** ([14]) *The subset $\Gamma_1$ of formulas $\in\mathcal{L}(E\text{–}G_nA_i^\omega)$ is defined inductively by*

*1) Quantifier–free formulas are in $\Gamma_1$.[6]*

*2) $A,B\in\Gamma_1\Rightarrow A\wedge B, A\vee B, \forall x\,A, \exists x\,A\in\Gamma_1$.*

*3) If $A$ is $\exists$–free and $B\in\Gamma_1$, then $(\exists\underline{x}A\to B)\in\Gamma_1$.*

**Definition 3.8** ([14]) *The independence–of–premise schema for $\exists$–free formulas is defined as*

$$IP_{\exists f}\ :\ (A\to\exists y^\rho B)\to\exists y^\rho(A\to B),$$

*where $A$ is $\exists$–free and does not contain $y$ as a free variable.*

**Remark 3.9** *Because of the fact that in our theories we can derive $\neg\neg P\leftrightarrow P$ for prime formulas $P$, IP$_\neg$ implies IP$_{\exists f}$. In the presence of AC also the converse implication holds.*

[5] Troelstra distinguishes between negative formulas which are built up from the double negation $\neg\neg P$ of prime formulas (instead of the arbitrary quantifier–free formulas in our definition) and $\exists$–free formulas where $P$ instead of $\neg\neg P$ may be used. Since our theories have only decidable prime formulas both notions coincide with our definition up to equivalence in E–$G_nA_i^\omega$.

[6] Note that in our theories quantifier–free formulas can be written a prime formulas $s=_0 t$.

**Theorem 3.10** *Let $A, D$ be $\in \Gamma_1$ and $B, C$ denote $\exists$–free formulas; $s, t \in G_nR^\omega$ $(n \geq 1)$. Then the following rule holds*

$$\begin{cases} E\text{–}G_nA_i^\omega + \forall x^\delta(A \to \exists y \leq_\rho sx\ B) + AC\text{+}IP_\neg \vdash \forall u^1 \forall v \leq_\gamma tu(C \to \exists w^2 D(w)) \\ \Rightarrow \exists \text{ (eff.) } \Psi \in G_nR_-^\omega[\Phi_1] \text{ such that} \\ E\text{–}G_nA_i^\omega + \exists Y \leq_{\rho\delta} s\forall x(A \to B(x, Yx)) \vdash \forall u^1 \forall v \leq_\gamma tu \exists w \leq_2 \Psi u(C \to D(w)) \\ \text{and therefore} \\ E\text{–}G_nA^\omega + b\text{-}AC + \forall x^\delta(A \to \exists y \leq_\rho sx\ B) \vdash \forall u^1 \forall v \leq_\gamma tu \exists w \leq_2 \Psi u(C \to D(w)). \end{cases}$$

*(If the type of $w$ is $0$ and $n = 2$ (resp. $n = 3$) $\Psi u$ is a polynomial (resp. a finitely iterated exponential function) in $u^M$).*
*An analogous result holds for E–$PRA_i^\omega$,$\widehat{PR}^\omega$, E–$PRA^\omega$ and E–$PA_i^\omega$, $T$, E–$PA^\omega$ instead of E–$G_nA_i^\omega$, $G_nR_-^\omega[\Phi_1]$, E–$G_nA^\omega$.*

***Proof:*** Since quantifier–free formulas can be transformed into formulas $t\underline{x} =_0 0$, we may assume that the $\exists$–free formulas $B, C$ do not contain $\vee$. The assumption of the theorem implies

$$(*)\ \mathcal{T} := \text{E–G}_n\text{A}_i^\omega + \exists Y \leq s\forall x^\delta(A \to B(x, Yx)) + \text{AC+IP}_\neg \vdash \forall u^1 \forall v \leq_\gamma tu(C \to \exists w^2 D(w)).$$

We now show that $\mathcal{T}$ has a monotone $mr$–interpretation in $\mathcal{T}^- := \mathcal{T} \setminus \{\text{AC,IP}_\neg\}$ by terms $\in$ $\text{G}_n\text{R}_-^\omega$. For E–$\text{G}_n\text{A}_i^\omega$ + AC+IP$_\neg$ this follows from the proof of the fact that E–HA$^\omega$ + AC+IP$_\neg$ has a $mr$–interpretation in E–HA$^\omega$ (see [14],[16]) combined with our remarks in §2 and prop.2.6 (The $mr$–interpretation of AC+IP$_\neg$ requires only terms built up from $\Pi, \Sigma$). Next we show that

$$\mathcal{T}^- \vdash \exists u\big(s^* \text{ s–maj } u \wedge u\ mr\ (\exists Y \leq s\forall x(A \to B(x, Yx)))\big) :$$

Since for $\exists$–free formulas $(\underline{x}\ mr\ B) \equiv B$ ($\underline{x}$ being the empty sequence) the $mr$–definition yields

$$u\ mr\ \big(\exists Y \leq s\forall x(A \to B(x, Yx))\big) \leftrightarrow u \leq s \wedge \forall x\big(\exists \underline{v}(\underline{v}\ mr\ A) \to B(x, ux)\big).$$

The right side of this equivalence is fulfilled by taking $u := Y$ since $\exists \underline{v}(\underline{v}\ mr\ A) \to A$ (because of the assumption $A \in \Gamma_1$). Hence $\mathcal{T}$ has a monotone $mr$–interpretation in $\mathcal{T}^-$ by terms $\in$ $\text{G}_n\text{R}_-^\omega$. Therefore $(*)$ implies the extractability of terms $\underline{\Psi} = \Psi_1, \ldots, \Psi_k \in \text{G}_n\text{R}_-^\omega$ such that

$$\exists \underline{\chi}\big(\underline{\Psi} \text{ s–maj } \underline{\chi} \wedge \underline{\chi}\ mr\ (\forall u \forall v \leq tu(C \to \exists w D(w)))\big).$$

The following chain of implications holds in E–$\text{G}_n\text{A}_i^\omega$:

$$\begin{aligned} &\underline{\chi}\ mr\ \big(\forall u \forall v \leq tu(C \to \exists w\ D(w))\big) \overset{C\ \exists\text{–free}}{\to} \\ &\forall u, v(v \leq tu \wedge C \to \chi_2 uv \ldots \chi_k uv\ mr\ D(\chi_1 uv)) \overset{D \in \Gamma_1}{\to} \\ &\forall u, v, (v \leq tu \wedge C \to D(\chi_1 uv)) \overset{\Psi_1 \text{ s–maj } \chi_1}{\to} \text{ (by lemma 2.4)} \\ &\forall u \forall v \leq tu(\lambda y^1.\Psi_1 u^M(t^* u^M)y^M \geq_2 \chi_1 uv \wedge (C \to D(\chi_1 uv)) \to \\ &\forall u \forall v \leq tu \exists w \leq_2 \Psi u(C \to D(w)), \end{aligned}$$

where $t^* \in \text{G}_n\text{R}_-^\omega$ such that E–$\text{G}_n\text{A}_i^\omega \vdash t^*$ s–maj $t$ and
$\Psi := \lambda u, y.\Psi_1 u^M(t^* u^M)y^M \in \text{G}_n\text{R}_-^\omega[\Phi_1]$.
As in the proof of the previous theorem one shows

$$\text{E–G}_n\text{A}^\omega\text{+ b-AC } \vdash \forall x(A \to \exists y \leq sx\ B) \to \exists Y \leq s\forall x(A \to B).$$

**Corollary 3.11 (to the proof)** *1) If $A \equiv \neg\tilde{A}$ is a negated (resp. $\exists$–free) formula, then the conclusion can be proved in E–$G_nA_i^\omega$+$IP_\neg$+ b-AC+$\forall x(A \to \exists y \leq sx\, B)$ (resp. E–$G_nA_i^\omega$+$IP_{\exists f}$+ b-AC+$\forall x(A \to \exists y \leq sx\, B)$).*

*2) If the variable $x$ is not present, i.e. if only axioms $A \to \exists y \leq sx\, B(y)$ are used ($A \in \Gamma_1, B$ $\exists$–free), then the conclusion can be proved without b-AC.*

*3) Instead of a single axiom $\forall x(A \to \exists y \leq sx\, B(y))$ we may also use a finite set of such axioms.*

**Remark 3.12** *For every $\exists$–free formula $A$ of our theories the equivalence $A \leftrightarrow \neg\neg A$ holds intuitionistically (since the prime formulas are stable). So the allowed axioms in theorem 3.3 include the axioms allowed in theorem 3.10.*

Although theorem 3.10 is weaker than theorem 3.3 in **some** respects (e.g. $A, D$ have to be in $\Gamma_1$) it is of interest for the following reason:
Despite the fact that the schema AC of full choice may be used in the proof of the assumption, the proof of the conclusion uses only b-AC instead of AC. This has the consequence that the conclusion is valid in the model $\mathcal{M}^\omega$ of all strongly majorizable functionals, if $\forall x(A \to \exists y \leq sx\, B)$ holds in $\mathcal{M}^\omega$ (although $\mathcal{M}^\omega \not\models$ AC, see [6] ). Let us e.g. consider the theory E–$\mathrm{G}_n\mathrm{A}_i^\omega$ + $F$+AC, where $F$ is the 'non-standard'-axiom studied in [10]:

$$F :\equiv \forall \Phi^{2(0)}, y^{1(0)} \exists y_0 \leq_{1(0)} y \forall k^0 \forall z \leq_1 yk\big(\Phi kz \leq_0 \Phi k(y_0 k)\big).$$

$F$ is valid in $\mathcal{M}^\omega$ (see [10] and also the proof of theorem 4.2 below) but does not hold in $\mathcal{S}^\omega$ (see [10]).
Since $F$ has the form $\forall x(A \to \exists y \leq sx\, B)$ (with $A :\equiv (0 = 0) \in \Gamma_1$ and $B$ $\exists$–free) of an allowed axiom in theorem 3.10 (and a fortiori in theorem 3.3 ) we can apply theorem 3.10 and obtain the following rule

$$\begin{cases} \text{E–G}_n\text{A}_i^\omega + F\text{+AC} \ \vdash \forall u^1 \forall v \leq_1 tu(C \to \exists w^2 D(w)) \\ \Rightarrow \exists \text{ (eff.) } \Psi \in \text{G}_n\text{R}_-^\omega[\Phi_1] \text{ such that} \\ \text{E–G}_n\text{A}_i^\omega + F\text{+ b-AC} \ \vdash \forall u^1 \forall v \leq_1 tu \exists w \leq_2 \Psi u(C \to D(w)). \end{cases}$$

The conlusion of this rule implies (see the proof of theorem 4.9 in [10])

$$\mathcal{M}^\omega \models \forall u^1 \forall v \leq_1 tu \exists w \leq_2 \Psi u(C \to D(w)).$$

If all positively occuring $\forall x^\rho$–quantifiers and all negatively occuring $\exists x^\rho$–quantifiers in this formula have types $\rho \leq 1$ and if all other quantifiers have types $\leq 2$, then we can conclude (since $\mathcal{M}_1 = \mathcal{S}_1$ and $\mathcal{M}_2 \subset \mathcal{S}_2$, for details see [10] (remark 4.10))

$$\mathcal{S}^\omega \models \forall u^1 \forall v \leq_1 tu \exists w \leq_2 \Psi u(C \to D(w)).$$

Hence the bound $\Psi$ is classically valid although it has been extracted from a proof in a theory which classically is inconsistent:

**Claim:** E–$\mathrm{G}_n\mathrm{A}^\omega$ + $F$+AC $\vdash 0 = 1$.

**Proof of the claim:** Consider

$$\forall f \leq_1 \lambda x.1 \exists n^0(\exists k^0(fk = 0) \to fn = 0),$$

which holds by classical logic. AC yields the existence of a functional $\Psi^{0(1)}$ such that

$$\forall f \leq_1 \lambda x.1(\exists k^0(fk=0) \to f(\Psi f)=0).$$

$F$ applied to $\Psi$ implies that $\Psi$ is bounded on $\{f^1 : f \leq_1 \lambda x.1\}$, hence

$$\exists n_0 \forall f \leq_1 \lambda x.1 \exists n \leq_0 n_0(\exists k^0(fk=0) \to fn=0),$$

which –of course– is wrong.

The (intuitionistically consistent) combination of $F$ and AC (instead of quantifier-free choice AC–qf only, which we have used in the classical setting of [10] in order to derive the principle $\Sigma^0_1$–UB of uniform boundedness for $\Sigma^0_1$–formulas) can be used to prove strengthened versions of various classical theorems which may have non–constructive counterexamples, but no constructive ones. These proofs rely on the fact that $F$ and AC prove a very general principle of uniform boundedness for **arbitrary formulas** $A$:

**Proposition 3.13**

$$E\text{–}G_nA_i^\omega + F + AC \vdash$$
$$\forall y^{1(0)}\big(\forall k^0 \forall x \leq_1 yk \exists z^0 A(x,y,k,z) \to \exists \chi^1 \forall k^0 \forall x \leq_1 yk \exists z \leq_0 \chi k\, A(x,y,k,z)\big),$$

*where $A$ is an arbitrary formula of $\mathcal{L}(E\text{–}G_nA^\omega)$ which may contain parameters of arbitrary type.*

***Proof:*** $\forall k^0 \forall x \leq_1 yk \exists z^0\, A(x,y,k,z)$ implies

$$\forall k^0 \forall x^1 \exists z^0\, A(\min_1(x,yk),y,k,z).$$

AC yields

$$\exists \Phi^{0(1)(0)} \forall k^0, x^1\, A(\min_1(x,yk),y,k,\Phi kx).$$

Hence by extensionality (E) (using that $x \leq_1 yk \to \min_1(x,yk) =_1 x$)

$$\exists \Phi^{0(1)(0)} \forall k^0 \forall x \leq_1 yk\, A(x,y,k,\Phi kx).$$

$F$ applied to $\Phi$ yields a function $\chi^1$ (namely $\chi k := \Phi k(y_0 k)$) such that

$$\forall k^0 \forall x \leq_1 yk \exists z \leq_0 \chi k\, A(x,y,k,z).$$

**Example 1: Pointwise convergence implies uniform convergence or 'Dini's theorem without monotonicity and continuity assumption'**[7]

$$\text{E–G}_2\text{A}_i^\omega + F\text{+AC} \vdash \forall \Phi_n, \Phi : [0,1]^d \to \mathbb{R}(\Phi_n \text{ converges pointwise to } \Phi \to$$
$$\Phi_n \text{ converges uniformly on } [0,1]^d \text{ to } \Phi \text{ and there exists a modulus of convergence}).$$

[7] This principle (with continuity assumption for $\Phi_n, \Phi$) has been studied in [1] in a purely intuitionistic context, i.e. without our (in general non–constructive) axioms $\forall x(A \to \exists y \leq sx \neg B)$,$\forall x(C \to \exists y \leq sx\, D)$ ($C \in \Gamma_1, D$ is $\exists$-free).

***Proof:*** By the assumption we have

$$\forall k^0 \forall x \in [0,1]^d \exists n^0 \forall l \geq_0 n\big(|\Phi x - \Phi_l x| \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

By prop.3.13 and the fact that '$\forall x \in [0,1]^d$' has the form '$\forall x \leq_1 M$' in our representation of $[0,1]^d$ in E–$\mathrm{G}_2\mathrm{A}_i^\omega$ (see [11],[12] for details) one obtains

$$\exists \chi^1 \forall k^0 \forall x \in [0,1]^d \exists n \leq_0 \chi k \forall l \geq_0 n\big(|\Phi x - \Phi_l x| \leq_{\mathbb{R}} \frac{1}{k+1}\big)$$

and therefore

$$\exists \chi^1 \forall k^0 \forall x \in [0,1]^d \forall l \geq_0 \chi k\big(|\Phi x - \Phi_l x| \leq_{\mathbb{R}} \frac{1}{k+1}\big).$$

**Remark 3.14** *1) The usual counterexamples to the theorem above do not occur in E–$G_n A_i^\omega$ since they use classical logic to verify the assumption of pointwise convergence: E.g. consider the well–known example $\Phi_n(x) := \max_{\mathbb{R}}(n - n^2|x - \frac{1}{n}|, 0)$ $(n \geq 1)$. The proof that $\Phi_n$ converges pointwise to 0 requires the instance '$\forall x \in [0,1](x =_{\mathbb{R}} 0 \vee x >_{\mathbb{R}} 0)$' of the tertium–non–datur schema, which cannot be proved in E–$G_n A_i^\omega$.*

*2) Note that in the classical setting (see [9],[12]) the monotonicity assumption of Dini's theorem is used just to eliminate the universal quantifier '$\forall l \geq_0 n$' which reduces the application of the general principle of uniform boundedness to an application of its restriction $\Sigma_1^0$–UB to $\Sigma_1^0$–formulas (since $\leq_{\mathbb{R}}$ can be replaced by $<_{\mathbb{R}}$), which follows from $F$ and* **quantifier–free** *choice.*

**Example 2: Heine–Borel property for $[0,1]^d$ and sequences of arbitrary (not necessarily open) balls**

$$\begin{aligned}&\text{E–G}_2\text{A}_i^\omega\text{+AC } + F \vdash \forall f : \mathbb{N} \to \mathbb{R}_+ \forall g : \mathbb{N} \to [0,1]^d \forall h^1\\ &\quad\big(\forall x \in [0,1]^d \exists k^0\big((hk =_0 0 \wedge \|x - gk\|_E <_{\mathbb{R}} fk) \vee (hk \neq 0 \wedge \|x - gk\|_E \leq_{\mathbb{R}} fk)\big) \to\\ &\quad\exists k_0 \forall x \in [0,1]^d \exists k \leq_0 k_0\big((hk =_0 0 \wedge \|x - gk\|_E <_{\mathbb{R}} fk) \vee (hk \neq 0 \wedge \|x - gk\|_E \leq_{\mathbb{R}} fk)\big)\big).\end{aligned}$$

***Proof:*** Similarly to the proof of example 1 using prop.3.13.

**Remark 3.15** *The restriction to open balls in the classical context of $G_2A^\omega$ is needed in order to restrict the use of uniform boundedness to $\Sigma_1^0$–UB (see [12] for details).*

**Examples of sentences having (in E–$\mathbf{G}_2\mathbf{A}_i^\omega$) the form $G \equiv \forall x(A \to \exists y \leq sx \neg B)$ or $H \equiv \forall x(C \to \exists y \leq sx\, D)$ where $D$ is $\exists$–free and $C \in \Gamma_1$:**

1) The attainment of the maximum of $f \in C([0,1]^d, \mathbb{R})$, the mean value theorem of integration, the Cauchy–Peano existence theorem, Brouwer's fixed point theorem and others can be expressed as axioms $H$ (and a fortiori as axioms $G$, see the remark below).

2) The generalization of the axiom $F$ to arbitrary types $\rho$:

$$F_\rho :\equiv \forall \Phi^{0\rho 0}, y^{\rho 0} \exists y_0 \leq_{\rho 0} y \forall k^0 \forall z \leq_\rho yk\big(\Phi kz \leq_0 \Phi k(y_0 k)\big), \text{ which still holds in } \mathcal{M}^\omega$$

(see the proof of theorem 4.2 below) has the form of an axiom $H$ (and so a fortiori can be written as $G$) since '$\forall k^0 \forall z \leq_\rho yk\big(\Phi kz \leq_0 \Phi k(y_0 k)\big)$' is $\exists$–free. Note that $F \equiv F_1$.

3) Our generalization

$$\mathrm{WKL}^2_{seq} :\equiv \begin{cases} \forall \Phi^{0010}\big(\forall k^0, x^0 \exists b \leq_1 \lambda n^0.1^0 \bigwedge\limits_{i=0}^{x} (\Phi k(\overline{b,i})i =_0 0) \\ \qquad\qquad \to \exists b \leq_{1(0)} \lambda k^0, n^0.1 \forall k^0, x^0 (\Phi k(\overline{bk,x})x =_0 0)\big) \end{cases}$$

of the binary König's lemma WKL from [10] has the form $H$ (and therefore can be written as $G$) since its implicative premise '$\forall k^0, x^0 \exists b \leq_1 \lambda n^0.1^0 \bigwedge\limits_{i=0}^{x} (\Phi k(\overline{b,i})i =_0 0)$' is in $\Gamma_1$.

4) The universal closure of each instance of the 'double negation shift' DNS : $\forall x \neg\neg A \to \neg\neg \forall x\, A$ has the form $G$.

5) The 'lesser limited principle of omniscience' is defined as:[8]

$$\mathrm{LLPO}\ :\ \forall f^1 \exists k \leq_0 1([k = 0 \to \forall n(f'(2n) = 0)] \wedge [k = 1 \to \forall n(f'(2n+1) = 0)]),$$

where

$$f'n := \begin{cases} 1, \text{ if } fn = 1 \wedge \forall k < n(fk \neq 1) \\ 0, \text{ otherwise.} \end{cases}$$

LLPO can be formulated also in the following equivalent form

$$\forall x^1, y^1 \exists k \leq_0 1([k = 0 \to x \leq_{\mathbb{R}} y] \wedge [k = 1 \to y \leq_{\mathbb{R}} x]).$$

LLPO has the form of an axiom $H$ and so can be written as an axiom $G$ (see [3] for a discussion of LLPO).

6) Comprehension for negated (resp. $\exists$–free) formulas:

$$CA^\rho_\neg :\ \exists \Phi \leq_{0\rho} \lambda x^\rho.1^0 \forall y^\rho \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big), \text{ where } A \text{ is arbitrary } (\Phi \text{ not free in } A),$$

$$CA^\rho_{\exists f} :\ \exists \Phi \leq_{0\rho} \lambda x^\rho.1^0 \forall y^\rho \big(\Phi y =_0 0 \leftrightarrow A(y)\big), \text{ where } A \text{ is } \exists\text{–free.}$$

By intuitionistic logic (and the decidability of prime formulas) we have

$$\neg\neg \forall y^\rho \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big) \leftrightarrow \forall y^\rho \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big).$$

Hence the universal closure of each instance of $CA^\rho_\neg$ is (equivalent to) an axiom $G$.

The universal closure of each instance of $CA^\rho_{\exists f}$ is an axiom $H$ since together with $A$ also $\forall y^\rho \big(\Phi y =_0 0 \leftrightarrow A(y)\big)$ is $\exists$–free.

[8] Usually one quantifies over all functions $\leq 1$ which are $=1$ in at most one point. This is achieved by our transformation $f \mapsto f'$.

**Remark 3.16** *1) Using a convenient representation of $C[0,1], \mathbb{R}, [0,1]^d$ etc. in $G_2A_i^\omega$ the theorems mentioned in 1) above can be expressed as sentences $\Delta$ (see [8]) and so a fortiori as sentences $H, G$. For $H, G$ even a much more simple representation suffices since $H, G$ are by far less restrictive than $\Delta$. Let us sketch the formalization of the assertion that every $f \in C[0,1]$ attains its maximum:*

*Real numbers (with fixed rate of convergence) can be represented by functions $x^1, y^1$ in $G_2A_i^\omega$ and $\leq_{\mathbb{R}} \in \Pi_1^0$, $-_{\mathbb{R}}, |\cdot|_{\mathbb{R}} \in G_2A_i^\omega$ represent the corresponding relations resp. operations on $\mathbb{R}$. Elements $x \in [0,1]$ can be represented by functions $x^1$ which are bounded by some fixed $M^1 \in G_2R^\omega$ (see [11]).*

*Hence our assertion can be expressed as follows:*

$$\forall \Phi^{1(1)}\big(\forall k^0, x^1(0 \leq_{\mathbb{R}} x \leq_{\mathbb{R}} 1 \to \exists n^0 \forall y^1(0 \leq_{\mathbb{R}} y \leq_{\mathbb{R}} 1 \wedge |x -_{\mathbb{R}} y|_{\mathbb{R}} \leq_{\mathbb{R}} \tfrac{1}{n+1} \to$$
$$|\Phi x -_{\mathbb{R}} \Phi y|_{\mathbb{R}} \leq_{\mathbb{R}} \tfrac{1}{k+1}) \to \exists x_0 \leq_1 M(0 \leq_{\mathbb{R}} x_0 \leq_{\mathbb{R}} 1 \wedge \forall x^1(0 \leq_{\mathbb{R}} x \leq_{\mathbb{R}} 1 \to \Phi x_0 \geq_{\mathbb{R}} \Phi x))\big),$$

*which clearly has the form $H$ (and a fortiori can be written as an axiom $G$). Moreover by this simple representation it is sufficient to assume that $\Phi$ represents a pointwise continuous function $[0,1] \to \mathbb{R}$ whereas the representation needed in order to express the assertion as an axiom $\in \Delta$ requires that $\Phi$ is endowed with a modulus of uniform continuity (In the classical setting of [10] this is no restriction since using $F^-$ (which can be eliminated from the proof of the verification of the extracted bound) and $AC$–qf one can prove that every pointwise continuous function $f : [0,1]^{(d)} \to \mathbb{R}$ possesses a modulus of uniform continuity, see [8],[12]. The same is true in the intuitionistic context of theorem 4.2 below but not for theorem 4.1 since $F$ is not an allowed axiom $\in \mathcal{A}$).*

*2) $WKL^2_{seq}$ does not have the form of an axiom $\in \Delta$ and therefore had to be derived from $F$ and $AC$–qf in the classical context of [10]. In $E$–$G_nA_i^\omega$ it can be treated directly as an axiom.*

*3) DNS and LLPO follow of course from classical logic but are not derivable in $E$-$G_nA_i^\omega$.*

*4) $F_\rho$ and $AC$ prove a principle of uniform boundedness for the type $\rho$:*

$$UB_\rho: \ \forall y^{\rho 0}\big(\forall k^0 \forall x \leq_\rho yk \exists z^0 A(x,y,k,z) \to \exists \chi^1 \forall k^0 \forall x \leq_\rho yk \exists z \leq_0 \chi k\, A(x,y,k,z)\big).$$

*5) One easily shows that LLPO is implied by $CA^1_{\exists f}$.*

*6) $CA^0_\neg$ added to $E$–$G_nA_i^\omega$ yields the axiom schema of induction for arbitrary negated formulas*

$$IA_\neg: \ \neg A(0) \wedge \forall x^0(\neg A(x) \to \neg A(x+1)) \to \forall x^0 \neg A(x):$$

*Apply (QF–IA) to the characteristic function of $\neg A(x^0)$ which exists by $CA^0_\neg$.*

*Likewise $E$–$G_nA_i^\omega + CA^0_{\exists f}$ proves induction for arbitrary $\exists$–free formulas ($IA_{\exists f}$) . Whereas in the classical theories $E$–$G_nA^\omega$ the restricted schemas $IA_\neg$ and $IA_{\exists f}$ are equivalent to the unrestricted schema of induction, which (for $n \geq 2$) makes every $\alpha(< \varepsilon_0)$–recursive function provably recursive, $IA_\neg$ and $IA_{\exists f}$ do not cause any growth of provable functionals when added to the* **intuitionistic** *theories $E$–$G_nA_i^\omega$.*

One limitation for applications of the theorems 3.3 and 3.10 is due to the fact that the Markov principle

$$M^\omega: \ \forall \underline{x}(A \vee \neg A) \wedge \neg\neg \exists \underline{x}\, A \to \exists \underline{x}\, A$$

is not an allowed axiom, not even in its weak form

$$M_{pr}: \ \neg\neg \exists x^0 A_0(x) \to \exists x^0 A_0(x),$$

where $A_0$ is a quantifier–free formula.
In fact the addition of $M_{pr}$ would make the theory E–$\mathrm{G}_n\mathrm{A}_i^\omega$+AC+$F$+IP$_\neg$ inconsistent:

$$\text{E–}\mathrm{G}_n\mathrm{A}_i^\omega + M_{pr} + \mathrm{IP}_\neg \vdash \forall f \leq_1 \lambda x.1 \exists k^0 (\neg\neg \exists n(fn = 0) \to fk = 0).$$

Together with AC and $F$ this gives a contradiction (as in the proof of the claim above).

As we have discussed in [9] many $\forall\exists$–sentences in classical analysis come from sentences

$$(1)\ \forall x \in X(Fx =_{\mathbb{R}} 0 \to Gx =_{\mathbb{R}} 0)$$

by prenexation to

$$(2)\ \forall x \in X \forall k^0 \exists n^0 \big(|Fx| \leq_{\mathbb{R}} \frac{1}{n+1} \to |Gx| <_{\mathbb{R}} \frac{1}{k+1}\big),$$

what intuitionistically just needs $M_{pr}$ (Here $X$ is a complete separable metric space and $F, G : X \to \mathbb{R}$ are constructive functions).
We now prove a theorem which covers $M^\omega$ but still allows the extraction of bounds for arbitrary $\forall\exists$–sentences. The price we have to pay for this is that the allowed axioms have to be restricted to the class $\Delta$ from the theorems in [10] (and that we can use only the quantifier–free rule of extensionality instead of (E)).

**Definition 3.17 ([14])**

$$IP_0^\omega: \ \forall \underline{x}(A \vee \neg A) \wedge (\forall \underline{x}\, A \to \exists y\, B) \to \exists y(\forall \underline{x}\, A \to B),$$

*where $y$ is not free in $A$.*

**Theorem 3.18** *Let $s, t \in G_nR^\omega$ $(n \geq 1)$, $A_0, B_0$ be quantifier–free and $C$ be an arbitrary formula (respecting the convention made before theorem 3.3). Then*

$$\begin{cases} G_nA_i^\omega{+}AC{+}IP_0^\omega + M^\omega + \forall x^\delta \exists y \leq_\rho sx \forall z^\gamma A_0 \vdash \forall u^1 \forall v \leq_\tau tu(\forall a^\eta B_0 \to \exists w^2 C) \\ \Rightarrow \ \textit{by monotone functional interpretation one can extract } \Psi \in G_nR_-^\omega[\Phi_1] \textit{ such that} \\ G_nA_i^\omega{+}AC{+}IP_0^\omega + M^\omega + \forall x^\delta \exists y \leq_\rho sx \forall z^\gamma A_0 \vdash \forall u^1 \forall v \leq_\tau tu \exists w \leq_2 \Psi u(\forall a^\eta B_0 \to C(w)). \end{cases}$$

*(If the type of $w$ is $0$ and $n = 2$ (resp. $n = 3$) $\Psi u$ is a polynomial (resp. a finitely iterated exponential function) in $u^M$).*
*An analogous result holds for $PRA_i^\omega$,$\widehat{PR}^\omega$ and $PA_i^\omega$, $T$ instead of $G_nA_i^\omega$, $G_nR_-^\omega[\Phi_1]$.*

***Proof:*** As an abbreviation we define $\mathcal{T}:=\mathrm{G}_n\mathrm{A}^\omega_i+\mathrm{AC}+\mathrm{IP}^\omega_0 + M^\omega + \forall x^\delta \exists y \leq_\rho sx \forall z^\gamma A_0$. By the assumption and $\mathrm{IP}^\omega_0$ we obtain

$$\mathcal{T} \vdash \forall u, v \exists w (v \leq tu \wedge \forall a\, B_0 \to C(w)).$$

Monotone functional interpretation extracts (using the proof of theorem 3.2.2 in [10] and the fact that the monotone interpretation of $\mathrm{AC}+\mathrm{IP}^\omega_0 + M^\omega$ is as trivial as their usual functional interpretation) a term $\tilde{\Psi} \in \mathrm{G}_n\mathrm{R}^\omega_-$ such that

$$\tilde{\mathcal{T}} := \mathcal{T} + \exists Y \leq s \forall x, z\, A_0(x, Yx, z) \vdash \\ \exists \chi \big(\tilde{\Psi} \text{ s–maj } \chi \wedge \forall u \forall v (v \leq tu \wedge \forall a\, B_0 \to C(\chi uv))^D\big).$$

By [14] (3.5.10) we have $\mathcal{T} \vdash A^D \leftrightarrow A$ for all formulas $A$. Hence

$$\tilde{\mathcal{T}} \vdash \exists \chi \forall u \forall v \leq tu \big(\underbrace{\lambda y^1.\tilde{\Psi} u^M (t^* u^M) y^M}_{\Psi u :=} \geq_2 \chi uv \wedge (\forall a\, B_0 \to C(\chi uv))\big),$$

and thus

$$\tilde{\mathcal{T}} \vdash \forall u \forall v \leq tu \exists w \leq_2 \Psi u \big(\forall a\, B_0 \to C(w)\big).$$

Since AC implies

$$\forall x^\delta \exists y \leq_\rho sx \forall z^\gamma A_0 \to \exists Y \leq_{\rho\delta} s \forall x^\delta, z^\gamma A_0(x, Yx, z),$$

the proof is finished.

# 4 Growth of functional dependencies for logically complex formulas in (non-constructive) analytical proofs relatively to the intuitionistic theories $\mathbf{E}$–$\mathbf{G}_n\mathbf{A}^\omega_i$

Let us summarize now the main consequences of the results obtained in this paper on the growth of uniform bounds which are extractable from **partially** constructive proofs in analysis:

Let $\mathcal{A}$ be the set of the following theorems and principles:[9]

1) Attainment of the maximum of $f \in C([a,b]^d, \mathbb{R})$

2) Mean value theorem for integrals

3) Cauchy–Peano existence theorem

4) Brouwer's fixed point theorem for continuous functions $f : [a,b]^d \to [a,b]^d$

5) The generalization $\mathrm{WKL}^2_{seq}$ of the binary König's lemma WKL

6) The 'double negation shift' DNS : $\forall x^\rho \neg\neg A \to \neg\neg \forall x^\rho\, A$ for all $\rho$

[9]Here and in the following $a, b \in \mathbb{R}$ such that $a < b$.

7) The 'lesser limited principle of omniscience'

$$\text{LLPO} \ : \ \forall x^1, y^1 \exists k \leq_0 1([k=0 \to x \leq_{\mathbb{R}} y] \wedge [k=1 \to y \leq_{\mathbb{R}} x])$$

8) Comprehension for negated formulas:

$$CA^{\rho}_{\neg} : \ \exists \Phi \leq_{0\rho} \lambda x^{\rho}.1^0 \forall y^{\rho} \big(\Phi y =_0 0 \leftrightarrow \neg A(y)\big), \ \text{where } A \text{ is arbitrary } (\Phi \text{ not free in } A).$$

**Theorem 4.1** *Let $\gamma \leq 2$, $n \geq 2$, $t \in G_nR^{\omega}$ and $C, D$ arbitrary formulas of $E\text{–}G_nA^{\omega}$ such that $\forall \underline{u}^1, \underline{k}^0 \forall v \leq_{\tau} t\underline{u}\,\underline{k}\big(\neg C \to \exists w^{\gamma} D(\underline{u}, \underline{k}, v, w)\big)$ is closed. Then the following rule holds*

$$\begin{cases} \textit{From a proof} \\ \text{E–G}_n\text{A}^{\omega}_i\text{+AC} \ + IP_{\neg} + \mathcal{A} \vdash \forall \underline{u}^1, \underline{k}^0 \forall v \leq_{\tau} t\underline{u}\,\underline{k}\big(\neg C \to \exists w^{\gamma} D(\underline{u}, \underline{k}, v, w)\big) \\ \textit{one can extract a bound } \Phi \in G_nR^{\omega}_{-}[\Phi_1] \textit{ such that} \\ \text{E–G}_n\text{A}^{\omega}\text{+AC} \ + \mathcal{A} \vdash \forall \underline{u}^1, \underline{k}^0 \forall v \leq_{\tau} t\underline{u}\,\underline{k} \exists w \leq_{\gamma} \Phi\underline{u}\,\underline{k}\big(\neg C \to D(\underline{u}, \underline{k}, v, w)\big)^{10} \\ \textit{and therefore} \\ \mathcal{S}^{\omega} \models \forall \underline{u}^1, \underline{k}^0 \forall v \leq_{\tau} t\underline{u}\,\underline{k} \exists w \leq_{\gamma} \Phi\underline{u}\,\underline{k}\big(\neg C \to D(\underline{u}, \underline{k}, v, w)\big). \end{cases}$$

*(For $\gamma = 0$ and $n = 2$ (resp. $n = 3$) $\Psi\underline{u}\,\underline{k}$ is a polynomial (resp. an finitely iterated exponential function) in $\underline{u}^M$ and $\underline{k}$).*
*An analogous result holds $E\text{–}PRA^{\omega}_i, \widehat{PR}^{\omega}$, $E\text{–}PRA^{\omega}$ and $E\text{–}PA^{\omega}_i, T$, $E\text{–}PA^{\omega}$ instead of $E\text{–}G_nA^{\omega}_i$, $G_nR^{\omega}_{-}[\Phi_1]$, $E\text{–}G_nA^{\omega}$.*

***Proof:*** The theorem follows from theorem 3.3 and the fact that the sentences in $\mathcal{A}$ can be expressed in the logical form $\forall x(A \to \exists y \leq sx \neg B)$ (using remarks 3.4,3.16 and the implication AC → b–AC).

Let $\mathcal{B}$ consist of the following theorems and principles:

1) Attainment of the maximum of $f \in C([a,b]^d, \mathbb{R})$

2) Mean value theorem for integrals

3) Cauchy–Peano existence theorem

4) Brouwer's fixed point theorem for continuous functions $f : [a,b]^d \to [a,b]^d$

5) The generalization $\text{WKL}^2_{seq}$ of the binary König's lemma WKL

6) The 'lesser limited principle of omniscience'

$$\text{LLPO} \ : \ \forall x^1, y^1 \exists k \leq_0 1([k=0 \to x \leq_{\mathbb{R}} y] \wedge [k=1 \to y \leq_{\mathbb{R}} x])$$

7) Comprehension for ∃–free formulas:

$$CA^{\rho}_{\exists f} : \ \exists \Phi \leq_{0\rho} \lambda x^{\rho}.1^0 \forall y^{\rho} \big(\Phi y =_0 0 \leftrightarrow A(y)\big), \ \text{where } A \text{ is } \exists\text{–free } (\Phi \text{ not free in } A)$$

[10] In fact the use of classical logic in the proof of this conclusion is very limited (as in theorem 3.3) and AC can be replaced by b-AC if it is not used in the assumption. For E-$G_nA^{\omega}$+AC the addition of $\mathcal{A}$ in the conclusion actually is redundant.

8) The generalization of the axiom $F$ to arbitrary types $\rho$:

$$F_\rho :\equiv \forall \Phi^{0\rho 0}, y^{\rho 0} \exists y_0 \leq_{\rho 0} y \forall k^0 \forall z \leq_\rho yk\big(\Phi kz \leq_0 \Phi k(y_0 k)\big)$$

9) Every pointwise continuous function $F : [a,b]^d \to \mathbb{R}$ is uniformly continuous (together with a modulus of uniform continuity)[11]

10) Every sequence of functions $F_n : [a,b]^d \to \mathbb{R}$ which converges pointwise to a function $F : [a,b]^d \to \mathbb{R}$ converges uniformly on $[a,b]^d$ (together with a modulus of convergence)

11) Every sequence of balls (not necessarily open ones) which cover $[a,b]^d$ contains a finite subcovering.

**Theorem 4.2** *Let $n \geq 2$, $\gamma, \tau \leq 2$, $C$ be $\exists$–free and $D \in \Gamma_1$ such that $\forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k}\big(C \to \exists w^\gamma D\big)$ is closed, where $t \in G_nR^\omega$. Suppose that all positively occuring $\forall x^\rho$ (resp. negatively occuring $\exists x^\rho$) in $C \to \exists w\, D$ have types $\leq 1$ and all other quantifiers have types $\leq 2$. Then the following rule holds:*

$$\begin{cases} \textit{From a proof} \\ \text{E–G}_n\text{A}^\omega_i\text{+AC } + IP_\neg + \mathcal{B} \vdash \forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k}\big(C \to \exists w^\gamma D(\underline{u}, \underline{k}, v, w)\big) \\ \textit{one can extract a bound } \Phi \in G_nR^\omega_-[\Phi_1] \textit{ such that} \\ \text{E–G}_n\text{A}^\omega\text{+b–AC } + \mathcal{B}^- \vdash \forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k} \exists w \leq_\gamma \Phi\underline{u}\,\underline{k}\big(C \to D(\underline{u}, \underline{k}, v, w)\big) \\ \textit{and} \\ \mathcal{S}^\omega \models \forall \underline{u}^1, \underline{k}^0 \forall v \leq_\tau t\underline{u}\,\underline{k} \exists w \leq_\gamma \Phi\underline{u}\,\underline{k}\big(C \to D(\underline{u}, \underline{k}, v, w)\big), \end{cases}$$

*where $\mathcal{B}^- := \mathcal{B} \setminus \{9), 10), 11)\}$.*
*(For $\gamma = 0$ and $n = 2$ (resp. $n = 3$) $\Psi\underline{u}\,\underline{k}$ is a polynomial (resp. an finitely iterated exponential function) in $\underline{u}^M$ and $\underline{k}$).*
*An analogous result holds E–PRA$^\omega_i$, $\widehat{PR}^\omega$, E–PRA$^\omega$ and E–PA$^\omega_i$, $T$, E–PA$^\omega$ instead of E–G$_n$A$^\omega_i$, $G_nR^\omega_-[\Phi_1]$, E–G$_n$A$^\omega$.*

***Proof:*** The first part of the theorem follows from theorem 3.10 (and remark 3.4), the fact that the principles 1)–8) from $\mathcal{B}$ have the logical form $\forall x\big(G \to \exists y \leq sxH\big)$ (where $G \in \Gamma_1$ and $H$ is $\exists$–free, see remark 3.16) and the fact that principles 9)–11) follow from AC and $F$ relatively to E–G$_2$A$^\omega_i$ (see above).
We now show $\mathcal{M}^\omega \models \mathcal{B}^-$ (and therefore $\mathcal{M}^\omega \models$ E–G$_n$A$^\omega$+b–AC+$\mathcal{B}^-$):
For 1)–4) this follows immediately from the representation of analytical objects given in [8] by which these principles can be expressed as sentences having the form (+) $\forall x^1 \exists y \leq_1 sx \forall z^{0/1} A_0(x,y,z)$ (where $A_0$ is quantifier–free). As in [10] (proof of 4.9, remark 4.10), the truth of (+) in $\mathcal{S}^\omega$ implies its truth in $\mathcal{M}^\omega$ (using $\mathcal{M}_1 = \mathcal{S}_1$).
For the more 'liberal' representation as indicated in remark 3.16.1) above this also is clear since $\mathcal{M}_2 \subset \mathcal{S}_2$ and the only quantifier of type $> 1$ '$\forall \Phi^{1(1)}$' occurs positively. The same is true for the corresponding formalization of 2),4). In 3) (in its naive formalization) one gets a positive $\exists$-quantifier

[11] This principle easily follows in E–G$_2$A$^\omega_i$ + $F$+AC using prop. 3.13.

of type 1(1) which however is bounded by a term $t \in \mathrm{G_2R}^\omega$ and therefore does not become stronger when restricted from $\mathcal{S}_{1(1)}$ to $\mathcal{M}_{1(1)}$ (since $\Phi \in \mathcal{S}_{1(1)} \wedge t^*$ s–maj $t \wedge t \geq_{1(1)} \Phi \rightarrow t^*$ s–maj $\Phi \in \mathcal{M}_{1(1)}$).
$\mathcal{M}^\omega \models 5)$ again follows from $\mathcal{S}^\omega \models \mathrm{WKL}^2_{seq}$ using $\mathcal{M}_2 \subset \mathcal{S}_2, \mathcal{M}_1 = \mathcal{S}_1$.
$\mathcal{M}^\omega \models 6)$ is trivial since we refer to classical truth in $\mathcal{M}^\omega$.
$\mathcal{M}^\omega \models 7)$ follows from $\Phi \in \mathcal{M}_0^{\mathcal{M}_\rho} \wedge \Phi \leq_{0\rho} \lambda x^\rho.1 \rightarrow \lambda x^\rho.1$ s–maj$_{0\rho}$ $\Phi \in \mathcal{M}_{0\rho}$.
$\mathcal{M}^\omega \models 8)$: $\Phi \in \mathcal{M}_{0\rho 0}, y \in \mathcal{M}_{\rho 0}$ implies the existence of $\Phi^* \in \mathcal{M}_{0\rho 0}, y^* \in \mathcal{M}_{\rho 0}$ such that $\Phi^*$ s–maj $\Phi$ and $y^*$ s–maj $y$ and therefore $\forall k^0(\Phi^* k$ s–maj $\Phi k \wedge y^* k$ s–maj $yk)$. Hence

$$\forall z \in \mathcal{M}_\rho(z \leq_\rho yk \rightarrow y^*k \text{ s–maj } z)$$

and therefore

$$\forall z \in \mathcal{M}_\rho(z \leq_\rho yk \rightarrow \Phi^*k(y^*k) \geq_0 \Phi kz).$$

Thus $\Phi k$ is bounded on $\{z \in \mathcal{M}_\rho : z \leq_\rho yk\}$. Hence the exists a $z_k \in \mathcal{M}_\rho, z_k \leq_\rho yk$ such that $\Phi k z_k \geq_0 \Phi kz$ for all $z \in \mathcal{M}_\rho$ with $z \leq_\rho yk$. Define now (using choice on the meta-level) $y_0 := \lambda k^0.z_k \in \mathcal{M}_\rho^{\mathcal{M}_0}$. Since $y^*$ s–maj$_{\rho 0}$ $y \wedge y \geq_{\rho 0} y_0$ it follows that $y^*$ s–maj$_{\rho 0}$ $y_0 \in \mathcal{M}_{\rho 0}$.
This concludes the proof of $\mathcal{M}^\omega \models \mathrm{E\text{–}G}_n\mathrm{A}^\omega$+b–AC+$\mathcal{B}^-$. Hence the conclusion of our theorem holds in $\mathcal{M}^\omega$ and so (because of $\tau, \gamma \leq 2$, the type-restrictions on $C, D$ and the implication $v \in \mathcal{S}_\tau \wedge v \leq_\tau t\underline{u}\,\underline{k} \rightarrow t^*\underline{u}^M\underline{k}$ smaj $v \in \mathcal{M}_\tau$ for $\tau \leq 2$) in $\mathcal{S}^\omega$.

**Remark 4.3** *As a special corollary of theorem 4.2 one obtains the consistency of* $E\text{–}G_nA_i^\omega$+$AC$+$IP_{\exists f} + \mathcal{B}$ *which is not obvious since (due to 9)–11)$\in \mathcal{B}$) the corresponding classical theory is inconsistent.*

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