Recent Progress in Proof Mining in Nonlinear Analysis

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Dedicated to the memory of Professor Georg Kreisel (1923-2015)

January 11, 2018

Abstract

During the last two decades the program of 'proof mining' emerged which uses tools from mathematical logic (so-called proof interpretations) to systematically extract explicit quantitative information (e.g. rates of convergence) from prima facie nonconstructive proofs (e.g. convergence proofs). This has been applied particularly successful in the context of nonlinear analysis: fixed point theory, ergodic theory, topological dynamics, convex optimization and abstract Cauchy problems. In this paper we give a survey on some of the results, both on the logical foundation of proof mining as well as its applications in nonlinear analysis, obtained since the monograph [65] appeared.

1 Introduction

Back in the 50's, Georg Kreisel's program of 'unwinding of proofs' asked for a re-orientation of proof theory by shifting the historical emphasis on foundational issues of Hilbert's program (consistency proofs) towards applications of proof-theoretic methods for well-defined mathematical goals. Whereas some of these goals, which were still close to foundational concerns, were much developed in the past 50 years (e.g. the classification of provably recursive function(al)s of various formal systems), applications to concrete problems from core mathematics remained rather scattered, two notably exceptions being C. Delzell's work on Hilbert's 17th problem (see e.g. [34]) and H. Luckhardt's extraction of a polynomial bound on the number of solutions in Roth's theorem from a proof of that theorem due to Esnault and Vieweg (see [103]).
Starting in [58], we engaged in giving Kreisel's ideas for a new form of an applied proof theory a fresh start. This time the focus was on applications in analysis since in this area proof theory can already contribute to the highly nontrivial issue of determining the correct representation of continuous objects in which then quantitative data such as effective bounds could be computed. Back in the 90's, the main emphasis was on the analysis of proofs that used Heine-Borel compactness in the form of the noneffective binary ('weak') König's lemma WKL and the proof-theoretic approach towards the elimination of WKL

from classes of proofs. One such class that turned out to be particularly fruitful was that of uniqueness proofs which led to new quantitative results concerning the issue of 'strong uniqueness' in best approximation theory (both w.r.t. the uniform norm, i.e. Chebycheff approximation, as well as w.r.t. the L^1-norm (see [65] for a book treatment of these developments and references to the literature).

In 2000-2003, we started to investigate strong convergence results for iterative procedures of nonexpansive and other classes of mappings in general normed spaces and (with L. Leuştean) hyperbolic spaces and succeeded in the extraction of highly uniform explicit bounds. Here 'highly uniform' refers to the fact that these bounds are essentially independent from the data of the abstract space such as the starting point of the iteration and the mapping used in the iteration, despite of the absence of any compactness assumption, but only depend on some general norm bounds and data from the concrete Polish spaces involved such as $C[0,1]$.

This led in 2003-05 to the discovery of the first so-called 'logical metatheorems' which explain these findings as instances of general proof-theoretic phenomena ([63, 44]). These logical results are called metatheorems since they take as assumption the existence of a proof of a theorem in some formal framework $\mathcal{A}^\omega[X,\ldots]$ and then assert the extractability of an effective uniform bound from such a given proof together with the verification that this bound is correct in any structure X that satisfies the axioms specified in $\mathcal{A}^\omega[X,\ldots]$. These logical metatheorems are based on certain proof-theoretic transformations ('proof interpretations') which are far reaching extensions and modifications of Gödel's famous functional ('Dialectica') interpretation ([65]). The main tool used is the so-called monotone functional interpretation due to [60]. All these proof interpretations prima facie only work for proofs that are based on a constructive ('intuitionistic' in the sense of Brouwer) logic. In order to apply them to proofs using ordinary ('classical') logic one uses - as a pre-processing step - an appropriate so-called negative translation (Gödel) which provides an embedding of the classical reasoning into an intuitionistic system. The crucial feature of the resulting combined interpretation is that important classes of theorems, e.g. theorems of the logical form $\forall x \exists y\, A_\exists(x,y)$, where $A_\exists$ is purely existential, survive this passage (since functional interpretations eliminate the Markov principle even in higher types) and get equipped with an explicit bound $\Phi(x^*)$ on $\exists y$ which only depends on some bounding ('majorizing') data x^* on x. Instead of giving the general type-inductive definition of the majorization relation $x^* \gtrsim x$ we just list the cases we need here: if $x \in \mathbb{N}$, then also $x^* \in \mathbb{N}$ and $x^* \geq x$, if $x \in \mathbb{N}^\mathbb{N}$, then also $x^* \in \mathbb{N}^\mathbb{N}$ and x^* is a nondecreasing upper bound on x. If $x \in X$, where X is an abstract metric space, then $x^* \in \mathbb{N}$ and we define $x^* \gtrsim x := x^* \geq d(a,x)$, where $a \in X$ is some fixed (for the definition of the majorizability relation) reference point. In the case where X is a normed space we always use $a := 0_X$. Finally, if $x \in X^\mathbb{N}$ or $x \in X^X$ resp., then $x^* \in \mathbb{N}^\mathbb{N}$ is nondecreasing with $x^*(n) \geq d(a, x(m))$, whenever $n \geq m$, or $x^*(n) \geq d(a, x(y))$, whenever $n \geq d(a,y)$, resp. While each x in $\mathbb{N}, \mathbb{N}^\mathbb{N}, X, X^\mathbb{N}$ is majorizable, this not always is the case for selfmaps $x \in X^X$. However, important classes of such mappings, e.g. the class of all nonexpansive mappings, have actually very simple majorants.

We now state a special case of our logical metatheorem for the context of abstract normed spaces:

Theorem 1.1 ([63, 44])**.** *Let P, K be Polish resp. compact metric spaces (explicitly representable in $\mathcal{A}^\omega$), $A_\exists$ be a purely existential formula, $\underline{z} := z_1, \ldots, z_k$ be variables ranging over X, $\mathbb{N} \to X$ or $X \to X$.*

From a proof $\mathcal{A}^\omega[X, \|\cdot\|] \vdash \forall x \in P\, \forall y \in K\, \forall \underline{z}\, \exists v^{\mathbb{N}} A_\exists(x, y, \underline{z}, v)$
one can extract a computable $\Phi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t.
$\forall y \in K\, \exists v \leq \Phi(r_x, \underline{z}^*)\, A_\exists(x, y, \underline{z}, v)$ holds in every normed space X,

where $r_x \in \mathbb{N}^{\mathbb{N}}$ is a representative of $x \in P$, and all $\underline{z}$ and $\underline{z}^* = z_1^*, \ldots, z_k^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $z_i^* \gtrsim z_i$ for $1 \leq i \leq k$.

The important point here is that Φ does not operate on the X-data $\underline{z}$ (in which case we not even could make sense of the term 'computable' unless X comes equipped with some notion of effectivity) but only on majorants $\underline{z}^*$ of $\underline{z}$. Moreover, Φ is not just 'computable' but of some restricted complexity which depends on the strength of the mathematical principles that are included in the deductive framework $\mathcal{A}^\omega[X, \ldots]$. In the application to a concrete proof, Φ reflects the computational content of that proof.

The development of the proof mining paradigm has been a back-and-forth movement from experimental case studies to the formulation of general theorems as the one above which explain the structure of the findings in these case studies as instances of a logical pattern which can then be used to systematically find new promising areas for case studies which in turn prompt new proof-theoretic results and so on. As discussed in the next section, theorems of the form above have been tailored towards numerous classes of metric and normed structures X and of mappings $T : X \to X$, where the input data often are enriched by appropriate moduli functions $\omega : \mathbb{N} \to \mathbb{N}$ such as (suitable forms) of moduli of uniform convexity or uniform smoothness for X or of uniform continuity of T etc.

In this paper we give a survey on these developments, both w.r.t. the logical foundations as well as to applications in the context of nonlinear analysis, since around 2010.

Notation: Throughout this paper, for $f \in \mathbb{N} \to \mathbb{N}$ we use $f^{(n)}(m)$ to denote the n-th iteration of f starting from m, i.e.

$$f^{(0)}(m) := m, \quad f^{(n+1)}(m) := f(f^{(n)}(m)).$$

For selfmappings $T : C \to C$ of some subset C of a metric or normed space, we simply use $T^n x$ (for $x \in C$) to denote the n-th iteration as here there is no danger to confuse this with the n-th power.

2 New Developments in the Logical Foundation of Proof Mining

In this section we survey the current stage on the logical methodology used in the proof-mining program as it is applied in nonlinear analysis.

2.1 Classes of abstract spaces admissible in proof mining

Whereas the main applications of proof mining to analysis before 2000 where in the context of specific Polish spaces, such as $C[0,1]$ only, since 2001 almost all applications concern situations where the theorem in question refers to some abstract class of spaces X in addition to such concrete ones. E.g. the mean ergodic theorem is a result formulated in the context of arbitrary (not necessarily separable) Hilbert spaces X. As discussed in detail in [65], the extractability of highly uniform bounds depending only on general metric bounds as input data rather than requiring compactness assumptions rests crucially on the fact that the proof being analyzed does not use the separability of X, as the uniform quantitative form of separability is total boundedness (and so in the presence of completeness implies boundedly compactness). In order to deal with such abstract (not assumed to be separable) classes of structures, the approach started in [63] has been to add X as a kind of atom to the formal systems at hand by including X as a new base type. To obtain a logical metatheorem on the extractability of uniform bounds for a class of metric structures X one needs the following requirements:

(a) the axioms used to axiomatize X have (possibly after being enriched with suitable quantitative moduli ω in $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$) a monotone functional interpretation by simple majorizing functionals,

(b) all the constants used to axiomatize X have effective majorants of low complexity.

These conditions usually follow from the fact that the axioms used to characterize X (when interpreted in the full set-theoretic model $\mathcal{S}^{\omega,X}$) can be written in purely universal form (once the various moduli are given; see [63, 44, 65]) or - more generally - in the form of axioms

$$\Delta :\equiv \forall \underline{a}^{\underline{\delta}} \exists \underline{b} \preceq_{\underline{\sigma}} \underline{r}\underline{a} \forall \underline{c}^{\underline{\gamma}} A_0(\underline{a},\underline{b},\underline{c}),$$

where A_0 is quantifier-free and does not contain any further free variables, $\underline{r}$ is a closed term (of suitable types) of $\mathcal{A}[X,\ldots]$, the types $\underline{\delta},\underline{\sigma},\underline{\gamma}$ satisfy some modest conditions (see [49]; for the finite types over $\mathbb{N}$ only such axioms were already considered in [59]). Here $\preceq$ is pointwise defined where in the normed case one takes $x^X \preceq_X y^X :\equiv \|x\| \leq \|y\|$.[1]

As shown in [49], such axioms cover all normed structures axiomatizable in positive bounded logic, both w.r.t. the normal strict interpretation as well as w.r.t. the weaker approximate satisfaction relation used in that context. Structures axiomatized in this weaker approximate sense are not only closed under ultraproducts but also ultraroots (see [51]). By adding a nonstandard axiom $F^X_{\preceq}$ of the form Δ, which is not true in the full set-theoretic model but which holds in the model of all strongly majorizable functionals $\mathcal{M}^{\omega,X}$ (in the sense of Bezem [17] extended to the types over $\mathbb{N}$ and X), one can prove (using quantifier-free choice QF-AC which does not $\mathcal{M}^{\omega,X}$) a uniform boundedness principle Σ^0_1-UB$^X_{\preceq}$ which implies that each sentence in positive bounded logic is equivalent to its approximations (written as a single sentence) so that the usual validity in a model and approximate validity coincide. This corresponds to the fact that the ultraproduct of X satisfies an axiom in positive

[1] This can be adapted to the metric case by taking $x^X \preceq_X y^{\mathbb{N}} =\equiv d(x,a) \leq y$ for some reference point $a \in X$.

bounded logic in the strong sense already when it satisfies it in the approximate sense. Since QF-AC gets eliminated by the functional interpretation and F^X_- can be eliminated from the verification of extractable bounds for $\forall\exists$-sentences by interpreting things in the model $\mathcal{M}^{\omega,X}$, one may freely use the strong reading of Δ when proving a theorem while the extracted bound will also be valid in the - in general - larger class of spaces X which are only required to satisfy the approximate version of Δ (see [49] for all this). In total, the following classes of spaces have been shown so far to satisfy appropriate logical metatheorems on the extractability of effective uniform bounds:

1. metric spaces (see [63] for the bounded case and [44] for the unbounded case),
2. W-hyperbolic and CAT(0)-spaces (see [63] for the bounded case and [44] for the unbounded case),
3. CAT(κ)-spaces for $\kappa > 0$ ([83]),
4. uniformly convex W-hyperbolic spaces with monotone modulus of uniform convexity, so called UCW-spaces (see [97, 100]), where then the bounds depend on a given modulus of uniform convexity,
5. δ-hyperbolic spaces and $\mathbb{R}$-trees (see [97]),
6. normed spaces, uniformly convex normed spaces and inner product spaces (see [63]),
7. uniformly smooth normed spaces (with single-valued and norm-to-norm uniformly continuous normalized duality mapping) (see [80]), where then the bounds depend on an appropriate concept of modulus of uniform smoothness,
8. totally bounded metric spaces (see [65] and [81]), where then the bounds depend on an appropriate notion of modulus of total boundedness,
9. metric completions of the spaces listed so far (see [65]),
10. Banach lattices (see [49]),
11. abstract L^p- and $C(K)$-spaces (see [49]),
12. bands in the $L^p(L^q)$-Banach lattice (see [49]).

If these conditions are satisfied, the contribution of the X-axioms to the extractable bounds mainly consists in the moduli ω referred to above. A further convenient, but not mandatory, requirement is that the axioms should imply the uniform continuity of the constants occurring in the axioms so that their extensionality (w.r.t. $x =_X y :\equiv d_X(x,y) =_{\mathbb{R}} 0$) can be inferred which must not be included as an =-equality axiom as the uniform quantitative interpretation of extensionality by monotone functional interpretation upgrades extensionality to uniform continuity (on bounded domains). However, whereas in the model-theoretic approaches to metric structures as in Chang and Keisler's continuous ([29]) or in Henson and Iovino's positive bounded logic ([51]), uniform continuity is a necessary part of the framework (used e.g. to define the ultrapower of X, see - however - the recent preprint [31] which aims at weakening this requirement), this is not case in the proof-theoretic treatment where one can avoid to use uniform continuity

(i) if the proof in question can be formalized with a weaker quantifier-free extensionality rule QF-ER instead of the extensionality axiom or

(ii) if a condition weaker than uniform continuity turns out to be sufficient to provide a uniform quantitative form of the special instances of the extensionality axiom used in the proof.

E.g. consider the definition of W-hyperbolic spaces X from [63] which is axiomatized (in addition to being a metric space) by four axioms (W1)-(W4) on a formal convexity operator $W : X \times X \times [0, 1] \to X$, where (W4), in particular, expresses the uniform continuity (and hence the extensionality) of W as a function in x, y (for fixed $\lambda \in [0, 1]$), whereas (W2) implies the uniform continuity (and hence extensionality) in λ. It turns out that many proofs do not use (W4) and the extraction of effective uniform bounds goes through without problems so that the extracted bounds are valid in the larger class of spaces axiomatized by (W1)-(W3) which coincides with the spaces of hyperbolic type from [46] (see e.g. Remark 3.25 in [76]).
If, however, one is in a situation where all the functions involved satisfy appropriate uniform continuity assumptions and the model theory for metric and normed spaces it applicable, then proof-theoretic bound extraction theorems may be viewed as constructive quantitative versions of qualitative uniformity results obtainable using ultraproducts (see [3]). Whereas for the latter, only the truth of the theorem in question in the respective class of structures is needed and only the existence of a uniform bound follows (though by results in effective model theory recently announced by J. Rute one may even conclude the existence of computable bounds obtained by blind unbounded search through some infinitary term language), the proof-theoretic approach uses the formalizability of a proof of the theorem in some suitable system and then extracts an explicit subrecursive bound reflecting the computational content of the given proof.

2.2 Classes of majorizable functions that are admissible in proof mining

The above comments on extensionality not only apply to the constants used in axiomatizing X but also to the classes $\mathcal{F}$ of functions $T : X \to X$ in the theorem

$$\forall T \in \mathcal{F}\, \exists n \in \mathbb{N}\, A_\exists(T, n)$$

for which we want to extract effective uniform bounds. E.g. if uniform continuity of T follows from $T \in \mathcal{F}$, then full extensionality is for free. Otherwise, the items (i),(ii) may apply: as an instance of the first item, one may refer to the nonlinear ergodic theorem due to [129] for selfmappings $T : K \to K$ of a subset K of a Hilbert space satisfying the Wittmann-condition

$$(W)\ \forall x, y \in K\, (\|Tx + Ty\| \leq \|x + y\|),$$

whose proof only uses QF-ER. The condition (W) trivially implies that T is majorized by $T^* := Id_{\mathbb{N}}$, while T in general will be discontinuous (see example 3.1 in [129]). In section 3.2 we will discuss the quantitative analysis of Wittmann's theorem given in [116].

Recently, the 2nd item above has been used substantially in the context of metric fixed point theory. There extensionality often is used only in the form

$$p \in Fix(T) \wedge x =_X p \to x \in Fix(T),$$

where $Fix(T)$ denotes the fixed point set of some selfmapping $T : X \to X$ of a metric space (X, d). This is a genuine use of the extensionality axiom that cannot be replaced by QF-ER. However, its uniform quantitative interpretation (as obtained via monotone functional interpretation) is satisfied by moduli $\delta_T, \omega_T : \mathbb{N} \to \mathbb{N}$ such that

$$(*) \left\{ \begin{array}{l} \forall x, p \in X\, \forall k \in \mathbb{N} \left(d(p, Tp) < \frac{1}{\delta_T(k)+1} \wedge d(x, p) < \frac{1}{\omega_T(k)+1} \right. \\ \qquad\qquad\qquad\qquad \left. \to d(x, Tx) \le \frac{1}{k+1} \right). \end{array} \right.$$

This condition (which is introduced in [81] under the name of '(moduli of) uniform closedness') can always be satisfied when T is uniformly continuous with a modulus of uniform continuity Ω_T by defining

$$\omega_T(k) := \max\{4k+3, \Omega_T(4k+3)\} \text{ and } \delta_T(k) := 2k+1$$

(see Lemma 7.1 in [81]), but is also satisfied for important classes of in general discontinuous functions: e.g. T satisfies the so-called condition (E) (introduced in [39] as a generalization of a condition (C) in [124]) if there exists a $\mu \ge 1$ such that

$$\forall x, y \in C \ (d(x, Ty) \le \mu d(x, Tx) + d(x, y)).$$

It is easy to see that in this case we may take (w.l.o.g. $\mu \in \mathbb{N}$)

$$\omega_T(k) := 4k+3 \text{ and } \delta_T(k) := 2\mu(k+1) - 1$$

(see [81])[2] while T may fail to be continuous. E.g., as shown in [39], $T : [-2, 1] \to [-2, 1]$ defined by

$$T(x) := |x|/2, \text{for } x \in [-2, 1), \text{ and } := -1/2, \text{ for } x = 1,$$

satisfies the condition (E) with $\mu := 3$.

The condition (E) also implies that T is majorizable (w.r.t. a reference point $a \in X$) by

$$T^*(n) := C_{a,\mu} + n,$$

where $\mathbb{N} \ni C_{a,\mu} \ge \mu \cdot d(a, Ta)$.

In addition to the issue that maybe only partial extensionality is available, the membership relation $T \in \mathcal{F}$ also has to satisfy the requirements $(a), (b)$ on the X-axioms in order to be able to design a bound-extraction theorem for the class $\mathcal{F}$. In the examples just discussed this is trivial by the existence of simple T-majorants and since both the condition (W)

[2] Correction to [81]: in the definition of the moduli χ, δ for mappings satisfying the condition (E) one has to use $\lceil \cdot \rceil$ to make the bounds χ, δ_T natural numbers.

as well as (E) (for given μ) are purely universal and hence admissible in logical bound-extraction theorems for these classes of mappings while in the case of (E) the extracted bound will additionally depend on the upper bound $C_{a,\mu}$ for some $a \in X$.

So far the following classes of mappings have been shown to have appropriate bound extraction theorems and have been used in actual unwindings of proofs:

1. nonexpansive, Lipschitz, Hölder-Lipschitz and uniformly continuous mappings ([44]),
2. weakly quasi-nonexpansive mappings (first implicitly introduced - without a name - in [75] and - independently as 'J-type mappings' [38] and as 'weakly quasi nonexpansive mappings' in [44]; used in proof mining in [75]),
3. asymptotically nonexpansive mappings (introduced in [45] and used in proof mining and discussed from a logical point of view in [75, 78]),
4. uniformly contractive mappings (introduced in [111] and used in proof mining and discussed from a logical point of view in [84, 43]),
5. uniformly generalized p-contractive mappings (introduced - as a uniform strengthening of a notion from [56] - and applied in proof mining in [20]; logically studied in [21]),
6. asymptotic contractions in the sense of Kirk (introduced in [57] and used in proof mining in [42, 19]; logically studied in [21]),
7. firmly (quasi-)nonexpansive mappings in geodesic and normed spaces (introduced for Hilbert spaces in [23], for general Banach spaces in [25], for the Hilbert ball in [47] and for W-hyperbolic spaces in [1] and used in proof mining in [108, 1, 2, 71, 82, 72]),
8. strongly (quasi-)nonexpansive mappings in geodesic and normed spaces (introduced in [28] in the normed setting and in the 'quasi'-form and for the Hilbert ball in [112, 15] and in metric spaces in [27]; used in proof mining and discussed from a logical point of view in [71, 72]),
9. pseudocontractive mappings in normed spaces (introduced by Browder [22] and used in proof mining in [89, 90, 88]),
10. strict pseudocontractions in Hilbert space (introduced in [24] and used in proof mining in [53, 81, 119]),
11. mappings satisfying Wittmann's condition (W) in Hilbert space (introduced in [129] and used in proof mining in [116, 67, 71]),
12. mappings satisfying condition (E) in geodesic spaces (introduced in [39] and used in proof mining in [81]; see above for a discussion from the point of logic),
13. majorizable mappings in metric spaces ([44])

as well as combinations thereof (e.g. Lipschitzian pseudocontractions).

Uniformly continuous selfmappings of geodesic spaces and Lipschitz continuous mappings on general metric spaces are always majorizable where an a-majorant can be computed in terms of a modulus of uniform continuity and an upper bound $b \geq d(a, Ta)$ for $a \in X$ (see the proof of Corollary 4.20 in [44]). Hence majorizability (and extensionality) follows for such mappings including mappings which are asymptotically nonexpansive, uniformly contractive, firmly or strongly nonexpansive or strict pseudocontractions.
Due to the lack of continuity one does not have extensionality for free in the other cases but majorizability holds for weakly quasi-nonexpansive mappings (see the proof of Corollary 4.20(5) in [44]) and so also for firmly quasi-nonexpansive and strongly quasi-nonexpansive mappings if the mappings possess a fixed point. Majorants can then be given in just an upper bound $b \geq d(a, p)$ for some reference point a and a respective fixed point p. For mappings satisfying the condition (E), majorizability has been shown above. As mentioned already, mappings satisfying condition (W) are trivially majorized by the identity map.
Let us discuss the issue of majorizability for asymptotic contractions in the sense of Kirk (the logical discussion in [21] is only in the context of bounded metric spaces where the majorizability is trivial but the results in [19] are proven for general metric spaces):
a function $T : X \to X$ is an asymptotic contraction in the sense of Kirk if there are continuous mappings $\phi, \phi_n : [0, \infty) \to [0, \infty)$ with $\phi(s) < s$ for $s > 0$ such that for all $n \in \mathbb{N}, x, y \in X$

$$d(T^n x, T^n y) \leq \phi_n(d(x, y))$$

and $\phi_n \to \phi$ uniformly on the range of d. While the continuity of the functions ϕ_n is not part of Kirk's original definition it is added as an assumption in his main result on these mappings in [57] and [19] officially included this condition in his definition of 'asymptotic contractions in the sense of Kirk'. If ϕ_1 (additionally) is assumed to be continuous, then the majorizability is shown as follows: let $a \in X$ be a reference point and $x \in X, n \in \mathbb{N}$ with $d(x, a) \leq n$. Define $T^*(n) := b + \sup_{y \in [0,n]} \phi_1(y)$, where $\mathbb{N} \ni b \geq d(a, Ta)$. Then

$$d(a, Tx) \leq d(a, Ta) + d(Ta, Tx) \leq b + \phi_1(d(a, x)) \leq T^*(n).$$

Hence T^* is a majorant for T w.r.t. a.
General pseudocontractions T, even in Hilbert spaces other than $\mathbb{R}$, do not seem to be majorizable in general. However, in the applications in proof mining they have either been used under the additional assumption of being Lipschitzian ([89]) or as selfmaps $T : C \to C$ of bounded convex subsets $C \subseteq X$, where majorizability is trivial ([90, 88]). An exception is Theorem 2.3 in [90] where, however, the assumption of the existence of a fixed point of T implies the boundedness of the path considered.

For all the function classes above, except for the 'quasi'-classes, the conditions become purely universal once the appropriate data and moduli are added to the definition. If one generalizes the classes of '(firmly, strongly) quasi-nonexpansive' mappings to their 'weakly' versions, where the condition in question is not claimed to hold for all of their fixed points but only for some fixed point, then these conditions have the form Δ and so are also amenable in logical metatheorems. It has been for this reason why we considered the class of weakly quasi-nonexpansive functions in [75] and also the use of firmly quasi-nonexpansive and strongly quasi-nonexpansive mappings in [71] is of this form.

2.3 New proof principles treated by proof interpretations

In the applications of proof mining in nonlinear analysis (mainly fixed point theory) up to 2008, functional analytic tools for the space X in question were only needed to a rather limited extent. This changed in connection with applications to nonlinear ergodic theory ([68, 70, 79] and convex optimization (see chapter 8 in [86]), where one had to rely on new finitary quantitative versions of appropriate projection and weak compactness arguments as well as the elimination of Banach limits (see [66, 69, 79]).
With respect to metric projections P_C of X onto a nonempty convex closed subset C (in the case of uniformly convex spaces X) one has to distinguish between the cases where C is an abstract convex closed subset just axiomatized to be so (for the convexity of C this is an obvious universal axiom while the closedness can axiomatized in a purely universal way using the completion operator from [65], see [48]). Then a projection P_C can be added equally axiomatic by the then universal axiom

$$\forall x \in X\, \forall y \in C\, (\|x - P_C x\| \leq \|x - y\|).$$

Given a modulus η of uniform convexity for X one can compute a modulus of uniqueness for being a metric projection (see [65], Proposition 17.4) which in turn gives a modulus of uniform continuity for P_C and hence full extensionality can be derived. A majorant of P_C is given by the function $P^*_C(n) := 2n + m$, where $\mathbb{N} \ni m \geq \|c\|$ for some $c \in C$ since

$$\|P_C x\| \leq \|x - P_C x\| + \|x\| \leq \|x - c\| + \|x\| \leq 2\|x\| + \|c\|$$

(see also [48]).
The situation is different if C is given by some formula $\varphi(x)$ in the language of $\mathcal{A}^\omega[X, \ldots]$ (which - if it expresses an $=_X$-extensional property of points in C - will never be quantifier-free)

$$\forall x \in X\, (x \in C \leftrightarrow \varphi(x)).$$

Then the existence of the unique best approximation of $x \in X$ by an element in C

$$\forall x \in X\, \exists! y \in C\, \forall z \in C\, (\|x - y\| \leq \|x - z\|)$$

can be proven e.g. in $\mathcal{A}^\omega[X, \|\cdot\|, \eta, \mathcal{C}]$ (which adds an abstract η-uniformly convex Banach space X to $\mathcal{A}^\omega$) using countable choice $\mathrm{AC}^{0,X}$ for points in X (see [66], Proposition 3.2 and Remark 4). As a consequence of this, the monotone functional interpretation (of the negative translation) of this principle has a solution by a bar recursive functional in Spector's calculus $T + BR$ (obtained by majorizing the bar recursive solution of $\mathrm{AC}^{0,X}$, see [44]).
The existence of ε-best approximations

$$(+)\ \forall k \in \mathbb{N}\, \forall x \in X\, \exists y \in C\, \forall z \in C\, (\|x - y\| \leq \|x - z\| + 2^{-k})$$

can be shown even without the uniform convexity of X and the convexity of C and without $\mathrm{AC}^{0,X}$ using only induction (see [66] Proposition 3.1) which then has a monotone functional interpretation by terms in Gödel's T alone.

A typical application of this in fixed point theory is the following: let $C \subset H$ be a nonempty bounded convex closed subset of a Hilbert space H and $T : C \to C$ be a nonexpansive selfmap. By a classical result due to Browder, Göhde and Kirk (independently), T has a fixed point. Hence the fixed point set $Fix(T)$ is nonempty and easily shown to be again closed and convex (all this holds also in general uniformly convex Banach spaces). Hence for each $v_0 \in H$ there exists a unique fixed point $u \in Fix(T)$ whose distance is closest to v_0 among all fixed points. When this is used in a proof of some concrete statement, e.g. expressing that a certain iteration procedure converges to u, proof mining usually reveals that a quantitative ε-version of this projection statement is all that is needed for the quantitative analysis of the convergence proof. Functional interpretation leads to the following result used in [68][3] for the quantitative analysis of a nonlinear ergodic theorem due to Wittmann [130] and recently used again in [86] in the analysis of an algorithm due to Yamada [132] in convex optimization (for technical reasons it is convenient in Hilbert space to use the square of the norms in (+)):

Proposition 2.1 ([68])**.** *: Let H, C, v_0, T as above and $\mathbb{N} \ni d \geq diam(C)$. Let $\varepsilon \in (0,1], \Delta : C \times (C \to (0,1]) \to (0,1]$ and $V : C \times (C \to (0,1]) \to C$. Then one can construct $u \in C$ and $\varphi : C \to (0,1]$ such that*

$$(1)\ \|u - T(u)\| < \Delta(u, \varphi)$$

and

$$(2)\ \begin{cases} \|T(V(u,\varphi)) - V(u,\varphi)\| < \varphi(V(u,\varphi)) \to \\ \|v_0 - u\|^2 \leq \|v_0 - V(u,\varphi)\|^2 + \varepsilon. \end{cases}$$

In fact, u, φ can be defined explicitly as functionals in ε, Δ, V (as well as in v_0, T and some fixed point $p \in C$ of U which we, however, do not mention as arguments as these are fixed parameters) as follows: for $i < n_\varepsilon := \left\lceil \frac{d^2}{\varepsilon} \right\rceil$ we define $\varphi_i : C \to (0,1]$ and $u_i \in C$ inductively by

$$\begin{array}{l} \varphi_0(v) := 1,\ \varphi_{i+1}(v) := \Delta(v, \varphi_i), \\ u_0 := p \in Fix(U),\ u_{i+1} := V(u_i, \varphi_{n_\varepsilon - i - 1}). \end{array}$$

Then for some $i < n_\varepsilon$ (that we may find by bounded search, see Remark 2.5 in [68]) we have that $u := u_i,\ \varphi := \varphi_{n_\varepsilon - i - 1}$ satisfy the claim.

Let us briefly discuss the monotone version of the above statement which translates majorants for V, Δ, p, v_0 into majorants for u, φ. Since C is assumed to be bounded, majorants for V, u, p, v_0 are trivial, namely given simply by a bound on the norm of the elements in C, in fact, since the whole argument only involves C and not H, only a bound d on $diam(C)$ is needed. So it suffices to majorize φ given a majorant $\Delta^* \gtrsim \Delta$ for Δ, i.e. a $\Delta^* : \mathbb{N} \to \mathbb{N}$ s.t.

$$\forall n \in \mathbb{N}\, v \in C\, (\forall w \in C(1/n \leq \varphi(w)) \to 1/\Delta^*(n) \leq \Delta(v, \varphi)).$$

It is now easy to see that for all $w \in C$

$$1/(\Delta^*)^{(i)}(1) \leq \varphi_i(w)$$

[3] Correction to [68]: in lemma 3.1 '$4dn(8dn+2)$' instead of '$4dn(4dn+2)$'.

and so the solution φ in the above proposition is majorized by

$$\max\{\,(\Delta^*)^{(i)}(1) : i < n_\varepsilon\,\}.$$

In addition to the use of the metric projection onto the fixed point set of T, Wittmann's proof in [130] also makes use of a weak sequential compactness argument. However, at the very end of the complete logical analysis of Wittmann's proof as given in [68] what remains from this use is a trivial lemma (namely Lemma 2.13 in [68], see the discussion after Lemma 5.1 in the same paper). In particular, as the use of weak compactness is totally eliminated, there is no contribution of the enormous complexity of the functionals needed to interpret the monotone functional interpretation of sequential compactness. The latter functionals are computed in [69] and make use of two nested applications of bar recursion $B_{0,1}$ of lowest type (corresponding to the usual strong sequential compactness of an appropriate compact Polish space and the proof of the Riesz representation theorem resp. both of which are used in the proof of weak compactness). As follows from [95], this complexity is optimal. The solution to the weak convergence of bounded sequences in Hilbert spaces was then used in the logical analysis of a proof of the famous nonlinear ergodic theorem of Baillon ([70], see also section 3.2 below) where it is applied again twice in a row. Since this bounds on the weak Cauchy property for the nonlinear ergodic theorem is of type 2 and the uses of $B_{0,1}$ are in a context which otherwise is in T_0, i.e. the fragment of Gödel's T with primitive recursion of type $\mathbb{N}$ only, it follows from the detailed analysis of the type-2 functionals definable in $T_0 + B_{0,1}$ (using results due to W.A. Howard and H. Schwichtenberg as well as a normalization argument due to the present author [62]) that this bound can be restated as a functional in T. This time it has not been possible to eliminate the use of weak compactness. But note that Baillon's theorem itself is only a weak convergence result (and strong convergence is known to fail in general). In the famous special case of odd nonexpansive operators, where the convergence is again strong, one can indeed avoid the use of weak compactness (see [129]) and the proof-theoretic analysis of the corresponding proof then gives a primitive recursive (in the ordinary sense, i.e. in T_0,) rate of metastability ([116], see also section 3.2 below).

This leads to the question of whether one can isolate certain conditions that guarantee that a use of weak compactness can be eliminated from a proof of strong convergence from proofs that satisfy these conditions. Related to this is the interesting topic to give quantitative versions of so-called 'weak-to-strong' principles used in convex optimization to ensure strong convergence by suitable changes in only weakly convergent iterative algorithms (see e.g. [13]).

Proof-theoretic elimination techniques for the use of a non-principal ultrafilter $\mathcal{U}$ in favor of arithmetical comprehension have been developed in [94, 128]. Kreuzer [94] uses functional interpretation combined with a normalization argument to first reduce the use of $\mathcal{U}$ (in proofs of theorems of a suitable logical form) to the uniform arithmetical comprehension functional

$$(E^2) : \exists E : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}\, \forall f : \mathbb{N} \to \mathbb{N}\, (E(f) = 0 \leftrightarrow \exists x \in \mathbb{N}(f(x) = 0)).$$

By a method due to Feferman [36] (based again on functional interpretation and normalization), (E) is then further reduced to arithmetic comprehension. The latter has a functional interpretation by bar recursion of lowest type ([122], see [65]). Towsner [128] reduces $\mathcal{U}$ directly to arithmetic comprehension by a syntactic forcing translation.

As mentioned already above, the results in [49] suggest that many uses of ultrafilters made in connection with the definition of ultrapowers of metric and normed structures might be replaced by proof-theoretic techniques involving a suitable nonstandard uniform boundedness principle which in contrast to (non-principal) ultrafilters can never contribute to the complexity of extractable bounds.
Another use of ultrafilters is the construction of Banach limits as the limit along $\mathcal{U}$ of the Cesàro mean $x_1 + \ldots x_n/n$ of bounded sequences (x_n) in $\mathbb{R}$ (see [91], alternatively one can apply Hahn-Banach's theorem to l^∞). In [79, 80], we extracted rates of metastability (see section 2.6) for Halpern iterations in CAT(0) ([79]) and uniformly smooth Banach spaces ([80]) from given convergence proofs that made use of Banach limits. Here, however, special Banach limits are used that are shown to exist by the Hahn-Banach theorem applied to l^∞, namely - given a fixed sequence $(a_n) \in l^\infty$ - one uses a Banach limit $\mu : l^\infty \to \mathbb{R}$ that satisfies $\mu \leq q$ with $\mu((a_k)) = q((a_k))$, where

$$q : l^\infty \to \mathbb{R}, \quad q((a_k)) := \limsup_{p\to\infty} \sup_{n\geq 1} \frac{1}{p} \sum_{i=n}^{n+p-1} a_i$$

is a sublinear functional that can be defined by the comprehension functional (E^2). The proof is then modified so that instead of μ only q is used which then in turn is eliminated in terms of an elementary lemma on the finite averages $C_{n,p}((a_k)) = \frac{1}{p}\sum_{i=n}^{n+p-1} a_i$ with an at most simple polynomial contribution to the final extracted bound. In the addendum to [79], we observed that the analysis given in [79, 80] could be trivially seen to avoid this polynomial contribution altogether so that no trace of the use of Banach limits remains. Subsequently, [102] extended the results from [79] to the technically very involved case of CAT(κ)-spaces for $\kappa > 0$ (since a CAT(0) space is CAT(κ) for every $\kappa > 0$, this is an - in fact far reaching - generalization).

Very recently, we applied proof mining to Bauschke's solution ([12]) of the so-called 'minimal displacement conjecture' which is a very concrete asymptotic regularity statement for compositions of arbitrary metric projections onto closed and convex subsets in Hilbert space. The proof, however, uses a large arsenal of prima-facie very noneffective abstract operator theory: Minty's theorem (Zorn's lemma), Brézis-Haraux theorem, Rockafellar's maximal monotonicity and sum theorems, Bruck-Reich theory of strongly nonexpansive mappings, conjugate functions, normal cone operator etc. Nevertheless, the computational contribution of the use of these principles turned out to be of very low complexity resulting in a rate of convergence which is a simple polynomial in the data (see [72]). We, therefore, believe that the abstract theory of maximally monotone operators should be studied more systematically from the perspective of proof mining.

2.4 Hybrid interpretations of partially constructive proofs

The logical metatheorems developed in [63, 44, 65] apply to theories $\mathcal{A}^\omega[X, \ldots]$ that are based on classical logic. As a consequence of this, the formula $A_\exists$ in bound extractions from proofs of theorems

$$\forall T \in \mathcal{F}\, \exists n \in \mathbb{N}\, A_{\exists}(T, n)$$

must be purely existential. If, however, one uses intuitionistic logic instead then

1. A may be a formula of arbitrary logical complexity,
2. one may add the axiom of full extensionality to this intuitionistic system and
3. one may add highly noneffective principle such as the schema of comprehension in all types for arbitrary negated formulas (resulting in a theory that is proof-theoretically of the same strength as classical simple type theory but in contrast to the latter has he same provably recursive functions as intuitionistic arithmetic HA)

$$\mathrm{CA}_{\neg} : \exists \Phi \leq_{\rho \to \mathbb{N}} \lambda x^{\rho}.1 \forall y^{\rho} \left(\Phi(y) =_{\mathbb{N}} 0 \leftrightarrow \neg A(y) \right).$$

   Note that $\mathrm{CA}_{\neg}$ implies the law-of-excluded-middle schema for negated formulas.

The extraction technique is then based on a monotone modified realizability interpretation.

Remark 2.2. *Alternatively, one may use Markov's principle in all types and König's lemma* KL *(instead of* CA_{neg}*) but then - again - one has to replace full extensionality by the quantifier-free extensionality rule. Here one uses plain monotone functional interpretation, i.e. without any preceding negative translation.*

These and many related results are proved in [43] (for theories without abstract spaces X already in [61]) and have been adapted to UCW-spaces in [100]. In [100], Leuştean gives a very interesting proof analysis in the context of fixed point theory in UCW-spaces (namely the extraction of a so-called rate of asymptotic regularity for the Ishikawa iteration of nonexpansive mappings in such spaces) in which certain parts of the proof, that are basically constructive, are analyzed by the semi-constructive approach from [43] while other parts of the proof that use more heavily classical logic are interpreted by the methods for classical systems from [44]. In fact, it is shown that the analysis given in [99] can be logically understood as such a hybrid approach combining two different proof-theoretic methods.
Another application of such a hybrid approach in nonlinear analysis has recently been given in [120].
Together with general logical theorems on related hybrid proof interpretations due to [52, 109], this suggests that this approach has a large potential for further applications.

2.5 Alternative proof interpretation with potential in proof mining

There are several new forms of proof interpretations related to the Gödel functional interpretation which have been proposed in recent years for the use of analyzing proofs in analysis. Similarly to the monotone (see [60, 65]) and bounded (see [41] and, for the extension to abstract normed spaces, [35]) functional interpretations which only aim after bounds rather than exact realizers, these methods typically also extract only some weaker partial information that is relevant in the case at hand. E.g. in [6], a version of functional interpretation related to [5] is used to bound the ordinal level sufficient in the transfinite hierarchy of so-called distal factors used in their analysis of an ergodic-theoretic proof due to Furstenberg and Katznelson of a multidimensional Szemerédi theorem.

In [16], a functional interpretation of certain nonstandard extensions of systems of arithmetic and analysis in the language of functionals of all finite types is developed. The approach is inspired by Nelson's internal set-theory as the system is based on a unary predicate symbol for 'being standard'. Formulas that do not contain this symbol are called 'internal' and the interpretation acts trivially on those while it extracts finite lists of witnessing candidates for external quantifiers.

In [40], a bounded functional interpretation (in the sense of [41]) is developed for another nonstandard extension of finite type arithmetic in which the 'finite lists of witnessing candidates' extracted by [16] are replaced by extracting majorants for the external quantifiers.

Recent papers by Sanders (see e.g. [117]) suggest that the techniques from [16, 40] can be used to extract computational information from proofs in nonstandard analysis and that, in particular, [40] can be applied also to purely standard proofs by translating them appropriately into the nonstandard framework. It remains to be seen whether this approach may lead to new results when carried out in sufficiently nontrivial case studies.

2.6 Logical aspects of convergence statements

Let (x_n) be a Cauchy sequence in a complete metric space (X, d), i.e.

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \geq n\ (d(x_i, x_j) \leq \frac{1}{k+1}) \in \forall\exists\forall.$$

Often a computable rate of convergence does not exist even for computable sequences (x_n). In fact, as shown in [121], there is a primitive recursive decreasing sequence $(x_n) \subset [0,1] \cap \mathbb{Q}$ with no computable Cauchy rate. The noneffectivity of the Cauchy property of monotone bounded sequences of reals corresponds precisely to the law-of-excluded-middle-principle for Σ^0_1-formulas Σ^0_1-LEM (see [127]). In fact, if only weaker forms of LEM such as either LEM for arbitrary negated formulas or, alternatively, the Markov Principle plus the so-called LLPO-principle ('lesser-limited-principle-of-omniscience') are used (relative to a suitable intuitionistic framework) in a proof of a Cauchy statement, then effective rates of convergence can be extracted. For finite type systems over $\mathbb{N}$ this follows from Corollary 7.7 resp. Theorem 9.3 in [65]: note that the '$\exists$'-quantifier in the Cauchy property is monotone, i.e. any any upper bound is already a witness so that these bound extraction theorems are applicable (as already mentioned, LEM for negated formulas follows immediately from $\mathrm{CA}_\neg$ and LLPO follows from WKL which can be written as an axiom Δ). For some extensions to theories with abstract spaces X see [43]. So at least Σ^0_1-LEM is needed to create a noneffective Cauchy statement (for a computable sequence). Conversely, it has been shown in [85] that over rather general intuitionistic frameworks $\mathcal{A}_i[X, \ldots]$, proofs of Cauchy statements can be transformed so that **only** Σ^0_1-LEM is needed: this follows by showing that $\mathcal{A}_i[X, \ldots] + \Sigma^0_1$-LEM is closed under the Σ^0_2-DNE rule, where

$$\Sigma^0_2\text{-DNE}: \ \neg\neg\exists x \in \mathbb{N}\, \forall y \in \mathbb{N}\, A_{qf}(x, y, \underline{a}) \to \exists x \in \mathbb{N}\, \forall y \in \mathbb{N}\, A_{qf}(x, y, \underline{a}),$$

(A_{qf} quantifier-free), suffices to prove that the negative translation of the Cauchy statement implies the original Cauchy statement.
So the intuitionistic reverse mathematics for Cauchy statements that do not admit a computable rate of convergence essentially is trivial as they will in all practical cases be equivalent to Σ^0_1-LEM. This also applies to the corresponding convergence statements which will follow from the Π^0_3-Cauchy property by applying countable choice for numbers (to create a fast converging sequence and hence a limit) which usually is included in intuitionistic frameworks. Classically, this need to apply Π^0_1-choice for numbers to create a limit will (in connection with the fact that the Π^0_3-property implies Σ^0_1-LEM) result in arithmetical comprehension being implied by the convergence theorem (and conversely, arithmetical comprehension suffices to prove the convergence from the Cauchy property). So to say something specific about the computational content of a concrete (noneffective) convergence statement one has to investigate the numerical content of the Cauchy statement once the latter is (classically equivalent) reformulated in such a way that it has a computational solution: the Cauchy property **noneffectively** is equivalent to

$$(*)\ \forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \in \mathbb{N}\, \forall i,j \in [n; n+g(n)]\ (d(x_i,x_j) < \frac{1}{k+1}) \in \forall\exists$$

which is the Herbrand normal form of the original Cauchy formulation and which has been - in the specific situation at hand - called metastability by Tao [125, 126].

We call a bound $\Phi(k,g)$ on '$\exists n$' in the latter formula a rate of metastability (this is essentially the Kreisel no-counterexample interpretation of the Cauchy statement in the sense of [92, 93]).

Since $(*)$ is (equivalent to a formula) of the form $\forall\exists$, the logical metatheorems discussed in section 2 above can be used to extract highly uniform such rates of metastability whose subrecursive complexity reflects the specific computational content of the proof.

Usually, convergence theorems not only state the plain convergence of some sequence (x_n) in X but also that the limit $x := \lim x_n$ satisfies some property which often can be written in the form $F(x) =_{\mathbb{R}} 0$, where $F : X \to \mathbb{R}$ is continuous. Then a rate of metastability should satisfy

$$(**)\ \forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \Phi(k,g)\, \forall i,j \in [n; n+g(n)]\ \left(d(x_i,x_j), |F(x_i)| \leq \frac{1}{k+1} \right).$$

This formulation is significant for the following reasons:

- $(**)$ is purely universal and so is a **real statement** in the sense of Hilbert (the universal quantifier behind the bounded ones hidden in $\leq$ can be avoided by switching to appropriate rational approximations of $d(x_i,x_j), |F(x_i)|$).
- By negative translation, $(**)$ always has a **constructive** proof.
- By classical logic (and QF-AC$^{\mathbb{N},\mathbb{N}}$, i.e. closure under recursion), $(**)$ mathematically trivially implies (by a fixed piece of proof) that (x_n) is Cauchy.

- By arithmetical comprehension, $(**)$ mathematically trivially implies (by a fixed piece of proof) that (x_n) is convergent and $F(\lim x_n) = 0$.
- Under certain conditions (e.g. uniqueness or monotonicity properties), rates of metastability even yield rates of convergence.
- The structure of Φ yields information on the learnability of a convergence rate and sometimes oscillation bounds ([85]).

Let us discuss the last item in some more detail (the other items following already by the preceding discussion or being self-explanatory):

Definition 2.3 ([85], Definition 2.4)**.** Consider a Σ^0_2 formula $\varphi \equiv \exists n \in \mathbb{N} \forall x \in \mathbb{N}\, \varphi_{qf}(x, n, \underline{a})$ (with quantifier-free φ_{qf} and all its free variables contained in $\underline{a}$) which is monotone in n, i.e.

$$\forall n \in \mathbb{N} \forall n' \geq n \forall x \in \mathbb{N} \left(\varphi_{qf}\ (x, n, \underline{a}) \to \varphi_{qf}(x, n', \underline{a})\right).$$

We call such a formula φ *(B,L)-learnable*, if there are function(al)s B and L such that the following holds:

$$\exists i \leq B(\underline{a})\ \forall x \in \mathbb{N}\, \varphi_{qf}(x, c_i, \underline{a}),$$

where

$$c_0 := 0,$$

$$c_{i+1} := \begin{cases} L(x, \underline{a}), & \text{for the } x \text{ with } \neg\varphi_{qf}(x, c_i, \underline{a})\ \wedge\ \forall y < x\ \varphi_{qf}(y, c_i, \underline{a}) \text{ if it exists} \\ c_i, & \text{otherwise.} \end{cases}$$

In [85](Theorem 2.11) it is shown that if the number of (parallel) instances of Σ^0_1-LEM used in a proof of a Cauchy statement is (implicitly) bounded then one can extract concrete terms (B, L) from the proof. Moreover, Proposition 2.16 (together with Proposition 2.5) in [85] shows that the (B, L) learnability of a Cauchy rate with majorizable B, L implies that the Cauchy statement has a rate of metastability given by (essentially)

$$\Phi(k, g, \underline{a}^*) = \left((\lambda x^{\mathbb{N}}.L^*(x, k, \underline{a}^*)) \circ \tilde{g}\right)^{(B^*(k,\underline{a}^*))}(0),$$

where B^*, L^* are majorants of B, L, and $\underline{a}^*$ are majorants for the parameters $\underline{a}$ used to define the sequence in question and $\tilde{g}(n) := \max\{n, \max_{m\leq n}\{g(m)\}\}$. By Remark 2.17 in [85] also a kind of converse of this holds, i.e. given a rate of metastability of the above form, then (essentially) a Cauchy rate is (B^*, L^*) learnable. With the notable exceptions of the rates in [70, 116], all rates of metastability extracted so far have this simple structure and so give rise to explicit learnability information on the rate of convergence of the respective sequence. While this in general is not sufficient to infer an effective bound on the number of ε-fluctuations of the Cauchy statement (see Propositions 4.7 and 4.11 in [85] for a counterexample), the latter does follow if an additional gap condition is satisfied by L (Proposition 5.1 and Remark 5.2 in [85]). This e.g. is the case for the rate of metastability extracted for the von Neumann mean ergodic theorem in uniformly convex spaces in [77] and explains why this could be strengthened to a bound on the number of ε-fluctuations in [4].

3 Recent Applications to Nonlinear Analysis

3.1 Metric Fixed Point Theory

Metric fixed point theory has been the area to which the proof mining methodology has been applied most extensively since 2000. Even the results since 2010 are too many to mention all of them here. Instead we only focus on a few developments.

3.1.1 Rates of asymptotic regularity for families of mappings and strongly and firmly nonexpansive mappings

In [96], the following iteration schema is considered: Let C be a nonempty convex subset of a Banach space X and $\{T_i : 1 \leq i \leq k\}$ be a finite family of nonexpansive self-mappings $T_i : C \to C$. Let $U_0 = Id$ be the identity mapping and $0 < \lambda < 1$, then using the mappings

$$\begin{aligned} U_1 &= (1-\lambda)Id + \lambda T_1 U_0 \\ U_2 &= (1-\lambda)Id + \lambda T_2 U_1 \\ &\vdots \\ U_k &= (1-\lambda)Id + \lambda T_k U_{k-1}, \end{aligned}$$

one defines

$$x_0 \in C,\ x_{n+1} := (1-\lambda)x_n + \lambda T_k U_{k-1} x_n, \quad n \geq 0. \tag{1.1}$$

The following result is implicit in [96]:

Theorem 3.1. *Let X be strictly convex, $C \subseteq X$ nonempty compact and closed, $T_1, \ldots, T_k : C \to C$ nonexpansive and $F := \bigcap_{i=1}^k Fix(T_i) \neq \emptyset$. Then for the sequence defined above one has the following asymptotic regularity result:*

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \ \ \textit{for all}\ 1 \leq i \leq k.$$

In [54], a quantitative version of this theorem is obtained: in order to apply the logical metatheorems discussed in the introduction, strict convexity needs to be upgraded to uniform convexity. As shown in [54], the compactness assumption can then be dropped and one obtains a full rate of convergence in the above asymptotic regularity result even in the case of uniformly convex W-hyperbolic spaces (more precisely the UCW-spaces mentioned in section 2.1).

Theorem 3.2 ([54])**.** *Let C be a nonempty convex subset of a UCW-space with a monotone modulus of convexity η and let $\{T_i\}_{i=1}^k$ be a finite family of nonexpansive self-mappings $T_i : C \to C$ with $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let $p \in F, x_0 \in C$ and $D > 0$ be such that $d(x_0, p) \leq D$. Then for the sequence (x_n) generated by Kuhfittig's schema above we have*

$$\forall \varepsilon \in (0,2]\, \forall n \geq \Phi_i(D, \varepsilon, \lambda, \eta) \ \ (d(x_n, T_i x_n) \leq \varepsilon) \quad \textit{for}\ 1 \leq i \leq k,$$

where

$$\Phi_i := \theta\left(\widehat{\eta}^{(k-i+\min(1,k-1))}\left(\frac{\varepsilon}{2}\right)\right)$$

with

$$\theta(\varepsilon) := \left\lceil \frac{D}{\widehat{\eta}(\varepsilon)} \right\rceil, \text{ where } \widehat{\eta}(\varepsilon) := \lambda(1-\lambda)\eta\left(D, \frac{\varepsilon}{D+1}\right)\varepsilon.$$

If $\eta(r,\varepsilon)$ can be written as $\eta(r,\varepsilon) := \varepsilon\tilde{\eta}(r,\varepsilon)$, where $\tilde{\eta}$ increases with ε (i.e. $\forall \varepsilon_2 \geq \varepsilon_1 > 0$ $(\tilde{\eta}(r,\varepsilon_2) \geq \tilde{\eta}(r,\varepsilon_1))$, then we can replace η by $\tilde{\eta}$ in the bound: $\Phi_i(D,\varepsilon,\lambda,\tilde{\eta})$. Using $N \in \mathbb{N}$ with $1/N \leq \lambda(1-\lambda)$ one can replace in the above bound the dependency of the bound on λ by that on N.
In the special case of CAT(0)*-spaces one may take $\tilde{\eta}(\varepsilon) := \varepsilon/8$.*

Let us briefly discuss the logical reason why one obtains in this case a full rate of convergence. The key result in [96] is that any fixed point of the nonexpansive mapping $S := T_k U_{k-1}$ is a common fixed point of $T_1, \ldots, T_k$ which prenexes as follows

$$\forall q \in C\, \forall \varepsilon > 0\, \exists \delta > 0\ \underbrace{\big(d(Sq,q) \leq \delta \to \bigwedge_{i=1}^{k}(d(T_i q, q) < \varepsilon\big)}_{\in \Sigma^0_1}.$$

By an appropriate logical metatheorem, one then extracts a uniform (positive lower) bound (and hence witness) $\Psi(D,\varepsilon,N,\eta)$ for δ which only depends on ε, some bound $D \geq d(x_0,p)$ for some $p \in F$, a modulus η of uniform convexity for X and λ (actually only an $N \in \mathbb{N}$ such that $1/N \leq \lambda(1-\lambda)$ is needed). Since Kuhfittig's iteration schema (x_n) is nothing else but a Krasnoselski-Mann iteration of S (with constant λ) one can use a previously extracted rate $\Theta(D,\varepsilon,N,\eta)$ of asymptotic regularity for the latter from [98] and simply put

$$\Phi(D,\varepsilon,N,\eta) := \Theta(D, \Psi(D,\varepsilon,N,\eta), N, \eta).$$

The extraction of a full rate Θ is possible since the sequence $(d(x_n, Sx_n))_{n\in\mathbb{N}}$ is nonincreasing so that $d(x_n, Sx_n) \to 0$ is in Π^0_2.
For a different type of iteration, an explicit rate of asymptotic regularity for compositions for nonexpansive mappings in general classes of geodesic spaces is obtained using proof mining in [101].
For so-called strongly (quasi-)nonxpansive mappings (see [28] for the definition of 'strongly nonexpansive' and [27] for 'strongly quasi-nonexpansive') $T_1, \ldots, T_k : S \to S$ with $\bigcap_{i=1}^k Fix(T_i) \neq \emptyset$, where $S \subseteq X$ is an arbitrary subset of a metric space X it is known ([27]) that $Fix(T) = \bigcap_{i=1}^k Fix(T_i)$, where $T := T_k \circ \ldots \circ T_1$.
In [71], we extracted from the proof of this fact a uniform bound Ψ such that

$$\forall \varepsilon > 0\ \left(d(Tx,x) \leq \Psi(\varepsilon) \to \bigwedge_{i=1}^{k} d(T_i x, x) < \varepsilon\right)$$

which, in addition to ε, only depends on k, a common modulus of strong quasi-nonexpansiveness, a bound $d \geq d(x,p)$ for some $p \in \bigcap_{i=1}^k Fix(T_i)$ and a common modulus of uniform continuity α_d for $T_1, \ldots, T_k$ on $S_d := \{y \in S : d(y,p) \leq d\}$ ([71], Proposition 4.15). Moreover, we extracted a rate of asymptotic regularity Φ for $(x_n := T^n x)$ in the case where the $T_1, \ldots, T_k$ are nonexpansive, in addition to being strongly quasi-nonexpansive, where Φ, in addition to ε, only depends on k, a common modulus of strong quasi nonexpansiveness and a bound $d \geq d(x,p)$ ([71], Theorems 4.6,4.7). Put together, this yields:

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(\Psi(\varepsilon)) \, \left(\bigwedge_{i=1}^{k} d(T_i x_n, x_n) < \varepsilon \right).$$

In UCW-spaces, so-called firmly nonexpansive mappings due to [25] are strongly quasi-nonexpansive and nonexpansive (in uniformly convex Banach spaces even strongly non-expansive) and so these results apply to the firmly nonexpansive mappings. The latter contain in the context of CAT(0)-spaces (and so in particular in Hilbert spaces) all metric projections onto closed convex subsets as well as resolvents of convex lower semicontinuous mappings (in Hilbert spaces even of general maximally monotone operators). Thus the most important mappings used in convex optimization are firmly nonexpansive. In [1], explicit rates of asymptotic regularity for the Picard iteration $T^n x$ of firmly nonexpansive mappings in UCW-space are extracted which become quadratic in the CAT(0)-case. In [71] we re-proved these rates as instances of the more general results for strongly quasi-nonexpansive mappings discussed above. The latter class of mappings covers even metric projections in CAT(κ)-spaces for $\kappa > 0$ which no longer are firmly nonexpansive and, in fact, not even nonexpansive. One then, however, can still obtain metastable versions of the above results which suffices to obtain rates of metastability for $(T^n x)$ in the compact case, where T is the composition of metric projections onto closed convex sets that have a nonempty intersection, in the context of CAT(κ)-spaces for $\kappa > 0$ (see section 3.1.2 below). This situation is studied in convex optimization under the label of convex feasibility problems and we will comment further on this in section 3.3.1.

Whereas strong nonexpansivity in uniformly Banach spaces is implied by being firmly non-expansive, this is false in general Banach spaces where these two concepts are independent. Asymptotic regularity of firmly nonexpansive mappings in general Banach spaces was first established in [113]. In [108], this is generalized to general W-hyperbolic spaces and an exponential rate of asymptotic regularity is extracted:

Theorem 3.3 ([108])**.** *Let X be a W-hyperbolic space and $C \subseteq X$ be a nonempty bounded subset and let b be an upper bound on the diameter of C. Let $T : C \to C$ be λ-firmly nonexpansive, i.e. (with $(1-\lambda)a \oplus \lambda b := W(a,b,\lambda)$)*

$$d(Tx,Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty), \quad \textit{for all } x,y \in C,$$

for some $\lambda > 0$ and $N \in \mathbb{N}$ with $N \geq 1/\lambda$. Then

$$\forall x \in C \, \forall \varepsilon > 0 \, \forall n \geq \Phi(\varepsilon, N, b) \, \left(d(T^n x, T^{n+1} x) \leq \varepsilon \right),$$

where

$$\Phi(\varepsilon, N, b) := M \left\lceil \frac{2b(1+e^{NM})}{\varepsilon} \right\rceil \quad \textit{with } M := \left\lceil \frac{4b}{\varepsilon} \right\rceil.$$

3.1.2 Rates of metastability for strong convergence theorems based on Fejér monotonicity

In this section we give a brief account of some of the results in [81]. In the following, (X,d) is a metric space.

Definition 3.4 ([81])**.** *1. Let $F_k \subseteq X$ and define $F := \bigcap_{k\in\mathbb{N}} F_k$. Points of $AF_k := \bigcap_{i\le k} F_i$ are called approximate F-points.*

2. A sequence (x_n) in X has approximate F-points if

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\ (x_n \in AF_k).$$

3. F is explicitly closed (w.r.t. (F_k)) if ($\overline{B}(p,\varepsilon)$ is the closed ε-ball with center p)

$$\forall p \in X\ \ (\forall N, M \in \mathbb{N}(AF_M \cap \overline{B}(p, 1/(N+1)) \neq \emptyset) \to p \in F).$$

The canonical examples one has in mind here are the following:

- **Fixed point sets:** Let $C \subseteq X$ and $T : C \to C$ and define

$$F_k := \left\{ x \in C\ :\ d(x,Tx) \le \frac{1}{k+1} \right\}.$$

  Then the F- (resp. AF_k)-points are the T-fixed points (resp. $1/(k+1)$-approximate fixed points of T). Note that if T is continuous, then F is always explicitly closed.

- **Zero sets of maximally monotone operators:** Let X be a real Hilbert space and (for $\gamma > 0$) $J_\gamma A := (Id + \gamma A)^{-1}$ be the resolvent of γA for a maximally monotone operator $A : X \to 2^X$. Let (γ_n) be a sequence in $(0,\infty)$ and define

$$F_k := \bigcap_{i\le k} \left\{ x \in H\ :\ \|x - J_{\gamma_i A} x\| \le \frac{1}{k+1} \right\}.$$

  Then $F = zer(A) := \{x \in X\ :\ 0 \in Ax\}$ (see also section 3.3.3 below for the significance of this example).

Definition 3.5. *$(x_n) \subset X$ is called Fejér monotone w.r.t. $F (\neq \emptyset)$ if*

$$\forall n \in \mathbb{N}\, \forall p \in F\ \ (d(x_{n+1}, p) \le d(x_n, p)).$$

Remark 3.6. [81] *actually considers a much more general form of Fejér monotonicity. In this definition one can also incorporate error terms $\delta_n \ge 0$ with $\sum \delta_n < \infty$: quasi-Fejér monotonicity.*

Proposition 3.7 ([81])**.** *Let X be a* **compact** *metric space and F be explicitly closed. If $(x_n) \subset X$ has approximate F-points and is Fejér monotone, then it converges to a point $x \in F$.*

The proof uses that sequences in X have convergent subsequences. Using results due to [107] it follows that for most of the usual iterations (x_n), the convergence (Cauchyness) above already in the case $X = [0,1]$ implies the convergence (Cauchyness) of monotone sequences in (x_n) and hence ACA (Σ^0_1-LEM). So effective rates of convergence are largely ruled out.

The main result of [81] is a quantitative metastable version of the above theorem. In order to state this we first have to introduce the appropriate uniform quantitative versions of the concepts involved in the above proposition:

$$\begin{aligned} \text{compactness} &\rightarrow \text{modulus } \gamma \text{ of total boundedness} \\ \text{explicit closedness} &\rightarrow \text{moduli } \omega, \delta \text{ of uniform closedness} \\ \text{approximate } F\text{-points} &\rightarrow \text{approximate } F\text{-point bound } \Phi \\ \text{Fejér monotonicity} &\rightarrow \text{modulus } \chi \text{ of uniform Fejér monotonicity.} \end{aligned}$$

We now give the definition of these concepts:

Definition 3.8 ([81]). *1. $\gamma : \mathbb{N} \to \mathbb{N}$ is a modulus of total boundedness for X if for all $k \in \mathbb{N}$ and any sequence (x_n) in X*

$$\exists i < j \leq \gamma(k) \left(d(x_i, x_j) \leq \frac{1}{k+1} \right).$$

2. F (more precisely (F_k)) is uniformly closed with moduli $\delta_F, \omega_F : \mathbb{N} \to \mathbb{N}$ if

$$\forall k \in \mathbb{N}\, \forall p, q \in X \left(q \in AF_{\delta_F(k)} \;\wedge\; d(p,q) \leq \frac{1}{\omega_F(k)+1} \to p \in AF_k \right)$$

(compare $(*)$ in section 2.2).

3. Let (x_n) be a sequence with approximate F-points. $\Phi : \mathbb{N} \to \mathbb{N}$ is an approximate F-bound bound for (x_n) if it is nondecreasing and

$$\forall k \in \mathbb{N}\, \exists N \leq \Phi(k)\ (x_N \in AF_k).$$

(x_n) has the $\liminf$-property with bound $\Phi : \mathbb{N}^2 \to \mathbb{N}$ if $\Phi(k,n)$ is monotone in both k, n and

$$\forall k, n \in \mathbb{N}\, \exists N \leq \Phi(k,n)\ (N \geq n \wedge x_N \in AF_k).$$

4. (x_n) is uniformly Fejér monotone w.r.t. F (more precisely: w.r.t. (F_k)) with modulus $\chi : \mathbb{N}^3 \to \mathbb{N}$ if for all $m, n, r \in \mathbb{N}$

$$\forall p \in X \left(p \in AF_{\chi(n,m,r)} \to \forall i \leq m \left(d(x_{n+i}, p) < d(x_n, p) + \frac{1}{r+1} \right) \right).$$

Theorem 3.9 ([81]). *Let X be totally bounded with modulus γ, (x_n) be uniformly Fejér monotone w.r.t. F with modulus χ and let (x_n) have approximate F-points with bound Φ. Then (x_n) is Cauchy and for all $k \in \mathbb{N}$ and all $g : \mathbb{N} \to \mathbb{N}$:*

$$\exists N \leq \Psi \; \forall i, j \in [N, N + g(N)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where $\Psi(k, g, \Phi, \chi, \gamma) := \Psi_0(P)$ with

$$\begin{aligned} &\Psi_0(0) := 0, \\ &\Psi_0(n+1) := \Phi\left(\chi_g^M(\Psi_0(n), 4k+3)\right), \end{aligned}$$

$$\chi_g^M(n, k) := \max_{i \leq n}\{\chi(i, g(i), k), \;\; P := \gamma(4k+3).$$

Additional results:

- If F is additionally uniformly closed with moduli δ_F, ω_F then

  $$\exists N \leq \tilde{\Psi} \, \forall i, j \in [N, N + g(N)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \text{ and } x_i \in AF_k \right),$$

  where $\tilde{\Psi}$ results from Ψ by replacing k and χ by

  $$k' := \max\{k, \lceil((\omega_F(k) - 1)/2\rceil\} \text{ and } \chi'(n, m, r) := \max\{\delta_F(k), \chi(n, m, r)\}.$$

- Theorem 3.9 can be adapted to uniformly quasi-Fejér monotone sequences if Φ is a lim inf-bound.

Using these quantitative results, rates of metastability have been obtained in the following situations (the first item was already obtained earlier and served as motivation for the general approach):

- Krasnoselski-Mann iterations of asymptotically nonexpansive mappings in uniformly convex spaces ([64]).
- Picard iterations of firmly nonexpansive mappings in uniformly convex W-hyperbolic ('UCW'-)spaces ([81, 1]).
- Ishikawa iterations of nonexpansive mappings in UCW-spaces ([81, 99]).
- Mann iterations of strict pseudo-contractions in Hilbert spaces ([81, 53], see also section 3.1.3 below).
- The proximal point algorithm for the zeroes of maximally monotone operators in Hilbert space ([81]; see also section 3.3.3 below).
- Mann iterations of mappings satisfying condition (E) ([81]).
- Convex feasibility problems in CAT(κ) spaces ([71]).
- Minimization problems for two maps ([82], see also section 3.3.2 below).

3.1.3 Rates of asymptotic regularity and metastability for pseudocontractive mappings

An important generalization of the class of nonexpansive mappings are the so-called pseudocontractive mappings that were introduced by Browder:

Definition 3.10 ([22])**.** *Let X be a normed linear space and $C \subseteq X$ be a nonempty convex subset of X. A mapping $T : C \to C$ is called pseudocontractive if it satisfies*

$$\forall u, v \in C\, \forall \lambda > 1\ ((\lambda - 1)\|u - v\| \leq \|(\lambda Id - T)(u) - (\lambda Id - T)(v)\|),$$

where Id denotes the identity mapping.

In a real Hilbert space X this is equivalent to

$$\forall u, v \in C\, (\langle Tu - Tv, u - v\rangle \leq \|u - v\|^2)$$

which in turn is equivalent to

$$\forall u, v \in C\, (\|Tu - Tv\|^2 \leq \|u - v\|^2 + \|u - Tu - (v - Tv)\|^2).$$

In [89], a rate of asymptotic regularity is extracted from a convergence proof due to [30] for an iteration schema due to [26] for Lipschitzian pseudocontractive mappings in general Banach spaces. In the case where C is bounded, the rate is polynomial. In the situation where X is a real Hilbert space, a rate of metastability for the strong convergence of that iteration is obtained in [90]. Finally, analyzing a proof in [26], Körnlein [88] recently extracted a rate of metastability for pseudocontractions that are only assumed to be demicontinuous.
Whereas pseudocontractive mappings (already on $\mathbb{R}$) in general are not continuous, the smaller class of strict pseudocontractions (introduced by Browder and Petryshyn in [24]) is even Lipschitzian:

Definition 3.11. *Let X be a real Hilbert space and $C \subseteq X$ be nonempty and convex. Let $T : C \to C$ and $0 \leq \kappa < 1$. Then T is called a κ-strict pseudocontraction if*

$$\forall u, v \in C\, (\|Tu - Tv\|^2 \leq \|u - v\|^2 + \kappa\|u - Tu - (v - Tv)\|^2).$$

In [53], the following rate of asymptotic regularity is extracted for Krasnoselski-Mann iterations

$$x_0 := x, \quad x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T x_n$$

of κ-strict pseudocontractions in real Hilbert spaces (see also [119] for related results):

Theorem 3.12 ([53])**.** *Let C be additionally bounded and $b \geq diam(C)$. If (λ_n) is a sequence in $(\kappa, 1)$ satisfying $\sum_{n=0}^{\infty}(\lambda_n - \kappa)(1 - \lambda_n) = \infty$ with rate of divergence $\theta : \mathbb{N} \to \mathbb{N}$, then*

$$\Phi(k, b, \theta) = \theta\left(\lceil b^2\rceil (k + 1)^2\right)$$

is a rate of asymptotic regularity for (x_n), i.e.

$$\forall k \in \mathbb{N}\, \forall n \geq \Phi(k, b, \theta)\ \left(\|x_n - Tx_n\| \leq \frac{1}{k + 1}\right).$$

3.2 Nonlinear Ergodic Theory

The classical von Neumann mean ergodic theorem, in the formulation due to Riesz, states that the sequence $(x_n)_{n\geq 0}$ of Cesàro means

$$x_n := \frac{1}{n+1}\sum_{i=0}^{n} T^i x$$

of a linear nonexpansive selfmapping $T : X \to X$ of a Hilbert space X starting from $x \in X$ strongly converges. In [77], a rate of metastability is extracted for a generalization of this result to uniformly convex Banach spaces from a proof due to G. Birkhoff which - as mentioned already at the end of section 2.6 - was subsequently improved to a bound on the number of ε-fluctuations in [4].

If the linearity of T is dropped then one has (in the case of Hilbert spaces) weak convergence by the famous Baillon nonlinear ergodic theorem ([8], also for $T : C \to C$, where $C \subseteq X$ is a closed convex subset) but strong convergence is known to fail.

In [70] we extracted from an alternative proof of Baillon's theorem in [18] an explicit rate φ of metastability for the weak Cauchy property of (x_n) (here $\mathbb{N} \ni b \geq \|x\|$)

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \forall w \in B_1(0)\, \exists n \leq \varphi(\varepsilon, b, g)\, \forall i, j \in [n; n + g(n)]\, \big(|\langle x_i - x_j, w\rangle| < \varepsilon\big)$$

which, however, is too complicated to state here in detail but whose complexity was already discussed in section 2.3.

In order to get strongly convergent nonlinear ergodic theorems one either has

(i) to add some weak form of linearity of T, e.g. being odd, or

(ii) to change the form of the sequence (x_n), where in the presence of full linearity of T this new sequence, nevertheless, coincides with the Cesàro means.

That (for symmetric C) the assumption of T being odd (in addition to being nonexpansive) implies the strong convergence of the sequence of Cesàro means was again shown by Baillon ([9]) and much later generalized by Wittmann in [129] to the situation, where K is an arbitrary subset of X and $T : K \to K$ any (not even necessarily continuous) mapping satisfying the already mentioned condition

$$(W) :\ \forall x, y \in K\ (\|Tx + Ty\| \leq \|x + y\|)$$

which is implied (for symmetric K) if T is nonexpansive and odd.

In [116], a rate of metastability is extracted from Wittmann's proof:

Theorem 3.13 ([116]). *Let $K \subseteq X$ be any subset of a Hilbert space and $T : K \to K$ any mapping satisfying the condition (W). Then the sequence (x_n) of Cesàro means of T starting from $x \in K$ with $\|x\| \leq b$ strongly converges with rate of metastability*

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists m \leq \Phi(k, b, g^M)\, \forall i, j \in [m, m + g(m)]\ (\|x_i - x_j\| \leq 2^{-k}),$$

where

$$\begin{aligned}
&\Phi(k,b,g) := (N(2k+7,g)+P(2k+7,g))\cdot b\cdot 2^{2k+8}+1,\\
&P(k,g) := P_0(k,F(k,g,N(k,g))),\\
&F(k,g,n)(p) := p+n+\tilde{g}((n+p)\cdot b\cdot 2^{k+1}),\\
&L(k,g)(n) := n+P_0(k,F(k,g,n))+\tilde{g}((n+P_0(k,F(k,g,n)))\cdot b\cdot 2^{k+1}),\\
&N(k,g) := (L(k,g))^{(b^2 2^{k+2})}(0),\\
&P_0(k,f) := \tilde{f}^{(b^2 2^k)}(0),\ \tilde{f}(n) := n+f(n),\ f^M(n) := \max_{i\le n+1} f(i).
\end{aligned}$$

For other metastability results for iterations of odd operators see [67, 71].

Instead of adding a weak form of linearity one may also change the sequence (x_n) in the nonlinear case to achieve strong convergence: let us consider the so-called Halpern iteration of $T: C\to C$

$$x_0 := x, \quad x_{n+1} := \lambda_{n+1}u + (1-\lambda_{n+1})Tx_n$$

with starting point $x\in C$ and the anchor $u\in C$, where $(\lambda_n)_{n\ge 1}$ is a sequence in $[0,1]$ (Halpern [50] only considered the case $u:=0$). In a celebrated paper [130], Wittmann for first time proved the strong convergence of (x_n) (X Hilbert space, $C\subseteq X$ closed and convex, $T: C\to C$ nonexpansive with $Fix(T)\neq\emptyset$ and $u=x$) under conditions on (λ_n) that permit the case $\lambda_n := 1/(n+1)$. With this choice of (λ_n), the sequence of Halpern iterates coincides with the sequence of Cesàro means if T is assumed to be linear. In [68], we extracted a rate of metastability for this strong convergence from Wittmann's original proof making use of the quantitative form of the projection to the fixed point set of T discussed already in section 2.3. [79] extracted such a rate from a rather different proof due to [115] for a generalization of Wittmann's result to CAT(0) spaces X (and general $u,x\in C$). For the aforementioned choice $\lambda_n := 1/(n+1)$ and bounded C the result is:

Theorem 3.14 ([79])**.** *Let (X,ρ) be a* CAT(0) *space, $C\subseteq X$ be convex, $diam(C)\le M$, (x_n) as above and $\varepsilon\in(0,1)$. Then (x_n) is strongly convergent with rate of metastability:*

$$\forall g:\mathbb{N}\to\mathbb{N}\,\exists k\le\Sigma(\varepsilon,g,M)\,\forall i,j\in[k,k+g(k)]\ \big(\rho(x_i,x_j)\le\ \varepsilon\big),$$

where[4]

$$\begin{aligned}
&\Sigma(\varepsilon,g,M) := \left\lceil\tfrac{12(M^2\chi^*_L(\varepsilon^2/12)+1)}{\varepsilon^2}\right\rceil-1,\ \textit{with } L := \widetilde{h^*}^{(\lceil M^2/\varepsilon_0^2\rceil)}(0)+\left\lceil\tfrac{1}{\varepsilon_0}\right\rceil,\\
&\chi^*_k(\varepsilon) := \left\lceil\tfrac{12M^2(k+1)}{\varepsilon}+\tfrac{288M^4(k+1)^2}{\varepsilon^2}\right\rceil-1,\ \varepsilon_0 := \varepsilon^2/24(d+1)^2,\\
&\Theta_k(\varepsilon) := \left\lceil\tfrac{3M^2(\chi^*_k(\varepsilon/3)+1)}{\varepsilon}\right\rceil-1,\ \Delta^*_k(\varepsilon,g) := \tfrac{\varepsilon}{3g_{\varepsilon,k}\left(\Theta_k(\varepsilon)-\chi^*_k(\varepsilon/3)\right)},\\
&g_{\varepsilon,k}(n) := n+g\left(n+\chi^*_k\left(\tfrac{\varepsilon}{3}\right)\right),\ h(k) := \max\left\{\left\lceil\tfrac{M^2}{\Delta^*_k(\varepsilon^2/4,g)}\right\rceil,k\right\}-k,\\
&h^*(k) := h\left(k+\left\lceil\tfrac{1}{\varepsilon_0}\right\rceil\right)+\left\lceil\tfrac{1}{\varepsilon_0}\right\rceil,\ \widetilde{h^*}(k) := k+h^*(k).
\end{aligned}$$

[4]Correction to [79]: on p.2534, line 6, 'ε' in the condition on D_k must be '$\varepsilon/2$' and hence the factor '2304' on line 19 (resp. '144' on line 1 of p.651 in the addendum to [79]) must be '4608' (resp. '288'). Also on p.2534, line 19 the factor '$48M$' must be '$48M^2$' and on p.2543, line 2 '$16M^2$' must be '$32M^2$'.

Further consequences of the analysis:

1. A quadratic rate of convergence for the asymptotic regularity $\rho(x_n, T(x_n)) \to 0$:

$$\forall \varepsilon > 0\, \forall n \geq \Psi(\varepsilon, M) := \frac{4M}{\varepsilon} + \frac{32M^2}{\varepsilon^2}\ (\rho(x_n, T(x_n)) \leq \varepsilon).$$

   This rate can be easily combined with the rate of metastability Σ from the previous theorem: define for $g : \mathbb{N} \to \mathbb{N}$ and $N \in \mathbb{N}$ a new function $g_N(n) := g(n+N)+N$ and put

$$\Sigma'(\varepsilon, g, M) := \Sigma(\varepsilon, g_{\Psi(\varepsilon,M)}, M) + \Psi(\varepsilon, M),$$

   then

$$\forall g \in \mathbb{N}^{\mathbb{N}}\, \exists k \leq \Sigma'(\varepsilon, g, M)\, \forall i, j \in [k, k+g(k)]\, \forall l \geq k\, \big(\rho(x_i, x_j), d(x_l, T(x_l)) \leq \varepsilon\big).$$

2. Let z_k^u be the unique fixed point of the contraction

$$T_k(x) := \frac{1}{k}u \oplus (1 - \frac{1}{k})T(x).$$

   Then the analysis yields primitive recursively in a given rate α of convergence of the resolvent (z_k^u) a rate of convergence of (x_n).

Similar results for so-called modified Halpern iterations are obtained in [118].

A highly nontrivial extension of the analysis to the much more general case of CAT(κ) spaces (with $\kappa > 0$) is given in [102].

In [80], a rate of metastability of (x_n) for the case of uniformly smooth Banach spaces is given relative to rate of metastability for the resolvent sequence (z_k^u) which is assumed to exist. Whereas, such a rate for the latter has been computed in [68] for Hilbert spaces and in [79] for CAT(0)-spaces, it is subject of ongoing research to achieve this even for L^p-spaces $(1 < p < \infty)$ other than L^2. The uniformly smooth case is also treated in [87] under a somewhat different set of conditions on (λ_n) due to [131] which also include the case $\lambda_n := 1/(n+1)$.

3.3 Convex Optimization

In this section we discuss some recent applications of proof mining in the area of convex optimization.

3.3.1 Convex feasibility problems

Let X be a real Hilbert space and $P_i : X \to C_i$ be the metric projections onto the closed and convex subsets $C_1, \ldots, C_r \subseteq X$ with $C := \bigcap_{i=1}^r C_i \neq \emptyset$.

The so-called convex feasibility problem (also called 'image recovery problem') is to find a point $p \in C$ in the 'image set' C.

For $1 \le i \le r$, define $T_i := Id + \lambda_i(P_i - Id)$ for $0 < \lambda_i \le 2$, $\lambda_1 < 2$ and put $T := \sum_{i=1}^{r} \alpha_i T_i$, where $\alpha_1, \ldots, \alpha_r \in (0,1)$ with $\sum_{i=1}^{r} \alpha_i = 1$.

By a result of Crombez [32, 33] one has $Fix(T) = C$ if C is nonempty. Moreover, T is asymptotically regular, i.e. $\|T^{n+1}x - T^n x\| \to 0$.

Define towards an ε-approximate version of the convex feasibility problem

$$C_{i,\varepsilon} := \bigcup\nolimits_{x \in C_i} B_\varepsilon(x),\ C_\varepsilon := \bigcap\nolimits_{i=1}^{r} C_{i,\varepsilon}.$$

By analyzing the proof of the (nontrivial) inclusion $Fix(T) \subseteq C$ one can extract a bound (compare with the discussion after Theorem 3.2) $\delta(D,\varepsilon) > 0$ such that (for $x \in X, p \in C$ and $D \in \mathbb{N}$ with $D \ge ||x - p||$)

$$\forall \varepsilon \in (0,1)\ (\|Tx - x\| \le \delta(D,\varepsilon) \to x \in C_\varepsilon)$$

(see Theorem 3.1(i) in [55]).

This bound is then combined with a rate of asymptotic regularity (Theorem 2.3 in [55]) to finally obtain the following quantitative solution of the approximate convex feasibility problem:[5]

Theorem 3.15 ([55])**.** *Let $x_0 \in X$ and $D > \|x_0 - p\|$ for some $p \in C$ and $N_1, N_2 \in \mathbb{N}$ s.t.*

$$\frac{1}{N_1} \le \min\{\alpha_i\lambda_i : 1 \le i \le r\},\ \frac{1}{N_2} \le \min\{\alpha_1, 2 - \lambda_1\}.$$

Then for $x_n := T^n x_0$ one has:

$$\forall \varepsilon \in (0,1)\, \forall n \ge \Psi(D, N_1, N_2, \varepsilon)\ (x_n \in C_\varepsilon),$$

where

$$\Psi(D, N_1, N_2, \varepsilon) := \left\lceil \frac{1936 \cdot N_1^6 \cdot (D+1)^4 (4N_1+1)^2 \cdot (2N_2+1)^2}{\pi \cdot \varepsilon^4} \right\rceil.$$

A similar result also holds for uniformly convex Banach spaces X, where then, however, one has to restrict λ_i to the interval $(0,1)$.

For quantitative versions of convex feasibility problems obtained in the context of CAT(κ) spaces ($\kappa > 0$) via general proof mining results for iterations of compositions of strongly quasi-nonexpansive mappings (as discussed in section 3.1.1), see [71].

3.3.2 Quantitative results for the composition of two mappings

Whereas in the convex feasibility problems discussed in the previous section one assumes that the intersection of the convex sets is nonempty and so that a common fixed point of the respective projections exists, the significance of the next theorem is that it only assumes the existence of a fixed point of the composition of two mappings:

[5] Correction to [55]: in Corollary 4.3(i),(ii) replace '$F(T) \neq \emptyset$' by '$C_0 \neq \emptyset$' and drop the dummy argument 'd' in Ψ.

Theorem 3.16 ([2])**.** *Let X be a* CAT(0)*-space and $T_1, T_2 : X \to X$ satisfying the condition (which for Hilbert spaces X is equivalent to being firmly nonexpansive)*

$$(P): \ 2d(T_ix, T_iy)^2 \le d(x, T_iy)^2 + d(y, T_ix)^2 - d(x, T_ix)^2 - d(y, T_iy)^2.$$

Let $Fix(T_2 \circ T_1) \neq \emptyset$ and consider sequences $(x_n), (y_n)$ in X with

$$d(y_n, T_1x_n) \le \varepsilon_n \text{ and } d(x_{n+1}, T_2y_n) \le \delta_n, \text{ for all } n \in \mathbb{N},$$

where $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \delta_n < \infty$. Then

$$\lim d(y_{n+1}, y_n) = \lim d(x_{n+1}, x_n) = 0.$$

The proof makes repeated use of the convergence of bounded monotone sequences in $\mathbb{R}$ and hence of arithmetical comprehension.

Motivation behind the theorem:

Consider two proper, convex and lower semi-continuous $f, g : X \to (-\infty, +\infty]$ and define (see [14] for the study of this problem in the context of Hilbert spaces and [11] for the generalization to CAT(0)-spaces)

$$\Phi(x, y) := f(x) + g(y) + \frac{1}{2\lambda} d(x, y)^2.$$

Then the resolvents $T_1 = J^g_\lambda, T_2 = J^f_\lambda$ of f, g satisfy (P), where

$$J^g_\lambda(x) := \underset{z \in X}{\operatorname{argmin}} \left[g(z) + \frac{1}{2\lambda} d(x, z)^2 \right]$$

(see [7] for the study of the resolvents in the context of CAT(0)-spaces).

Computing sequences $(x_n), (y_n)$ as above (which only requires to know the resolvents up to some error) provides ε-solutions for the minimization problem

$$\underset{(x,y) \in X \times X}{\operatorname{argmin}} \ \Phi(x, y).$$

For this particular case, a quadratic rate of asymptotic regularity is extracted in [2] in the absence of error terms (i.e. $\delta_n = \varepsilon_n = 0$) and extended to the situation with error terms in [82] (see Remark 3.4 in that paper).

In the general situation of Theorem 3.16 one has an exponential bound:

Theorem 3.17 ([82])**.** *Let α be a Cauchy-rate for $\sum_{n=0}^{\infty} \gamma_n$, where $\gamma_n := \varepsilon_n + \delta_n$ and let $B \in \mathbb{N}, b > 0$ be such that $\sum \gamma_n \le B$ and $d(x_0, u) \le b$ for some $u \in Fix(T_2 \circ T_1)$.*
Then

$$\forall n \ge \Phi(\varepsilon, b, B, \alpha) \ (d(x_n, x_{n+1}) \le \varepsilon),$$

where

$$\Phi(\varepsilon, b, B, \alpha) := \alpha(\varepsilon/3) + k \left\lceil \frac{12(1 + 2^k)(b + B)}{\varepsilon} \right\rceil + 1, \ k := \left\lceil \frac{12(b + B)}{\varepsilon} \right\rceil.$$

Similarly for (y_n) with $\Phi'(\varepsilon, b, B, \alpha) := \Phi(\varepsilon/2, b, B, \alpha)$.

The sequences $(x_n), (y_n)$ are easily be seen to be uniformly quasi-Fejér monotone. Hence using Theorem 3.9 (adapted to quasi-monotonicity) one obtains rates of metastability in the case where X is totally bounded:

Theorem 3.18 ([82]). *Let X additionally be totally bounded with a modulus γ. Then*

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Psi(k,g) \forall i,j \in [n, n+g(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where

$$\Psi(k,g) := \Psi_0(P),\ P := \gamma(8k+7)+1,\ \xi(k) := \alpha(1/(k+1)),$$
$$\chi_g^M(n,k) := (\max_{i\leq n} g(i)) \cdot (k+1),$$

and using ($\widehat{\Phi}(k,N) := \max\{N, \Phi(1/2(i+1)) : i \leq k\}$ with Φ from Theorem 3.17)

$$\Psi_0(0) := 0,\ \Psi_0(n+1) := \widehat{\Phi}\left(\chi_g^M\left(\Psi_0(n), 8k+7\right), \xi(8k+7)\right).$$

A similar bound holds for (y_n).

Note that asymptotic regularity in the above situation is just the special case of metastability when $g(n) := 1$ for all $n \in \mathbb{N}$. The fact that the proof of asymptotic regularity does not use the total boundedness of X is reflected by the above rate of metastability: if $g(n)$ does not depend on n, then also $\chi_g^M(n,k)$ does not depend on n and so the recursive definition of $\Psi(n)$ becomes constant from $n = 1$ on. Hence the bound does not depend on the modulus of total boundedness γ (as this only enters into P) and the bound (essentially) collapses to the rate of metastability Φ from Theorem 3.17.

3.3.3 Proximal point algorithm

Let X be real Hilbert space and $A : X \to 2^X$ be a maximally monotone, $J_{\gamma A} = (Id + \gamma A)^{-1}$ be the resolvent of γA for $\gamma > 0$ and $(\gamma_n) \subset (0, \infty)$.
The famous proximal point algorithm (due in this setting to [114] but formulated in the important case of resolvents of convex lower semi-continuous functions already in [104]) is given by (see the 2nd example after Definition 3.4 above)

$$x_0 \in X, \qquad x_{n+1} := J_{\gamma_n A} x_n.$$

Proposition 3.19 ([81]).

1. *(x_n) is uniformly Fejér monotone (w.r.t. $F = \bigcap_{k\in\mathbb{N}} F_k$, where $F_k := \bigcap_{i\leq k} \left\{ x \in H \ :\ \|x - J_{\gamma_i A} x\| \leq \frac{1}{k+1} \right\}$) with modulus*

$$\chi(n,m,r) := \max\{n+m-1, m(r+1)\}.$$

2. *If $\sum \gamma_n^2 = \infty$ with rate of divergence θ, then*

$$\Phi_{b,\theta}(k) := \theta(\lceil b^2(M_k+1)^2 \rceil) \lceil b^2(M_k+1)^2 \rceil - 1,$$

where $M_k := \lceil (k+1)(2 + \max_{0\leq i\leq k} \gamma_i) \rceil - 1$ and $\|x_0 - p\| \leq b$ for some $p \in zer(A)$, is an approximate F-point bound for (x_n).

Hence, Theorem 3.9 can be applied to obtain a rate of metastability in the finite dimensional case.

3.3.4 The hybrid steepest descent method

Let X be real Hilbert space and consider a mapping $\Theta : X \to \mathbb{R}$. The goal is to solve $\min \Theta$ over a closed convex subset $S \subseteq X$.

Let the gradient $\mathcal{F} := \Theta'$ of Θ be κ-Lipschitzian and η-strongly monotone and $S = Fix(T)$ for some nonexpansive $T : X \to X$. Then the above goal is equivalent to solving the following variational inequality problem:

$$\text{VIP: Find } u^* \in S \text{ s.t. } \langle v - u^*, \mathcal{F}(u^*)\rangle \geq 0 \text{ for all } v \in S.$$

In [132], Yamada showed that under suitable conditions on (λ_n) the scheme (with $\mu := \eta/\kappa^2$)

$$u_{n+1} := T(u_n) - \lambda_{n+1}\mu\mathcal{F}T(u_n)$$

converges strongly to a solution of VIP.

Very recently, Körnlein [86] extracted an explicit and highly uniform effective rate of metastability for (u_n). In fact, [86] does this also for a generalization to finite families of mappings $T_1, \ldots, T_N : C \to C$ (that is also considered by [132]) and for general τ-contractions $\mathcal{G} : C \to C$ (with $\tau \in (0,1)$ and C closed and convex, instead of $\mathcal{G} := Id - \mu\mathcal{F}$ only)

$$(*)\ u_{n+1} := (1 - \lambda_{n+1})T_{[n+1]}(u_n) + \lambda_{n+1}\mathcal{G}T_{[n+1]}(u_n),$$

where $[n]$ is the 'modulo N' function (note that, for $N = 1$, this schema has also been known as Moudafi's viscosity method, see [106]). The conditions on (λ_n) are those considered in [130] which allow for the choice of $\lambda_{n+1} := 1/(n+1)$ (see the discussion before Theorem 3.14). [106] (and also Yamada [132] in his first proof for the case $N = 1$) used stronger conditions which do not permit this choice. However, in his proof for general N, which even for $N = 1$ is different from his first proof, Yamada needs only the Wittmann conditions (and also the viscosity method has later been studied under these conditions by various authors).

3.4 Nonlinear Semigroups and Abstract Cauchy Problems

3.4.1 Proof Mining in nonlinear semigroup theory

Let X be a Banach space, $C \subseteq X$ a nonempty convex subset and $\lambda \in (0,1)$.

Definition 3.20. *A family $\{T(t) : t \geq 0\}$ of nonexpansive mappings $T(t) : C \to C$ is a nonexpansive semigroup if*

(i) $T(s+t) = T(s) \circ T(t) \quad (s,t \geq 0)$,
(ii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous.

Theorem 3.21 ([123])**.** *Let $0 < \alpha < \beta$ be such that α/β is irrational. Then any fixed point $p \in C$ of*

$$S := \lambda T(\alpha) + (1-\lambda)T(\beta)$$

is a common fixed point of $T(t)$ for all $t \geq 0$, i.e. (note that trivially '$\supseteq$')

$$Fix(S) = \bigcap_{t \geq 0} Fix(T(t)).$$

Suzuki's proof uses weak König's lemma WKL in the form that a continuous function $[0, M] \to \mathbb{R}$ on a compact interval $[0, M]$ attains its maximum. General logical metatheorems for the extraction of uniform bounds become applicable once we assume additionally that $t \mapsto T(t)x$ is equicontinuous on norm-bounded subsets of X with a modulus $\omega : \mathbb{N}^3 \to \mathbb{N}$. This condition (which is usually satisfied in practice) is only needed to make the bound Φ (discussed below) independent of the point $p \in C$. Let $f_\gamma : \mathbb{N} \to \mathbb{N}$ be a modulus of irrationality (called effective irrationality measure in number theory) for $\gamma := \alpha/\beta$, $\Lambda, N, D \in \mathbb{N}$ be s.t. $1/\Lambda \leq \lambda, 1-\lambda$ and $1/N \leq \beta \leq D$. Then one can extract a bound $\Phi(\varepsilon, M, d) := \Phi(\varepsilon, M, d, N, \Lambda, D, f_\gamma, \omega)$ s.t. for all $M, d \in \mathbb{N}$:

$$\forall p \in C \, \forall \varepsilon > 0 \, \big(\|p\| \leq d \wedge \|S(p) - p\| < \Phi(\varepsilon, M, d) \to \forall t \in [0, M] \, (\|T(t)p - p\| < \varepsilon)\big).$$

Let $x_{n+1} := \frac{1}{2}x_n + \frac{1}{2}Sx_n$ be a d-bounded Krasnoselski iteration of S with rate of asymptotic regularity $\Psi(\varepsilon, d)$, then

$$\forall n \geq \Psi(\Phi(\varepsilon, M, d), d) \; \forall t \in [0, M] \, (\|T(t)x_n - x_n\| < \varepsilon).$$

In the case at hand, the optimal rate Ψ is known. Combining this with the explicitly extracted Φ the following final rate is obtained:

Theorem 3.22 ([74])**.** *Under the previous assumptions:*

$$\forall M \in \mathbb{N} \; \forall m \in \mathbb{N} \; \forall n \geq \Omega(m, M, d) \, \forall t \in [0, M] \;\; (\|T(t)x_n - x_n\| < 2^{-m})$$

with

$$\Omega(m, M, d) = \frac{2^{2m+8} d^2 ((\sum_{i=1}^{\phi(k, f_\gamma)-1} \Lambda^i + 1)(1 + MN))^2}{\pi},$$

where $d \geq \|x_0 - Sx_n\|, \|x_n\|$ for all n, $k := D2^{\omega_{D,b}(3+[\log_2(1+MN)]+m)+1}$ and

$$\phi(k, f) := \max\{2f(i) + 6 \;\; : \;\; 0 < i \leq k\}.$$

Example: $\alpha = \sqrt{2}, \beta = 2, \lambda = 1/2$. Then we may take $\Lambda = 2, N = 1, D = 2, f_\gamma(p) = 4p^2$.

Remark 3.23. *A bound d on either of the sequences $(\|x_0 - Sx_n\|), (\|x_n\|)$ can be transformed into one of the other: let $b \geq \|x_n\|$ for all $n \in \mathbb{N}$. Then for $B \geq \|x_0 - Sx_0\|$ and using that $(\|x_n - Sx_n\|)$ is nonincreasing one gets that*

$$\|x_0 - Sx_n\| \leq \|x_0 - x_n\| + \|x_n - Sx_n\| \leq 2b + B.$$

Conversely, $b \geq \|x_0 - Sx_n\|$ for all $n \in \mathbb{N}$ implies for $B' \geq \|x_0\|$

$$\|x_n\| \leq \|x_0\| + \|x_0 - Sx_n\| + \|Sx_n - x_n\| \leq \|x_0\| + \|x_0 - Sx_n\| + \|Sx_0 - x_0\| \leq b + B + B'.$$

3.4.2 Cauchy problems and set-valued accretive operators

Let X be a real Banach space and let $D(A)$ denote the domain of a set-valued operator A. $A : D(A) \to 2^X$ is accretive if

$$\forall (x,u),(y,v) \in A \ \big(\langle u-v, x-y\rangle_+ \geq 0\big),$$

where $\langle y,x\rangle_+ := \max\{\langle y,j\rangle : j \in J(x)\}$ for the normalized duality map J of the Banach space X.

A with $0 \in Az$ is uniformly accretive at zero with modulus $\Theta : \mathbb{N}^2 \to \mathbb{N}$ if, moreover,

$$\forall k, K \in \mathbb{N}\, \forall (x,u) \in A \left(\|x-z\| \in [2^{-k}, K] \to \langle u, x-z\rangle_+ \geq 2^{-\Theta_K(k)}\right)$$

([73]). E.g. this holds for m-ψ-strongly accretive operators or even for ϕ-accretive operators in the sense of García-Falset [38] if ϕ has some normal form (which is the case in many applications).

Consider the following homogeneous Cauchy problem for an accretive A (satisfying the so-called range condition):

$$(1) \ \begin{cases} u'(t) + A(u(t)) \ni 0, & t \in [0,\infty) \\ u(0) = x_0, \end{cases}$$

which has a unique integral solution for $x_0 \in \overline{D(A)}$ given by the Crandall-Liggett formula

$$u(t) := S(t)(x_0) := \lim_{n\to\infty} (Id + \frac{t}{n}A)^{-n}(x_0).$$

A continuous $v : [0,\infty) \to \overline{D(A)}$ is an almost-orbit of the nonexpansive semigroup S if

$$\lim_{s\to\infty} \big(\sup_{t\in[0,\infty)} \|v(t+s) - S(t)v(s)\|\big) = 0.$$

Theorem 3.24 ([37]). *Let A be a ϕ-accretive at zero operator with the range condition s.t.* (1) *has a strong solution for each $x_0 \in D(A)$. Then every almost-orbit (for the semigroup S generated by $-A$) strongly converges to the zero z of A.*

Theorem 3.25 ([73]). *Same as above but A uniformly accretive at zero with modulus Θ. Then*

$$\forall k \in \mathbb{N}\, \forall \overline{g} : \mathbb{N} \to \mathbb{N}\, \exists \overline{n} \leq \Psi\, \forall x \in [\overline{n}, \overline{n} + \overline{g}(\overline{n})] \left(\|v(x) - z\| < 2^{-k}\right),$$

where

$$\begin{aligned} &\Psi(k, \overline{g}, B, \Phi, \Theta) := \Phi(k+1, g) + h(\Phi(k+1,g)), \text{ with} \\ &h(n) := (B(n)+2)\cdot 2^{\Theta_{K(n)}(k+2)+1}, \ g(n) := \overline{g}(n+h(n)) + h(n), \\ &K(n) := \left\lceil \sqrt{2(B(n)+1)} \right\rceil, \ B(n) \geq \tfrac{1}{2}\|v(n)-z\|^2 \ \ (B(n) \textit{ nondecreasing}), \end{aligned}$$

and Φ is rate of metastability for v, i.e.

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k,g)\, \forall t \in [0, g(n)] \ (\|v(t+n) - S(t)v(n)\| \leq 2^{-k}).$$

Consider now the nonhomogeneous Cauchy problem (A as before):

$$(2) \begin{cases} u'(t) + A(u(t)) \ni f(t), & t \in [0, \infty) \\ u(0) = x, \end{cases}$$

where $f \in L^1(0, \infty, X)$.
Then for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of (2) is an almost-orbit ([105]).

Proposition 3.26 ([73])**.**

$$\Phi_M(k, g) := \tilde{g}^{(M \cdot 2^{k+1})}(0)$$

with

$$\tilde{g}(n) := n + g(n), \ M \geq \int_0^\infty \|f(\xi)\| d\xi$$

is a rate of metastability of u (and so can be used as Φ in the previous theorem).

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