

Pointwise hereditary majorization and some applications

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Abstract

A pointwise version of the Howard–Bezem notion of hereditary majorization is introduced which has various advantages, and its relation to the usual notion of majorization is discussed. This pointwise majorization of primitive recursive functionals (in the sense of Gödel’s T as well as Kleene/Feferman’s \widehat{PR}) is applied to systems of intuitionistic and classical arithmetic (H and H^c) in all finite types with full induction as well as to the corresponding systems with restricted induction $\hat{H}\downarrow$ and $\hat{H}\downarrow^c$.

- 1) H and $\hat{H}\downarrow$ are closed under a generalized fan–rule. For a restricted class of formulae this also holds for H^c and $\hat{H}\downarrow^c$.
- 2) We give a new and very perspicuous proof that for each $\Phi^2 \in T(\widehat{PR})$ one can construct a functional $\tilde{\Phi}^2 \in T(\widehat{PR})$ such that $\tilde{\Phi}\alpha$ is a modulus of uniform continuity for Φ on $\{\beta^1 | \forall n(\beta n \leq \alpha n)\}$. Such a modulus can also be obtained by majorizing any modulus of pointwise continuity for Φ .
- 3) The type structure \mathcal{M} of all pointwise majorizable set–theoretical functionals of finite type is used to give a short proof that quantifier–free “choice” with uniqueness $(AC!)^{1,0}$ –qf. is not provable within classical arithmetic in all finite types plus comprehension (given by the schema $(C)^\rho : \exists y^{0\rho} \forall x^\rho (yx = 0 \leftrightarrow A(x))$ for arbitrary A), dependent ω –choice and bounded choice. Furthermore \mathcal{M} separates several μ –operators.

1 Introduction

Howard (1973) introduced the notion “ $x^* \text{ maj } x$ ” where x^* and x are functionals of a finite type ρ and showed that each primitive recursive functional Φ^ρ (in the sense of Gödel’s T) can be majorized by a suitable primitive recursive functional

$\Phi^{*\rho}$. As an application he proved that no primitive recursive functional satisfies the functional interpretation of the axiom of extensionality for objects of type 2.

A variant of this notion with the property that each majorant majorizes itself was defined in Bezem (1985) who called this notion “strong majorization”. Bezem proved that the corresponding type structure \mathcal{M}^s of all strongly majorizable set-theoretical functionals of finite type is a model of Spector’s calculus $T + BR$ of bar recursive functionals, although \mathcal{M}^s contains discontinuous functionals.

In this paper we first introduce a pointwise version maj of Bezem’s strong majorization where $x^* maj_{\rho} x$ is defined by $\forall n(x^* n maj_{\rho} xn)$. The majorizable functionals are the same for both notions since, for each x^* which majorizes x in our sense, one can compute, primitive recursively in x^* , a strong majorant \hat{x} for x and vice versa (2.14). In particular the type structure \mathcal{M}^s and the corresponding type structure \mathcal{M} defined by pointwise majorization coincide. However the majorization relations differ for $\rho > 0$ and our modification has several advantages:

- 1) For types $\rho \leq 2$, maj_{ρ} can be described by the natural inequality relation \geq_{ρ} (2.4,2.5). This allows mathematical applications some of which are given below. More substantial ones will follow in subsequent papers.
- 2) The proof that \mathcal{M} is a model for $T + BR$ is technically simpler than Bezem’s proof for \mathcal{M}^s (see also the proof of 3.16 below).
- 3) The pointwise definition is extremely convenient for the purpose of generalizing \mathcal{M} to models $\mathcal{M}^{\infty,\sigma}$ of the calculi of bar recursive functionals of infinite type $T^{\infty,\sigma} + BR$; these were introduced in Friedrich (1985) in order to carry out the functional interpretation of higher classical analysis with analytical comprehension over objects of arbitrary type (via the game quantifier translation developed in Friedrich (1984)).

The models $\mathcal{M}^{\infty,\sigma}$ will be constructed in a subsequent paper where 2) and 3) will become apparent.

In this paper, we only utilize the first property of maj for applications.

Notation

The set \tilde{T} of finite types is defined inductively:

- 1) $0 \in \tilde{T}$,
- 2) $\rho, \tau \in \tilde{T} \Rightarrow \tau(\rho) \in \tilde{T}$.

The subset $\tilde{P} \subset \tilde{T}$ of pure types is given by

- 1) $0 \in \tilde{P}$,
- 2) $\rho \in \tilde{P} \Rightarrow 0(\rho) \in \tilde{P}$

Pure types are often denoted by natural numbers: $0(n) := n + 1$. Intuitively each object of type $\tau(\rho)$ is a mapping which assigns objects of type ρ to objects of type τ .

In the following, we omit brackets that are uniquely determined, e.g. we write $o(o)$ instead of $o(o(o))$.

$E-HA^\omega$ denotes the system of extensional intuitionistic arithmetic in all finite types as defined in Troelstra (1973), 1.6.12, where only equality between objects of type 0 is primitive and higher equality is inductively defined as

$$z =_{\rho\tau} u := \forall y^\tau (zy =_\rho uy).$$

$WE-HA^\omega$ is obtained from $E-HA^\omega$ by replacing the extensionality axiom by a quantifier-free rule of extensionality

$$ER\text{-qf. } \frac{A_0 \rightarrow s =_\rho t}{A_0 \rightarrow r[s] =_\tau r[t]},$$

where A_0 is quantifier-free.

The corresponding systems with classical logic are denoted by $E-PA^\omega$ and $WE-PA^\omega$ resp. The calculus T is defined to be the quantifier-free part of $WE-HA^\omega$ (Troelstra (1973), 1.6.13.). The theories above all contain 0^0 (zero), S^{00} (successor), constants $\Pi_{\rho,\tau}$, $\Sigma_{\delta,\rho,\tau}$ and recursor constants R_ρ with the defining axioms $\Pi_{\rho,\tau} x^\rho y^\tau =_\rho x^\rho$, $\Sigma_{\delta,\rho,\tau} xyz =_\tau xz(yz)$ (where $x \in \tau\rho\delta$, $y \in \rho\delta$, $z \in \delta$) and

$$\begin{cases} R_\rho 0yz =_\rho y \\ R_\rho (Sx^0)yz =_\rho z(R_\rho xyz)x, \text{ where } y \in \rho, z \in \rho 0\rho. \end{cases}$$

If the constants R_ρ are replaced by elementary recursor operators \hat{R}_ρ characterized by

$$\begin{cases} \hat{R}_\rho 0yz\underline{v} =_0 y\underline{v} \\ \hat{R}_\rho (Sx^0)yz\underline{v} =_0 z(\hat{R}_\rho xyz\underline{v})x\underline{v}, \end{cases}$$

where y, z as above and $\underline{v} = v_1^{\rho_1} \dots v_k^{\rho_k}$ such that $y\underline{v}$ is of type 0, and if the schema of full induction is replaced by the axiom of quantifier-free induction

(IA)-qf. $\vdash \forall f^1(f0 = 0 \wedge \forall x(fx = 0 \rightarrow f(Sx) = 0) \rightarrow \forall x(fx = 0))$,

then one obtains the restricted systems $E - \widehat{HA}^\omega \downarrow, WE - \widehat{HA}^\omega \downarrow$, with quantifier-free part \widehat{PR} due to Feferman (1977) (The functionals of \widehat{PR} are essentially the primitive recursive functionals in the sense of Kleene (1959)).

Each of the systems defined so far permit us to construct a term $\lambda x.t[x]$, for each term t^ρ , such that $(\lambda x.t[x])(t') =_\rho t[t']$ (see Troelstra (1973), 1.6.8).

Furthermore we need the following schemata ($A \in \mathcal{L}(E - HA^\omega)$)

$(AC)^{\rho,\tau} : \forall x^\rho \exists y^\tau A(x, y) \rightarrow \exists Y^{\tau\rho} \forall x^\rho A(x, Yx)$ (choice),

$(AC)^{\rho,\tau}$ -qf. $\vdash \forall x^\rho \exists y^\tau A_0(x, y) \rightarrow \exists Y^{\tau\rho} \forall x^\rho A_0(x, Yx)$,

where $A_0 \in \mathcal{L}(E - HA^\omega)$ is quantifier-free (quantifier-free choice),

$(MP^\omega) : \forall \underline{x}(A(\underline{x}) \vee \neg A(\underline{x})) \wedge \neg \neg \exists \underline{x} A(\underline{x}) \rightarrow \exists \underline{x} A(\underline{x})$,

(Markov-principle)

$(IP_0^\omega) : \forall \underline{x}(A(\underline{x}) \vee \neg A(\underline{x})) \wedge (\forall \underline{x} A(\underline{x}) \rightarrow \exists \underline{y} B(\underline{y})) \rightarrow \exists \underline{y} (\forall \underline{x} A(\underline{x}) \rightarrow B(\underline{y}))$

(independence-of-premiss schema). Here \underline{x} denotes a finite tuple of variables.

“ $\forall x \leq_\rho y A$ ” and “ $\exists x \leq_\rho y A$ ” are used as abbreviations for “ $\forall x(x \leq_\rho y \rightarrow A)$ ” and “ $\exists x(x \leq_\rho y \wedge A)$ ” respectively. “ $\forall x; \tilde{x} \leq_\rho sx A$ ” stands for “ $\forall x \forall \tilde{x} \leq_\rho sx A$ ”, but “ $\forall x, \tilde{x} \leq_\rho y A$ ” stands for “ $\forall x \leq_\rho y \forall \tilde{x} \leq_\rho y A$ ”.

2 Basic definitions and results about *maj*

The results 2.1–2.14 can be proved within $WE - \widehat{HA}^\omega \downarrow$.

2.1 Definition (see Bezem (1985))

The relation $x^* s - maj_\rho x$ (x^* strongly majorizes x) between functionals of type ρ is defined inductively as follows:

$$\begin{cases} x^* s - maj_0 x := x^* \geq x, \\ x^* s - maj_{\rho\tau} x := \forall y^*, y (y^* s - maj_\tau y \rightarrow x^* y^* s - maj_\rho x^* y, xy) . \end{cases}$$

(Here \geq denotes the usual primitive recursively defined inequality relation for objects of type 0).

Our pointwise variant of $s - maj$ is defined as follows

2.2 Definition

For x^*, x of type ρ , $x^* maj_\rho x$ is given by

$$\begin{cases} x^* maj_0 x & :\equiv x^* \geq x, \\ x^* maj_{\rho 0} x & :\equiv \forall n^0 (x^* n maj_\rho x n), \\ x^* maj_{\rho \tau} x & :\equiv \forall y^*, y (y^* maj_\tau y \rightarrow x^* y^* maj_\rho x^* y, xy) \\ & (\tau \neq 0) \end{cases}$$

“ $x^* maj_\rho x$ ” is read as “ x^* pointwise strongly majorizes x ”.

2.3 Remark

Howard’s definition of majorization results if the clause $x^* y$ in 2.1 is deleted. As a consequence of this clause in the definition of $s - maj_\rho$ and maj_ρ , one has

- 1) $x^* s - maj_\rho x \rightarrow x^* s - maj_\rho x^*$ (Bezem (1985)) and
- 2) $x^* maj_\rho x \rightarrow x^* maj_\rho x^*$ (induction on the type ρ).

2.4 Definition

For functionals of type ρ we define a natural inequality relation \geq_ρ by

$$\begin{cases} x_1 \geq_0 x_2 & :\equiv x_1 \geq x_2, \\ x_1 \geq_{\rho \tau} x_2 & :\equiv \forall y^\tau (x_1 y \geq_\rho x_2 y); \quad x_1 \leq_\rho x_2 :\equiv x_2 \geq_\rho x_1. \end{cases}$$

The following lemma establishes the strong connection between maj_ρ and \geq_ρ :

2.5 Lemma

- 1) $\forall x^*, x (x^* \text{maj}_1 x \leftrightarrow x^* \geq_1 x)$, in particular $\forall x^1(x \text{maj}_1 x)$;
- 2) $\forall x^*, x (x^* \text{maj}_2 x \leftrightarrow x^* \geq_2 x \wedge \forall y_1, y_2 (y_1 \leq_1 y_2 \rightarrow x^* y_1 \leq_0 x^* y_2))$;
- 3) $\forall x^*, \tilde{x}, x (x^* \text{maj}_\rho \tilde{x} \wedge \tilde{x} \geq_\rho x \rightarrow x^* \text{maj}_\rho x)$;
- 4) $\forall z^{\rho}, \tilde{x}^{\rho\tau}, x^{\rho\tau} (\forall y^\tau (z^* \text{maj}_\rho \tilde{x} y \wedge \tilde{x} y \geq_\rho xy) \rightarrow \lambda y^\tau . z^* \text{maj}_{\rho\tau} x)$
(In 3,4 $\rho, \tau \in \tilde{T}$ are arbitrary).

Proof: 1) follows immediately from the definition of pointwise majorization

2) “ \rightarrow ”: Assume $x^* \text{maj}_2 x$. By 1) we have $\forall y^1 (y \text{maj}_1 y)$ and therefore $\forall y^1 (x^* y \geq_0 xy)$, i.e. $x^* \geq_2 x$. Let $y_1 \leq_1 y_2$; then, again by 1), $y_2 \text{maj}_1 y_1$ which implies $x^* y_2 \geq_0 x^* y_1$.

“ \leftarrow ”: Assume $y^* \text{maj}_1 y$; then by 1) $y^* \geq_1 y$, and therefore, by the assumption, $x^* y^* \geq_0 x^* y$. Again by the assumption one has $x^* y \geq_0 xy$ and hence $x^* y^* \geq_0 x^* y, xy$ which implies $x^* \text{maj}_2 x$.

3) This is proved by induction on the type.

4) Consider first the case $\tau \neq 0$. Then from $y^* \text{maj}_\tau y$ it follows by 3) and 2.3.2 that

$$(\lambda y^\tau . z^*)(y^*) = z^* \text{maj}_\rho z^* (= (\lambda y^\tau . z^*)(y)), xy \text{ ,i.e. } \lambda y^\tau . z^* \text{maj}_{\rho\tau} x.$$

The case $\tau = 0$ is treated analogously.

2.6 Remark

2.5.1, 2.5.2 are **not valid** for Bezem’s or Howard’s notion of majorization. 3) and 4) hold for their notions as well.

2.7 Notation

$$x^* \underline{\text{maj}}_\rho x := \begin{cases} x^* =_0 x & \text{if } \rho = 0, \\ x^* \text{maj}_\rho x & \text{otherwise.} \end{cases}$$

2.8 Lemma

Let $\rho = \tau \rho_k \dots \rho_1$. Then

$x^* \text{maj}_\rho x \longleftrightarrow$

$$\forall y_1^*, y_1, \dots, y_k^*, y_k \left(\bigwedge_{i=1}^k y_i^* \underline{\text{maj}}_{\rho_i} y_i \rightarrow x^* y_1^* \dots y_k^* \text{maj}_\tau x^* y_1 \dots y_k, x y_1 \dots y_k \right).$$

(This holds also for $s - \text{maj}$ if $\underline{\text{maj}}_{\rho_i}$ is replaced by $s - \text{maj}_{\rho_i}$).

Proof: By induction on k using 2.3.

2.9 Definition

Let $\rho = 0\rho_k \dots \rho_1$. Then Max_ρ is defined by

$$\text{Max}_\rho(x^{\rho_0}) := \lambda n^0, y^{\rho_1}, \dots, y^{\rho_k}. \text{max}(x 0 y_1 \dots y_k, \dots, x n y_1 \dots y_k),$$

where $\text{max}(u_1^0, \dots, u_n^0)$ is primitive recursively defined in the usual way.

2.10 Remark

One easily verifies that Max_ρ can be defined in \widehat{PR} .

2.11 Notation

We abbreviate $\text{Max}_\rho(x^{\rho_0})$ by x^M .

2.12 Lemma

For each $\rho \in \tilde{T}$ one has

$$x^* \text{maj}_{\rho_0} x \longrightarrow \forall m^0, n^0 (m \geq n \rightarrow x^{*M} m \text{maj}_\rho x^M n, x n);$$

in particular $\text{Max}_\rho \text{maj}_{\rho_0(\rho_0)} \text{Max}_\rho$.

Proof: Let $\rho = 0\rho_k \dots \rho_1$, $x^* \text{maj}_{\rho_0} x$ and $y_i^* \underline{\text{maj}}_{\rho_i} y_i$ ($i = 1, \dots, k$). Using only quantifier-free induction (IA)-qf. on m , one proves that

$$\forall m \left(\forall n \leq m (x^{*M} m y_1^* \dots y_k^* \geq_0 x^M n y_1 \dots y_k, x n y_1 \dots y_k) \right).$$

By 2.8 and 2.3 this implies the lemma.

2.13 Definition

By MAX we denote the set of all terms of \widehat{PR} , which are constructed from only O^0 , S^{00} , Π , Σ and Max_ρ .

2.14 Proposition

For each $\rho \in \tilde{T}$, there are effectively closed terms s_ρ and $t_\rho \in MAX$ of type $\rho\rho$ such that

$$\forall x^*, x \left((x^* s - \text{maj}_\rho x \rightarrow s_\rho x^* \text{maj}_\rho s_\rho x, x) \wedge (x^* \text{maj}_\rho x \rightarrow t_\rho x^* s - \text{maj}_\rho t_\rho x, x) \right).$$

Proof:

Induction on the type $\rho : \underline{\rho = 0} : s_0 := t_0 := \lambda x^0 . x$.

$\rho = \delta 0 : s_{\delta 0} x^{\delta 0} := \lambda n^0 . s_\delta(xn), t_{\delta 0} x^{\delta 0} := \lambda n^0 . t_\delta(x^M n) :$

(i) Assume $x^* s - \text{maj}_{\delta 0} x$. We show that $s x^* \text{maj}_{\delta 0} s x, x$, i.e.

(*) $\forall n \left((s x^*) n \text{maj}_\delta (s x) n, x n \right)$. The assumption implies

$\forall n (x^* n s - \text{maj}_\delta x n)$. The induction hypothesis applied to s_δ yields (*).

(ii) Assume $x^* \text{maj}_{\delta 0} x$. We have to show: $t(x^*) s - \text{maj}_{\delta 0} t(x), x$.

By 2.12, the assumption, and 2.3.2, one has

$$\forall m^0, n^0 (m \geq n \rightarrow x^{*M} m \text{maj}_\delta x^{*M} n, x^M n, x n)$$

and hence, by the induction hypothesis applied to t_δ

$$\forall m, n \left(m \geq n \rightarrow t_\delta(x^{*M} m) s - \text{maj}_\delta t_\delta(x^{*M} n), t_\delta(x^M n), x n \right).$$

$\rho = \delta \tau \ (\tau \neq 0)$

$$s_{\delta \tau} x^{\delta \tau} := \lambda y^\tau . s_\delta(x(t_\tau y)),$$

$$t_{\delta \tau} x^{\delta \tau} := \lambda y^\tau . t_\delta(x(s_\tau y)).$$

(i) Assume $x^* s - \text{maj}_{\delta \tau} x$. We show that $s x^* \text{maj}_{\delta \tau} s x, x$, i.e.

$$\forall y^*, y (y^* \text{maj}_{\tau} y \rightarrow s x^* y^* \text{maj}_\delta s x^* y, s x y, x y) :$$

$$y^* \text{maj}_{\tau} y \xrightarrow{\text{ind.hypoth.}} t_\tau y^* s - \text{maj}_\tau t y, y \xrightarrow{\text{assumpt. 2.3.1}}$$

$$x^*(t y^*) s - \text{maj}_\delta x^*(t y), x(t y), x y \xrightarrow{\text{ind.hypoth.}}$$

$$s_\delta(x^*(t y^*)) \text{maj}_\delta s_\delta(x^*(t y)), s_\delta(x(t y)), x y.$$

(ii) $x^* \text{maj}_{\delta \tau} x \rightarrow t x^* s - \text{maj}_{\delta \tau} t x, x$: The proof is similar to the proof of (i) above.

2.15 Proposition

- 1) Let $t^\tau[x_1^{\rho_1}, \dots, x_k^{\rho_k}] \in T$ be a term whose free variables x_i are all of type $\rho_i \leq 1$. Then one can effectively compute a term $t^{*\tau}[x_1, \dots, x_k] \in T$ having at most the free variables of t such that

$$WE - HA^\omega \vdash \forall x^{\rho_1}, \dots, x^{\rho_k} \left(t^*[x_1, \dots, x_k] \text{maj}_\tau t^*[x_1, \dots, x_k], t[x_1, \dots, x_k] \right).$$

In particular, for each closed term $t^\tau \in T$, there exists a corresponding closed term $t^{*\tau} \in T$ such that $WE - HA^\omega \vdash t^* \text{maj}_\tau t^*, t$.

2) For $t^\tau[x_1, \dots, x_k] \in \widehat{PR}(MAX)$, 1) holds with $t^{*\tau} \in \widehat{PR}(MAX)$ provable in $WE - \widehat{HA}^\omega$.

Proof:

1) We show that for each constant $c^\tau \in T$ there exists a closed term $c^{*\tau} \in T$ such that $WE - HA^\omega \vdash c^* \text{maj}_\tau c$ (Since $x_i \text{maj}_{\rho_i} x_i$ for $\rho_i \leq 1$, and $t_1^* \text{maj}_{\rho\tau} t_1 \wedge t_2^* \text{maj}_\tau t_2 \rightarrow t_1^* t_2^* \text{maj}_\rho t_1 t_2$, this implies the proposition).

It is trivial that $0^0 \text{maj}_0 0^0$ and $S \text{maj}_{00} S$. By 2.8 $\Pi_{\rho,\tau} \text{maj}_{\rho\tau\rho} \Pi_{\rho,\tau}$ and for $\rho \neq 0$ $\Sigma_{\delta,\rho,\tau} \text{maj} \Sigma_{\delta,\rho,\tau}$. Using 2.8 and 2.12 one shows that

$$\Sigma_{\delta,0,\tau}^* := \lambda x, y, z. (xz)^M (yz) \text{maj} \Sigma_{\delta,0,\tau}.$$

Ad R_ρ : (i) $\rho \neq 0$: By induction one easily shows that

$$\forall x^0 (R_\rho x \text{maj} R_\rho x) \text{ and therefore } R_\rho \text{maj} R_\rho.$$

(ii) $\rho = 0$: $R_0^* := \lambda x, y, z. R_0 xy(z^M)$. By induction and 2.12 it follows that

$$\forall x (R_0^* x \text{maj} R_0 x).$$

2) Define $\hat{R}_\rho^* := \lambda x, y, z, v. \hat{R}_\rho xy(z^M)v$.

Assume $\rho = 0\rho_k \dots \rho_1$, $y^* \text{maj} y$, $z^* \text{maj} z$ and $v_i^* \text{maj}_{\rho_i} v_i$ ($1 \leq i \leq k$): By (IA)-qf. using 2.12 one shows

$$\forall x (\hat{R}_\rho^* xy^* z^* v^* \geq_0 \hat{R}_\rho^* xyzv, \hat{R}_\rho^* xyzv). \text{ By 2.8 this yields } \hat{R}_\rho^* \text{maj} \hat{R}_\rho. \text{ It is clear that } \hat{R}_\rho^* \in \widehat{PR}, \text{ since } \lambda z^{\rho^0}. z^M \in \widehat{PR} \text{ (2.10).}$$

For MAX the proposition follows from 2.12.

2.16 Definition

The extensional type structure \mathcal{M} of all hereditarily pointwise strongly majorizable set-theoretical functionals of finite type is defined as

$$\left\{ \begin{array}{l} M_0 := \omega, \quad n \text{maj}_0 m := n \geq m \wedge n, m \in \omega, \\ x^* \text{maj}_{\rho 0} x := x^*, x \in M_\rho^{M_0} \wedge \forall n (x^* n \text{maj}_\rho xn), \\ x^* \text{maj}_{\rho\tau} x := x^*, x \in M_\rho^{M_\tau} \wedge \forall y^*, y \in M_\tau (y^* \text{maj}_\tau y \rightarrow x^* y^* \text{maj}_\rho x^* y, xy) \\ (\tau \neq 0) \\ M_{\rho\tau} := \{x \in M_\rho^{M_\tau} \mid \exists x^* \in M_\rho^{M_\tau} : x^* \text{maj}_{\rho\tau} x\} \quad (\rho, \tau \in \tilde{T}). \end{array} \right.$$

(Here $M_\rho^{M_\tau}$ denotes the set of all total set-theoretical mappings from M_τ into M_ρ).

$$\mathcal{M} := \bigcup_{\rho \in \tilde{T}} M_\rho.$$

2.17 Remark

$x^* \text{maj}_\rho x \rightarrow x \in M_\rho \wedge x^* \text{maj}_\rho x^* \rightarrow x^*, x \in M_\rho$ (induction on ρ).

Similar to the proof of 2.15 one shows

2.18 Proposition

$\mathcal{M} \models T$.

2.19 Remark

By the proof of 2.14 it is clear that $\mathcal{M} = \mathcal{M}^s$, where \mathcal{M}^s is Bezem's type structure, which is based on $s\text{-maj}$ instead of maj (see Bezem (1985)).

3 Applications

Let $A(\alpha, n) \in \mathcal{L}(E - HA^\omega)$ be a formula whose free variables are all of type $0, 1$. The usual fan-rule for systems such as $H := E - HA^\omega, WE - HA^\omega$ in the literature (see e.g. Troelstra (1974),(1977)) states that

$$(*) \left\{ \begin{array}{l} H \vdash \forall \alpha^1 \exists n^0 A(\alpha, n) \Rightarrow \\ H \vdash \forall \beta; \alpha \leq_1 \beta \exists m^0 \forall \gamma \leq_1 \beta (\bar{\gamma}(t_M \beta) =_0 \bar{\alpha}(t_M \beta) \rightarrow A(\gamma, m)), \end{array} \right.$$

where $t_M \in T$ is a suitable term, and $\bar{\gamma}k$ denotes a primitive recursive coding for $(\gamma 0, \dots, \gamma(k-1))$ (see Troelstra (1973))

Since there are only finitely many initial sequents $\bar{\alpha}(t_M \beta)$ a bound can be given for the number m (depending only on β) and using closure of H under choice-rule one can compute such a bound from a term $t \in T$:

$$(**) H \vdash \forall \beta; \alpha \leq_1 \beta \exists m \leq_0 t \beta \forall \gamma \leq_1 \beta (\bar{\gamma}(t_M \beta) =_0 \bar{\alpha}(t_M \beta) \rightarrow A(\gamma, m)).$$

Thus in particular, omitting the continuity part of $(**)$, one concludes

$$H \vdash \forall \beta; \alpha \leq_1 \beta \exists m \leq_0 t \beta A(\alpha, m).$$

This boundedness property can be generalized from type 1 to arbitrary finite types using 2.15:

3.1 Application (Generalized “FAN”–Rule)

- 1) Let H be $E - HA^\omega$, $WE - HA^\omega$ or $WE - HA^\omega + (MP^\omega) + (IP_0^\omega) + AC$ and $A(x^1, \tilde{x}^\rho, y^\tau) \in \mathcal{L}(H)$ a formula with no other free variables than x^1, \tilde{x}^ρ and y^τ . Assume that $\tau \leq 2$ and that $\rho \in \tilde{T}$ is arbitrary; let $s^{\rho^1} \in T$ be a closed term. Then the following rule holds

$$\left\{ \begin{array}{l} H \vdash \forall x^1; \tilde{x} \leq_\rho s x \exists y^\tau A(x, \tilde{x}, y) \Rightarrow \\ \exists \text{ a closed term } t^{\tau^1} \in T \text{ such that} \\ H \vdash \forall x^1; \tilde{x} \leq_\rho s x \exists y \leq_\tau t x A(x, \tilde{x}, y). \end{array} \right.$$

- 2) Analogous for the corresponding restricted systems $\hat{H} \upharpoonright$ and \widehat{PR} instead of H and T .

Proof:

- 1) Let H be $E - HA^\omega$ or $WE - HA^\omega$ and assume $H \vdash \forall x; \tilde{x} \leq_\rho s x \exists y^\tau A(x, \tilde{x}, y)$. Using mq-realizability (see Troelstra (1973) 3.4.2–3.4.5) one finds closed terms $t_1, \dots, t_n \in T$ such that

$$H \vdash \forall x, \tilde{x} \left(t_1 x \tilde{x}, \dots, t_n x \tilde{x} \text{ mq } (\tilde{x} \leq_\rho s x \rightarrow \exists y^\tau A(x, \tilde{x}, y)) \right)$$

which implies

$$H \vdash \forall x, \tilde{x} \left(\tilde{x} \leq_\rho s x \rightarrow t_1 x \tilde{x}, \dots, t_n x \tilde{x} \text{ mq } \exists y^\tau A(x, \tilde{x}, y) \right)$$

since $\tilde{x} \leq_\rho s x$ is purely universal. Therefore

$$(i) \quad H \vdash \forall x; \tilde{x} \leq_\rho s x A(x, \tilde{x}, t_1 x \tilde{x})$$

by the definition of mq-realizability. By 2.15 we can construct closed terms $t_1^*, s^* \in T$ such that

$H \vdash t_1^* \text{maj } t_1 \wedge s^* \text{maj } s$. Define $t := \lambda x.t_1^*x(s^*x)$ if $\rho > 0$ and $t := \lambda x.(t_1^*x)^M(s^*x)$ otherwise. Using 2.5.1,2.5.3,2.8 and 2.12 it follows that

$$H \vdash \forall x; \tilde{x} \leq_\rho sx (tx \text{maj}_\tau t_1x\tilde{x})$$

and therefore by 2.5.2

$$(ii) H \vdash \forall x; \tilde{x} \leq_\rho sx (tx \geq_\tau t_1x\tilde{x}).$$

(i) and (ii) imply the conclusion.

Now let H be $WE - HA^\omega + (MP^\omega) + (IP_0^\omega) + AC$ and assume

$$H \vdash \forall x; \tilde{x} \leq_\rho sx \exists y^\tau A(x, \tilde{x}, y)$$

By (IP_0^ω) it follows that

$$H \vdash \forall x; \tilde{x} \exists y^\tau (\tilde{x} \leq_\rho sx \rightarrow A(x, \tilde{x}, y)).$$

Using functional interpretation (Gdel (1958), Luckhardt (1973), Troelstra (1973) 3.5.10) one extracts a closed term $t_1^{\rho^1} \in T$ such that

$$H \vdash \forall x; \tilde{x} \leq_\rho sx A(x, \tilde{x}, t_1x\tilde{x}),$$

since $H \vdash F \leftrightarrow F^D$ for every formula $F \in \mathcal{L}(H)$, where F^D denotes the functional interpretation of F .

Reasoning as before one proves that

$$H \vdash \forall x; \tilde{x} \leq_\rho sx \exists y \leq_\tau tx A(x, \tilde{x}, y).$$

- 2) One has only to show the soundness of mq-realizability resp. functional interpretation for the restricted systems. This is verified by an easy modification of the corresponding proofs for H since relativ to $\hat{H} \upharpoonright$ the axiom (IA) -qf. is equivalent to

$$\forall f, x (f0 = 0 \wedge \forall y < x (fy = 0 \rightarrow f(Sy) = 0) \rightarrow fx = 0)$$

and hence to a purely universal sentence (since the bounded quantification can be expressed in a quantifier-free form in $\hat{H} \upharpoonright$). Therefore it is mq-realized and functional interpreted by itself (up to intuitionistic logical equivalence).

3.2 Corollary to the proof of 3.1

- 1) The proof of 3.1 generalizes immediatly to the situation where one has tupels $\underline{x} := x_1^{\delta_1}, \dots, x_k^{\delta_k}$ ($\delta_i \leq 1$ for $i = 1, \dots, k$), $\tilde{x} := \tilde{x}_1^{\rho_1}, \dots, \tilde{x}_m^{\rho_m}$, $s_1^{\rho_1 \delta_k \dots \delta_1}, \dots, s_m^{\rho_m \delta_k \dots \delta_1}$ and $y_1^{\tau_1}, \dots, y_n^{\tau_n}$ ($\tau_i \leq 2$ for $i = 1, \dots, n$) instead of $x^1, \tilde{x}^\rho, s^{\rho^1}, y^\tau$. Thus

$$\left\{ \begin{array}{l} H \vdash \forall \underline{x}; \tilde{x}_1 \leq s_1 \underline{x} \dots \forall \tilde{x}_m \leq s_m \underline{x} \exists y_1^{\tau_1}, \dots, y_n^{\tau_n} A(\underline{x}, \tilde{x}, y_1, \dots, y_n) \\ \Rightarrow \exists \text{ closed terms } t_1, \dots, t_n \in T \text{ such that} \\ H \vdash \forall \underline{x}; \tilde{x}_1 \leq s_1 \underline{x} \dots \forall \tilde{x}_m \leq s_m \underline{x} \exists y_1 \leq_{\tau_1} t_1 \underline{x} \dots \exists y_n \leq_{\tau_n} t_n \underline{x} A(\underline{x}, \tilde{x}, y_1, \dots, y_n). \end{array} \right.$$

The following results also generalize to finite tupels. For notational simplicity we formulate them only for tupels of length 1.

- 2) If $\tau \in \tilde{T}$ is arbitrary then 3.1 holds with “ $\exists y(tx \text{ maj}_\tau y \wedge A)$ ” instead of “ $\exists y \leq_\tau tx A$ ”. Furthermore, if one has “ $\exists y^2; \tilde{y}^\tau A$ ” ($\tau \in \tilde{T}$ arbitrary) instead of “ $\exists y^2 A$ ” then it is still possible to compute a bound for y : $\exists y \leq_2 tx; \tilde{y}^\tau A$.

In order to get a version of 3.1, which holds for **classical** arithmetic, we need the following application of functional interpretation:

3.3 Lemma

- 1) Let $s^{\rho\delta} \in T$ denote a closed term and let $A_0 \in \mathcal{L}(WE-HA^\omega)$ be a quantifier-free formula whose free variables are x^δ, \tilde{x}^ρ and y^τ where δ, ρ and τ are arbitrary types. Then the following rule holds

$$\left\{ \begin{array}{l} WE - PA^\omega + AC\text{-qf.} \vdash \forall x^\delta; \tilde{x} \leq_\rho s x \exists y^\tau A_0(x, \tilde{x}, y) \Rightarrow \\ \exists \text{ a closed term } \Phi^{\tau\rho\delta} \in T \text{ such that} \\ WE - HA^\omega \vdash \forall x^\delta; \tilde{x} \leq_\rho s x A_0(x, \tilde{x}, \Phi x \tilde{x}). \end{array} \right.$$

Φ can be extracted from any given proof of the assumption with the use of functional interpretation (combined with negative translation).

- 2) 1) holds also for $WE \widehat{-} PA^\omega \upharpoonright, \widehat{PR}$ and $WE \widehat{-} HA^\omega \upharpoonright$ instead of $WE - PA^\omega, T, WE - HA^\omega$.

Proof:

- 1) Assume $\rho = 0\rho_k \dots \rho_1$.

$$WE - PA^\omega + AC\text{-qf.} \vdash \forall x^\delta; \tilde{x} \leq_\rho sx \exists y^\tau A_0(x, \tilde{x}, y) \stackrel{\leq_\rho\text{-def.}}{\implies} 2.4$$

$$WE - PA^\omega + AC\text{-qf.} \vdash \forall x^\delta; \tilde{x}^\rho \exists v_1^{\rho_1}, \dots, v_k^{\rho_k}, y (\tilde{x}v \leq_0 sxv \rightarrow A_0(x, \tilde{x}, y)).$$

Using functional interpretation (combined with negative translation), one extracts a closed term $\Phi \in T$ such that

$$WE - HA^\omega \vdash \forall x; \tilde{x} \exists v_1, \dots, v_k (\tilde{x}v \leq_0 sxv \rightarrow A_0(x, \tilde{x}, \Phi x \tilde{x})).$$

By intuitionistic logic this implies the lemma.

- 2) Analogous.

3.4 Corollary

- 1) Let $A_0(x, \tilde{x}, y) \in \mathcal{L}(WE - PA^\omega)$ be a quantifier-free formula with the free variables $x^1, \tilde{x}^\rho, y^\tau$. Assume that $\tau \leq 2$ and that $s^{\rho_1} \in T$ is a closed term. Then the following rule holds

$$\left\{ \begin{array}{l} WE - PA^\omega + AC\text{-qf.} \vdash \forall x^1; \tilde{x} \leq_\rho sx \exists y^\tau A_0(x, \tilde{x}, y) \implies \\ \exists \text{ a closed term } t^{\tau_1} \in T \text{ such that} \\ WE - HA^\omega \vdash \forall x^1; \tilde{x} \leq_\rho sx \exists y \leq_\tau tx A_0(x, \tilde{x}, y). \end{array} \right.$$

- 2) Analogous for $\widehat{WE - PA^\omega} \upharpoonright, \widehat{PR}$ and $\widehat{WE - HA^\omega} \upharpoonright$ instead of $WE - PA^\omega, T$ and $WE - HA^\omega$.

Proof: Analogous to the proof of 3.1 using 3.3 instead of mq-realizability.

As a second application of majorization (which is in fact an application of 3.4) we show how one can extract a primitive recursive modulus of uniform continuity for a closed term $t^2 \in T(\widehat{PR})$ from extensionality proofs for t . We first recall a standard proof of the extensionality of t^2 from Troelstra (1973):

Define $x_1^\rho \approx_\rho x_2^\rho$ by

$$\begin{cases} x_1 \approx_0 x_2 := (x_1 =_0 x_2), \\ x_1 \approx_{\rho\tau} x_2 := \forall y_1^\tau, y_2^\tau (y_1 \approx_\tau y_2 \rightarrow x_1 y_1 \approx_\rho x_2 y_2) \end{cases}$$

(Troelstra (1973),2.7.2).

One easily shows that for any $t^\rho \in T$ built up from constants, type 1 and type 0 variables

$$WE - HA^\omega \vdash t \approx_\rho t \text{ (W.A. Howard, Troelstra (1973),2.7.3)}$$

and

$$WE \widehat{-} HA^\omega \upharpoonright \vdash t \approx_\rho t \text{ if } t \in \widehat{PR}.$$

As a corollary one gets:

3.5 Lemma

- 1) (Troelstra (1973),2.7.4 (ii)): Let $t^2 \in T$ be a term whose free variables are of type 0 and 1. Then

$$WE - HA^\omega \vdash \forall \alpha^1, \beta^1 (\forall n (\alpha n =_0 \beta n) \rightarrow t\alpha =_0 t\beta).$$

- 2) 1) holds also for $WE \widehat{-} HA^\omega \upharpoonright$ instead of $WE - HA^\omega$ if $t \in \widehat{PR}$.

3.6 Application

Let $t \in T(\widehat{PR})$ be as in 3.5. From any given proof of the extensionality of t one can extract a modulus $t_M^2 \in T(\widehat{PR})$ of uniform continuity for t , i.e.

$$WE - HA^\omega \vdash \forall \gamma^1; \alpha, \beta \leq_1 \gamma (\bar{\alpha}(t_M \gamma) =_0 \bar{\beta}(t_M \gamma) \rightarrow t\alpha =_0 t\beta)$$

$$(WE \widehat{-} HA^\omega \upharpoonright \vdash (...)).$$

The free variables of t_M are among the free variables of t .

Proof: Assume that

$$WE - HA^\omega \vdash \forall \gamma; \alpha, \beta \leq_1 \gamma (\forall n (\alpha n =_0 \beta n) \rightarrow t\alpha =_0 t\beta).$$

By 3.4 and 3.2.1 one can extract a term $\Phi \in T$ (whose free variables are among the free variables of t) such that

$$WE - HA^\omega \vdash \forall \gamma; \alpha, \beta \leq_1 \gamma \left(\forall n \leq_0 \Phi \gamma (\alpha n =_0 \beta n) \rightarrow t\alpha =_0 t\beta \right).$$

Therefore $t_M \gamma := 1 + \Phi \gamma$ fulfils the claim.

Analogous for $t \in \widehat{PR}$ and $\widehat{WE - HA^\omega}$ instead of $WE - HA^\omega$.

It should be noted that the above proof of the existence of a primitive recursive modulus of uniform continuity (provable in $WE - HA^\omega$ or $\widehat{WE - HA^\omega}$) does not make any reference to (the formalization of) reduction sequences or to the computability of t .

Another way to get a modulus of uniform continuity for t^2 which gives a slightly stronger result is to majorize a modulus of pointwise continuity \tilde{t}^2 for t i.e.

$$(+) WE - HA^\omega \vdash \forall \alpha^1, \beta^1 \left(\bar{\alpha}(\tilde{t}\alpha) =_0 \bar{\beta}(\tilde{t}\alpha) \rightarrow t\alpha =_0 t\beta \right).$$

The existence of such a $\tilde{t} \in T$ for $t \in T$ is proved e.g. in Troelstra (1973) or Schwichtenberg (1973). Troelstra's proof can be modified to yield the corresponding result for $t \in \widehat{PR}$ with $\tilde{t} \in \widehat{PR}$ provable in $\widehat{WE - HA^\omega}$ (see Kohlenbach (1990)).

3.7 Application

- 1) Let $t^2 \in T$ be a term whose free variables are all of type 0,1 and $\tilde{t}^2 \in T$ be a modulus of pointwise continuity for t (in the sense of (+)) whose free variables are among those of t . Then for any $\tilde{t}^* \in T$ such that $WE - HA^\omega \vdash \tilde{t}^* \text{maj}_2 \tilde{t}$ the following holds

$$WE - HA^\omega \vdash \forall \gamma^1, \alpha^1, \beta^1 \left(\forall n < \tilde{t}^* \gamma (\alpha n = \beta n \leq \gamma n) \rightarrow t\alpha =_0 t\beta \right).$$

(By 2.15 such a \tilde{t}^* whose free variables are among those of \tilde{t} can be constructed).

- 2) 1) holds also with $\widehat{PR}, \widehat{WE - HA^\omega}$ instead of $T, WE - HA^\omega$.

Proof: Assume that $t, \tilde{t}, \tilde{t}^*$ fulfil the assumptions. By 2.5.1 it follows that

$$WE - HA^\omega \vdash \forall \gamma; \alpha \leq_1 \gamma (\tilde{t}^* \gamma \geq_0 \tilde{t}\alpha)$$

and hence by \tilde{t} -definition

$$(++) \quad WE - HA^\omega \vdash \forall \gamma; \alpha \leq_1 \gamma; \beta^1 \left(\bar{\alpha}(\tilde{t}^* \gamma) =_0 \bar{\beta}(\tilde{t}^* \gamma) \rightarrow t\alpha =_0 t\beta \right).$$

Now we assume that $\forall n < \tilde{t}^* \gamma (\alpha n = \beta n \leq \gamma n)$ and define $\hat{\alpha} := \bar{\alpha}(\tilde{t}^* \gamma) * \lambda k. \gamma(k + \tilde{t}^* \gamma)$, where

$$(\bar{\alpha}k * \delta)(n) := \begin{cases} \alpha n & \text{if } n < k, \\ \delta(n - k) & \text{otherwise.} \end{cases}$$

Clearly $\hat{\alpha} \leq \gamma$ and $\bar{\hat{\alpha}}(\tilde{t}^* \gamma) = \bar{\alpha}(\tilde{t}^* \gamma) = \bar{\beta}(\tilde{t}^* \gamma)$. Therefore $(++)$ applied twice (to $\hat{\alpha}, \alpha$ and $\hat{\alpha}, \beta$) yields $t\hat{\alpha} =_0 t\alpha$ and $t\hat{\alpha} =_0 t\beta$.

An analogous proof can be given for the case where T and $WE - HA^\omega$ are replaced by \widehat{PR} and $\widehat{WE} - HA^\omega \upharpoonright$.

As a corollary to 3.6 or 3.7 one can derive the closure of the systems $H, \hat{H} \upharpoonright$ under the usual fan-rule (*). 3.7 allows a slight strengthening of (*):

3.8 Corollary

- 1) (See also Troelstra (1977), 3.6 Remarks (ii)) Let H be $WE - HA^\omega$ or $E - HA^\omega$ and $A(\alpha^1, n^0) \in \mathcal{L}(WE - HA^\omega)$ be a formula whose free variables are all of type 0,1. Then the following rule holds

$$\left\{ \begin{array}{l} H \vdash \forall \alpha \exists n^0 A(\alpha, n) \implies \\ \exists \text{ terms } t_M^2, t^{*2} \in T \text{ whose free var. are among those of } A_0 \text{ without } \alpha, n \text{ such that} \\ H \vdash \forall \beta^1; \alpha \leq_1 \beta \exists n \leq_0 t^* \beta \forall \gamma^1 (\bar{\gamma}(t_M \beta) =_0 \bar{\alpha}(t_M \beta) \rightarrow A(\gamma, n)). \end{array} \right.$$

- 2) 1) holds also for $\hat{H} \upharpoonright$ and \widehat{PR} instead of H and T .

Proof:

- 1) Assume $H \vdash \forall \alpha \exists n^0 A(\alpha, n)$. As in the proof of 3.1 one constructs a term $t \in T$ whose free variables are among those of A without α, n such that $H \vdash \forall \alpha A(\alpha, t\alpha)$. Let \tilde{t}^* be as in 3.7 and $t^* \in T$ satisfy the condition $H \vdash t^* \text{ maj } t$. Then the corollary holds with $t_M := \tilde{t}^*$ and t^* .

- 2) is proved analogously.

We conclude this paper with two applications of the type structure \mathcal{M} :

3.9 Notation

1) Natural numbers n are coded into higher types as follows:

$$n^0 := n, \quad n^{\rho\tau} := \lambda x^\tau . n^\rho.$$

2)

$$\overline{(y^{\rho 0}, n^0)}(k) := \begin{cases} yk & \text{if } k < n, \\ O^\rho & \text{otherwise;} \end{cases}$$

$$\overline{(yn * 1^{\rho 0})}(k) := \begin{cases} yk & \text{if } k < n, \\ 1^\rho & \text{otherwise.} \end{cases}$$

3.10 Definition

By the principle of **bounded choice** we mean the schema

$$(b - AC)^{\rho, \tau} : \forall Z^{\tau\rho} \left(\forall x^\rho \exists y \leq_\tau Zx A(x, y, Z) \rightarrow \exists Y \leq_{\tau\rho} Z \forall x A(x, Yx, Z) \right).$$

Further notations are

$$(DC)^\rho : \forall x^\rho \exists y^\rho A(x, y) \rightarrow \forall x^\rho \exists z^{\rho 0} (z0 =_\rho x \wedge \forall z_1^0 A(z z_1, z(z_1 + 1)))$$

(dependent ω -choice),

$$(C)^\rho : \exists y^{0\rho} \forall x^\rho (yx =_0 0 \leftrightarrow A(x)) \text{ (comprehension).}$$

$$b - AC := \bigcup_{\rho, \tau \in \tilde{T}} \{(b - AC)^{\rho, \tau}\}, \quad DC := \bigcup_{\rho \in \tilde{T}} \{(DC)^\rho\}, \quad C := \bigcup_{\rho \in \tilde{T}} \{(C)^\rho\}.$$

$\mathcal{T} := E - PA^\omega + C$. The theory \mathcal{T} is a formalization of simple type theory in the language of functionals of finite type without choice (see also Luckhardt (1975)).

$$(AC!)^{1,0}\text{-qf.} \vdash \forall x^1 \exists! y^0 A_0(x, y) \rightarrow \exists Y^{01} \forall x A_0(x, Yx).$$

3.11 Lemma

- 1) $WE - HA^\omega + AC \vdash b - AC$,
- 2) $WE - PA^\omega + b - AC \vdash C$.

Proof: 1) is trivial.

2) $WE - PA^\omega \vdash \forall x^\rho \exists n \leq 1(n =_0 0 \leftrightarrow A(x)) \Rightarrow$
 $WE - PA^\omega + (b - AC)^{\rho,0} \vdash \exists Y \leq_{0\rho} \lambda v^\rho. 1 \forall x^\rho (Yx = 0 \leftrightarrow A(x)).$

3.12 Application

- 1) $\mathcal{M} \models \mathcal{T} + b - AC + DC$,
- 2) $\mathcal{M} \not\models (AC!)^{1,0}\text{-qf.}$,
- 3) $\mathcal{T} + b - AC + DC \not\models (AC!)^{1,0}\text{-qf.}$

Proof: 1) By 2.18 \mathcal{M} is a model of $E - PA^\omega$.

We show $\mathcal{M} \models b - AC$: Assume $Z \in \mathcal{M}$ and

$\forall x \in M_\rho \exists y \in M_\tau (y \leq_\tau Zx \wedge [A]_{\mathcal{M}}(x, y))$. Using choice on the meta-level it follows that

$$\exists Y \in M_\tau^{M_\rho} \forall x \in M_\rho (Yx \leq_\tau Zx \wedge [A]_{\mathcal{M}}(x, Yx)).$$

$Z \in M_\tau \Rightarrow \exists Z^* \in M_\tau : Z^* \text{ maj } Z \xrightarrow{Y \leq Z} Z^* \text{ maj } Y \in M_\tau$.

$\mathcal{M} \models DC$ holds since $M_\rho^{M_0} = M_\rho$ (By DC on the meta-level).

2) Assume $\mathcal{M} \models (AC)^{1,0}\text{-qf.}$ By $(C)^{00}$ let $v \in \mathcal{M}$ be such that

(i) $\exists z^0 (x^{00}z = 0) \Leftrightarrow vx = 0$ for all x^{00} . $(AC)^{1,0}\text{-qf.}$ applied to $\forall x^{00} \exists z^0 (vx = 0 \Rightarrow xz = 0)$ yields

(ii) $\exists y \in M_{0(00)} \forall x^{00} [vx = 0 \Rightarrow x(yx) = 0]$. Since $y \in M_{0(00)}$ there exists a $y^* \in M_{0(00)}$ such that $y^* \text{ maj}_{0(00)} y$. From (i) and (ii) it follows that

$\forall m \in \omega (y(\overline{1, m}) \geq m)$. On the other hand one has $1^{00} \text{ maj}_{00} \overline{1, m}$ and therefore $y^* 1^{00} \geq y(\overline{1, m}) \geq m$ for all m , which is a contradiction. $(AC!)^{1,0}\text{-qf.}$ implies $(AC)^{1,0}\text{-qf.}$ (relative to $WE - PA^\omega$): Replace $A_0(x, y)$ by $A'_0(x, y) := A_0(x, y) \wedge \forall \tilde{y} < y (\neg A_0(x, \tilde{y}))$

(A'_0 can also be expressed in a quantifier-free form in $WE - PA^\omega$). Therefore 2) follows.

3) follows immediatly from 1) and 2).

3.13 Remark

In the context of set theory in a language with set variables instead of functional variables where functionals are identified with their graphs, $(AC!)^{1,0}$ is provable from the schema of comprehension formulated as the existence of the comprehending set (instead of its characteristic functional). In Feferman/Levy (1963) it is shown that dependent choice is not provable within ZF . This result uses P.Cohen's method of forcing.

As a final application, we show how \mathcal{M} separates different μ -operators:

3.14 Definition

$${}^\rho\mu_s x^{0(\rho 0)} y^{\rho 0} := \begin{cases} \min n[x(\overline{y, n}) < n] \text{ if } \exists n(x(\overline{y, n}) < n) \\ \text{undefined otherwise ,} \end{cases}$$

$${}^\rho\mu_1 x^{0(\rho 0)} y^{\rho 0} := \begin{cases} \min n[x(\overline{y, n}) =_0 xy] \text{ if } \exists n(x(\overline{y, n}) =_0 xy), \\ \text{undefined otherwise ,} \end{cases}$$

$${}^\rho\mu_2 x^{0(\rho 0)} y^{\rho 0} := \begin{cases} \min n[x(\overline{y, n}) =_0 x(\overline{yn} * 1)] \text{ if } \exists n(x(\overline{y, n}) =_0 x(\overline{yn} * 1)), \\ \text{undefined otherwise .} \end{cases}$$

3.15 Remark

The subscript s in the definition of ${}^\rho\mu_s$ refers to Spector since $x(\overline{y, n}) < n$ is the bar condition in his schema of bar recursion; see Spector (1962).

3.16 Application

${}^\rho\mu_s \in \mathcal{M}$, but ${}^\rho\mu_1, {}^\rho\mu_2 \notin \mathcal{M}$ for each $\rho \in \tilde{T}$.

Proof: ${}^\rho\mu_s xy$ is defined for all $x \in M_{0(\rho 0)}, y \in M_{\rho 0}$:

By \mathcal{M} -definition and 2.17 there exists $x^* \in M_{0(\rho 0)}, y^* \in M_{\rho 0}$ such that $x^* \text{ maj } x$ and $y^* \text{ maj } y$. Since $\forall n \in \omega(y^* \text{ maj}_{\rho 0} \overline{y, n})$ it follows that $x^* y^* \geq x(\overline{y, n})$ for all $n \in \omega$ and hence $x(\overline{y, n_0}) < n_0$ for $n_0 := x^* y^* + 1$ (see also Bezem (1985)).

We now show that ${}^\rho\mu_s \text{maj } \mu_s$: Assume again that $x^* \in M_{0(\rho_0)}, y^* \in M_{\rho_0}$ such that $x^* \text{maj } x, y^* \text{maj } y$. Then $\overline{y^*, n} \text{maj}_{\rho_0} \overline{y, n}$ since maj_{ρ_0} is pointwise defined (!) and therefore $x^*(\overline{y^*, n}) \geq x(\overline{y, n})$ for all n .

One concludes that ${}^\rho\mu_s x^* y^* \geq {}^\rho\mu_s x y$ and therefore ${}^\rho\mu_s \text{maj } {}^\rho\mu_s \in \mathcal{M}$.

On the other hand, $\lambda x^\rho. \mu_i x 1^{\rho_0} \notin M_0^{M_\rho} \supset M_{0\rho}$ for $i=1,2$:

Assume $\rho = 0\rho_k \dots \rho_1$ and $x \in M_0^{M_\rho}$ be defined by

$$xy := \begin{cases} 0 & \text{if } \exists n \in \omega (yn O^{\rho_1} \dots O^{\rho_k} =_0 0), \\ 1 & \text{otherwise,} \end{cases}$$

for $y \in M_{\rho_0}$. Clearly $x \in M_{0(\rho_0)}$ since $1^{0(\rho_0)} \text{maj } x$, and $x 1^{\rho_0} =_0 x(\overline{1n} * 1) =_0 1$ but $x(\overline{1^{\rho_0}, n}) =_0 0$ for all $n \in \omega$. Therefore ${}^\rho\mu_1 x 1^{\rho_0}$ and ${}^\rho\mu_2 x 1^{\rho_0}$ are undefined, but \mathcal{M} contains only total functionals.

3.17 Corollary

${}^\rho\mu_1$ and ${}^\rho\mu_2$ are not definable within $WE - HA^\omega + \mu_s$ where $WE - HA^\omega + \mu_s$ denotes the enlargement of $WE - HA^\omega$ by new constants ${}^\tau\mu_s$ together with the defining axioms

$x(\overline{y, \mu_s xy}) < \mu_s xy$ and $x(\overline{y, n}) < n \rightarrow \mu_s xy \leq n$ ($x^{0(\tau_0)}, y^{\tau_0}$) for all types τ .

3.18 Remark

In Kohlenbach (1990) it is proved that ${}^\rho\mu_s, {}^\rho\mu_i$ are definable in $WE - HA^\omega + {}^\tau\mu_j$ where $i, j = 1, 2$ and $\rho, \tau \in \tilde{T}$ arbitrary.

3.19 Proposition

There is no set-theoretic functional $\Phi^{0(0(00))}$ which majorizes $\lambda x^{0(00)}. {}^0\mu_1 x 1^{00}$ ($= \lambda x^{0(00)}. {}^0\mu_2 x 1^{00}$) on all primitive recursive arguments $x^{0(00)}$ (on which ${}^0\mu_1$ and ${}^0\mu_2$ are always defined!). In particular ${}^0\mu_{1,t} \notin \mathcal{M}$ where

$${}^0\mu_{1,t} x^{0(00)} y^{00} := \begin{cases} \min n [x(\overline{y, n}) =_0 xy] & \text{if } \exists n (x(\overline{y, n}) =_0 xy), \\ 0^0 & \text{otherwise.} \end{cases}$$

Proof: Assume that $\Phi \text{maj}_{0(0(0))} \lambda x. {}^0\mu_1 x 1^{00}$ on prim. rec. arguments.
Define

$$z_n^{0(00)} x^{00} := \begin{cases} 0 & \text{if } xn = 0, \\ 1 & \text{otherwise.} \end{cases}$$

z_n is primitive recursive in n (in the sense of \widehat{PR}) and $1^{0(00)} \text{maj} z_n$ for all $n \in \omega$.
Therefore $\Phi 1^{0(00)} \geq {}^0\mu_1 z_n 1^{00}$ for all $n \in \omega$. On the other hand, one has
 $\lim_{n \rightarrow \infty} {}^0\mu_1 z_n 1 = \infty$ since $\mu_1 z_n 1 = n + 1$ which is a contradiction.

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