PROOF THEORY AND NONSMOOTH ANALYSIS

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Abstract. We develop a general proof-theoretic framework for various classes of set-valued operators, including maximally as well as cyclically monotone and rectangular operators and we discuss a treatment for sums of set-valued operators A, B in that context such that all of the previous fits into logical metatheorems on bound extractions. In particular, we introduce quantitative forms for A being (weakly) uniformly rectangular with witnessing moduli. Based on this we give quantitative forms of the Brezis-Haraux theorem which use such moduli as input. It turns out that a modulus for weak uniform rectangularity, which can be extracted even from noneffective proofs of rectangularity, is sufficient while the bound gets simpler in the case of a modulus for A being uniform rectangular which can be extracted from semi-constructive proofs. We use our results to explain recent proof minings in the context of Bauschke's solution to the zero displacement conjecture and its extensions to other classes of functions than metric projections as instances of logical metatheorems.

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1. Introduction

During the past 15-20 years, proof-theoretic methods have been used to extract explicit effective and uniform bounds from numerous proofs in mainly nonlinear analysis ('proof mining', see e.g. [18] and - for a more recent survey - [21]). In most of the resulting papers abstract metric structures are used alongside with concrete represented Polish spaces such as $\mathbb{R}$ or $C[0,1]$. General logical metatheorems have been established which explain why these bounds only depend on general metric upper estimates on the data from the abstract metric structures even in the absence of any compactness assumptions (see e.g. [10, 12, 17, 18]). This is due to the fact that the proofs being analyzed do not use the separability of these structures.

In recent years, some case studies have been carried out which make use of set-valued monotone or accretive operators on Hilbert or more general Banach spaces and their resolvents (see e.g. [22, 23, 24]) and use properties such as being maximally monotone, cyclic or rectangular and facts like the fundamental theorem of Minty for maximally monotone operators or the Brezis-Haraux theorem for rectangular operators (as in [20, 34]).

The case studies were successful in extracting low complexity uniform bounds even though the existing frameworks of proof mining did not include set-valued operators, general resolvents and the tools to prove the aforementioned theorems. A first important step to extend things to this setting was done by the second author in [31], where (maximally) monotone and accretive operators together with their resolvents were axiomatized in an admissible way so that the existing logical metatheorems can be adapted.

In this paper, building upon [31], we for the first time include the theory of cyclic and rectangular operators in this setting of logical metatheorems, add suitable axioms for treating the sum $A+B$ of two monotone operators A, B as well as for treating conditions such as $\mathrm{dom}A \subseteq \mathrm{dom}B$ with necessary moduli such that all the previous is admissible in these extraction theorems. All this then plays a crucial role in a quantitative treatment of the Brezis-Haraux theorem which in turn depends on appropriate quantitative moduli witnessing the property of an operator being rectangular. W.r.t. the latter we introduce two new notions of being 'uniformly rectangular' and 'weakly uniformly rectangular' with respective moduli. If A is provably rectangular, then one can always extract a modulus for A being weakly uniformly rectangular and, if the proof can be carried out in some semi-constructive framework, one can even extract a modulus for A being uniformly rectangular. We then establish two quantitative versions of the Brezis-Haraux theorem. The first one uses that one has a modulus for A being uniformly rectangular which is the case in the recent applications [20, 34] and explains these applications as instances of a general logic metatheorem. The second quantitative version only uses that one has a modulus of A being weakly uniformly rectangular on the expense of a more complicated bound but is applicable even in

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situations where the property of A being rectangular is only established noneffectively and may pave the way towards new applications.

2. Logical systems and bound extraction theorems for monotone operators

To give a formal framework for the treatment of set-valued operators and resolvents, we follow the approach taken in [31]. We focus on the new aspects of the logical systems introduced there and rely on the formal setup presented in detail in [18] for background, in particular on the there presented language and systems of finite type arithmetic[1] and the representation of real numbers in them by objects of type $\mathbb{N}^{\mathbb{N}} = (\mathbb{N} \to \mathbb{N})$, resulting in respective formulas $<_{\mathbb{R}}, =_{\mathbb{R}}$ and so on. In particular, let $\mathcal{A}^\omega = \text{WE-PA}^\omega + \text{DC} + \text{QF-AC}$ be weakly extensional classical analysis formulated in the language of functionals of finite type with the axiom schemas of dependent choice and of quantifier-free choice in all types and $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$ be its extension with a new base type X with operations $+_X, -_X, \cdot_X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X$ for a real abstract inner product space together with corresponding characterizing axioms, both as defined in [17] (see also [18]).

Remark 2.1. As shown in [18], one can also add the assumption that X is complete (and hence a Hilbert space) to $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$ in a way so that the logical metatheorems on uniform bound extractions from [18] apply and this also holds for the extensions considered in [31] and this paper, but we refrain from doing so since completeness is mostly not needed in the context of proof mining which exhibits quantitative results on approximations.

A set-valued operator $A : X \to 2^X$ (which may be identified with a subset of $X \times X$) is treated via introducing a constant for its characteristic function χ_A which we consider to be of type $X \to (X \to \mathbb{N})$. In that vein, we write $u \in Ax$ or $(x, u) \in \text{gra}A$ as an abbreviation for $\chi_A(x, u) =_{\mathbb{N}} 0$. In that language, monotonicity of A is immediately expressed via the universal axiom

$$\forall x^X, y^X, u^X, v^X\, (u \in Ax \wedge v \in Ay \to \langle x -_X y, u -_X v\rangle_X \geq_{\mathbb{R}} 0)\,.$$

Further, the resolvent $J^A_\gamma := (Id + \gamma A)^{-1}$ of a monotone operator A is a single-valued function $J^A_\gamma : \text{ran}(Id + \gamma A) \to \text{dom}(A)$ and is treated via a constant J^{χ_A} of type $\mathbb{N}^{\mathbb{N}} \to (X \to X)$ with the parameter of type $\mathbb{N}^{\mathbb{N}}$ representing the real index $\gamma > 0$ in J^A_γ. If $\text{ran}(Id + \gamma A)$ is strictly contained in X, we call J^A_γ partial.

2.1. Maximal monotone operators.

We treat maximal operators by employing Minty's theorem:

Theorem 2.2 (essentially [28])**.** *Let X be a (real) Hilbert space. Then $A : X \to 2^X$ is maximally monotone if and only if its resolvent $J^A_\gamma := (Id+\gamma A)^{-1}$ is single-valued, firmly nonexpansive and $\text{dom}(J^A_\gamma) = \text{ran}(Id+\gamma A) = X$ for some (or any) $\gamma > 0$.*

So we can avoid the non-universal maximality statement by instead posing totality of the resolvent (see [31] for more background information on all of this). Since J^{χ_A} is of type $\mathbb{N}^{\mathbb{N}} \to (X \to X)$, any interpretation of the constant J^{χ_A} is total just by the type and this provides the basis for our treatment of operators with total resolvents: If we can provide a suitable resolvent axiom expressing the defining equality $J^A_\gamma := (Id + \gamma A)^{-1}$, then the interpretation of this constant will be forced to be a total resolvent which in turn forces the operator to be maximally monotone by Minty's theorem.

Following [31], one suitable choice for such an axiom is

$$\forall \gamma^{\mathbb{N}\to\mathbb{N}}, x^X \left(\gamma >_{\mathbb{R}} 0 \to \gamma^{-1}(x -_X J^A_\gamma x) \in A(J^A_\gamma x)\right)$$

which is essentially an intensional variant of one direction of the resolvent equality. In particular, this axiom is universal. The other direction follows from the provable uniqueness of J^A_γ, see [31, Proposition 3.5].

The system for a monotone operator $A : X \to 2^X$ with total resolvents over an inner product space X then takes the following form:

Definition 2.3. The system $\mathcal{T}^\omega$ is the extension of $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$ with the previously discussed constants together with the following axioms:

(I) $\forall x^X, y^X (\chi_A xy \leq_{\mathbb{N}} 1)$,
(II) $\forall \gamma^{\mathbb{N}\to\mathbb{N}}, x^X \left(\gamma >_{\mathbb{R}} 0 \to \gamma^{-1}(x -_X J^A_\gamma x) \in A(J^A_\gamma x)\right)$,
(III) $\forall x^X, y^X, u^X, v^X\, (u \in Ax \wedge v \in Ay \to \langle x -_X y, u -_X v\rangle_X \geq_{\mathbb{R}} 0)$.

[1] Throughout, we write $\mathbb{N}$ for the base type and given types ρ, τ, we denote their function type by $\rho \to \tau$.

Axiom (I) formalizes the fact that χ_A codes a characteristic function and, further, immediately results in χ_A being majorizable. Axiom (III) is the immediate formalized version of monotonicity for A mentioned before. For further discussions on the intuition and particularities of this axiomatization, we refer to [31].

Remark 2.4. For technical reasons (see [31]), the system $\mathcal{T}^\omega$ actually contains three additional constants and a further axiom: $\tilde{\gamma}$ of type $\mathbb{N} \to \mathbb{N}$, $m_{\tilde{\gamma}}$ of type $\mathbb{N}$ and c_X of type X as well as the axiom

(IV) $\tilde{\gamma} \geq_{\mathbb{R}} 2^{-m_{\tilde{\gamma}}}$.

Their purpose lies in the majorization of the constant J^{χ_A}: bounding $\left\|x - J^A_\gamma x\right\|$ for *some* x and *some* $\gamma > 0$ suffices to majorize J^{χ_A} and $c_X, \tilde{\gamma}$ designate such an arbitrary point and index. We use $m_{\tilde{\gamma}}$ to stipulate $\tilde{\gamma} > 0$ in a universal way via the axiom (IV). In our paper we will, however, always use $\tilde{\gamma} := 1, m_{\tilde{\gamma}} := 0$ and mostly set $c_X := 0_X$ (with some necessary exceptions in the context of the systems for operators with partial resolvents introduced later, see Remark 2.6).

By $\mathcal{T}^\omega_-$ we denote the fragment of $\mathcal{T}^\omega$ where DC is dropped.

Our system $\mathcal{T}^\omega$ and all its variants and extensions based on classical logic only contain a weak quantifier-free extensionality rule instead of the full extensionality axiom (see [10, 17, 18] for extensive discussions on this). For the operator A this means that only from a proof of $F_{qf} \to s_1 =_X t_1 \wedge s_2 =_X t_2$ we can conclude that $F_{qf} \to ((s_1, t_1) \in \mathrm{gra}A \to (s_2, t_2) \in \mathrm{gra}A)$, where F_{qf} is a quantifier-free formula. By the full extensionality of A we mean that $x_1 =_X x_2 \wedge y_1 =_X y_2 \to ((x_1, y_1) \in \mathrm{gra}A \to (x_2, y_2) \in \mathrm{gra}A)$ which is not provable in our systems.

Although totality of the resolvent as well as its extensionality (both in x w.r.t. $=_X$ as well as in $\gamma > 0$ w.r.t. $=_{\mathbb{R}}$) is provable in $\mathcal{T}^\omega$ (see [31, Proposition 3.1]), the maximality statement remains unprovable as it turns out to be equivalent to the extensionality statement of A:

Theorem 2.5 ([31])**.** *Over $\mathcal{T}^\omega$, the following are equivalent:*

(1) Extensionality of A, i.e.

$$\forall x_1^X, x_2^X, y_1^X, y_2^Y \; (x_1 =_X x_2 \wedge y_1 =_X y_2 \to \chi_A x_1 y_1 =_{\mathbb{N}} \chi_A x_2 y_2).$$

(2) Maximal monotonicity of A, i.e.

$$\forall x^X, u^X \left(\forall y^X, v^X \left(v \in Ay \to \langle x -_X y, u -_X v\rangle_X \geq_{\mathbb{R}} 0\right) \to u \in Ax\right).$$

This in particular has an impact on the quantitative treatment of proofs using the maximality statement for A in an essential way (e.g. by using that the graph of a maximally monotone operator is closed) as then a quantitative version of the extensionality statement for A has to be included, e.g. via the inclusion of a modulus of uniform continuity for A (as e.g. defined in [29], see in particular the discussions in [24, 30, 31]).

Akin to [9], one can also introduce a 'semi-constructive' version $\mathcal{T}^\omega_i$ of the above system $\mathcal{T}^\omega$ which can be defined as the extension, in the spirit of the above, not of the system $\mathcal{A}^\omega$ as in the case of $\mathcal{T}^\omega$ but of $\mathcal{A}^\omega_i = \text{E-HA}^\omega + \text{AC}$ defined as in [9]. This resulting system $\mathcal{T}^\omega_i$, potentially extended with a large class of non-constructive axioms, allows for bound extraction results which lift the severe restrictions one has to place on the proof and theorem if we would be working in the classical system $\mathcal{T}^\omega$. This will be discussed in more detail later on.

2.2. **Monotone operators with partial resolvents.** We modify the previous approach to treat non-maximal operators. This requires us to treat partial resolvents which can be achieved as follows: By definition, the domain of the resolvent fulfills

$$\mathrm{dom}J^A_\gamma = \mathrm{ran}(Id + \gamma A)$$

and thus inclusion in $\mathrm{dom}J^A_\gamma$ can be existentially expressed via

$$x \in \mathrm{dom}J^A_\gamma := \exists y^X \left(\frac{1}{\gamma}(x -_X y) \in Ay\right).$$

Now, we modify the previous resolvent axiom (II) so that it just specifies the behavior of the resolvent on the points in the domain:

$$\text{(II}'\text{)} \qquad \forall \gamma^{\mathbb{N}\to\mathbb{N}}, x^X \left(\gamma >_{\mathbb{R}} 0 \wedge x \in \mathrm{dom}J^A_\gamma \to \gamma^{-1}(x -_X J^A_\gamma x) \in A(J^A_\gamma x)\right).$$

The systems $\mathcal{T}^\omega_p / \mathcal{T}^\omega_{i,p}$ are derived from $\mathcal{T}^\omega / \mathcal{T}^\omega_i$, respectively, by replacing the previous (II) with the above (II$'$).

Remark 2.6. Again for technical reasons, the constant c_X previously used to specify an arbitrary but fixed point in the space for majorization is now actually assumed to specify a common element of the domains of all J^A_γ which we treat by adding another axiom:

$$\forall \gamma^{\mathbb{N}\to\mathbb{N}}(\gamma >_{\mathbb{R}} 0 \to \gamma^{-1}(c_X -_X J^A_\gamma c_X) \in A(J^A_\gamma c_X)). \tag{V}$$

2.3. **Majorizable operators.** In many situations occurring in nonlinear and nonsmooth analysis (see [7]), and in the theory of set-valued operators in particular, specific functions selecting a certain element from Ax for $x \in \mathrm{dom}A$ appear.

One specific and motivating example for such a *selection functional* is the operator $A^\circ x := P_{Ax}(0)$ selecting the element of minimal norm from Ax (which is well-defined for maximally monotone operators over Hilbert spaces, see Proposition 20.36 and Theorem 3.16 in [2]). In particular $A^\circ x \in Ax$ for any $x \in \mathrm{dom}A$. This operator is central, e.g., for the formulation of the dual version of the well-known Bundle method (see e.g. [33]), selecting elements of minimal norm from (subsets of) generalized subgradients.

For a general function $a : X \to X$, being a selection functional for A is characterized by the following axiom

$$\forall x^X(x \in \mathrm{dom}A \to ax \in Ax) \tag{NE}$$

and such an a then requires majorizing data if used in proofs which are to be analyzed using the bound extraction theorems. Here, a majorant for a functional $a : X \to X$ is a function $f : \mathbb{N} \to \mathbb{N}$ which is non-decreasing and satisfies

$$n \geq \|x\| \to fn \geq \|ax\|.$$

We consider the following definition from [31]:

Definition 2.7. An operator $A : X \to 2^X$ is called *majorizable* if there exists a choice for a satisfying (NE) which is majorizable.

In the sense of the discussion above, this notion of majorizability for A is the minimal setup for treating proofs involving selection functionals for the operator A and, in particular, A is majorizable in this sense if, and only if, A° is majorizable.
Corresponding to this, we consider the following stronger notion of majorizability from [31]:

Definition 2.8. An operator $A : X \to 2^X$ is called *uniformly majorizable* if there exists an $A^* : \mathbb{N} \to \mathbb{N}$ majorizing any functional a satisfying (NE) with $ax = 0$ for $x \notin \mathrm{dom}A$.

This notion will feature later in the discussion on formal treatments of the Brezis-Haraux theorem but can also be recognized as a proof-theoretical version of a ubiquitous analytical notion from monotone operator theory:

Proposition 2.9 ([31])**.** *An operator A is uniformly majorizable if and only if A is* bounded on bounded sets*, i.e.*

$$A(\overline{B}_r(0)) = \bigcup_{x\in\overline{B}_r(0)} Ax \text{ is bounded for any } r > 0.$$

2.4. **Treating sums of monotone operators in the logical metatheorems.** The sum $A + B$ of two monotone operators A, B can be defined in terms of its graph by stipulating that

$$(w, h) \in \mathrm{gra}(A + B) \leftrightarrow \exists h_1 \in Aw, h_2 \in Bw(h =_X h_1 +_X h_2).$$

In order for this to be an admissible axiom, one needs to bound the norms of h_1, h_2 in terms of norm bounds on w, h.
To be more precise, if two operators $A, B : X \to 2^X$ are added to our formal systems, we obtain the extension of the system by $A + B$ when adding (in addition to χ_{A+B} representing the characteristic function of $\mathrm{gra}(A + B)$) a constant $\xi : \mathbb{N} \to \mathbb{N}$ and the two axioms

$$\forall (w, h_1) \in \mathrm{gra}A, (w, h_2) \in \mathrm{gra}B\, ((w, h_1 + h_2) \in \mathrm{gra}(A + B)) \tag{i}$$

and

$$\begin{cases} \forall n^{\mathbb{N}}, (w, h) \in \mathrm{gra}(A + B)\, (\|w\|_X, \|h\|_X <_{\mathbb{R}} n \\ \quad \to \exists h_1^X, h_2^X (\|h_1\|_X, \|h_2\|_X \leq_{\mathbb{R}} \xi(n) \wedge (w, h_1) \in \mathrm{gra}A \wedge (w, h_2) \in \mathrm{gra}B \wedge h =_X h_1 +_X h_2)). \end{cases} \tag{ii}$$

Such a ξ is e.g. available if at least one of A or B, say B, is uniformly majorizable, say by B^*: for $n \geq \|w\|, \|h\|$, where $(w,h) \in \mathrm{gra}(A+B)$, we then get

$$\exists h_1 \in Aw, h_2 \in Bw\, (\|h_2\| \leq B^*(n) \wedge h = h_1 + h_2),$$

but then $\|h_1\| \leq \|h_1 + h_2\| + \|h_2\| \leq n + B^*(n) =: \xi(n)$.
This situation is given in the implicit uses of instances of Proposition 3.1 in [20] and [34] discussed in Section 3.2 below, where - in [20] - $B := M$ is single-valued, $\mathrm{dom}(M) = X$ and M is 2-Lipschitzian while in [34] both A and B are single-valued and Lipschitz continuous.

In the Brezis-Haraux theorem we consider the operator $A + B$ under the assumption that $\mathrm{dom}A \subseteq \mathrm{dom}B$. For two operators A, B to state $\mathrm{dom}A \subseteq \mathrm{dom}B$ in a form which makes this an admissible axiom in our bound extraction theorems we have to use the following witnessing modulus:

Definition 2.10. We say that $\beta : \mathbb{N} \to \mathbb{N}$ is a modulus witnessing uniformly that $\mathrm{dom}A \subseteq \mathrm{dom}B$ if

$$(\beta) \qquad \forall (w, \tilde{w}) \in \mathrm{gra}A\, \forall n \in \mathbb{N}\, (\|w\|, \|\tilde{w}\| < n \to \exists v\, ((w, v) \in \mathrm{gra}B \wedge \|v\| \leq \beta(n)))\,.$$

Remark 2.11. Let in the situation of the definition above B be majorizable with a (not necessarily uniform) majorant B^*, then we can take the axiom (β) formulated with $\beta := B^*$.

2.5. **Cyclically monotone operators.** A particular stronger notion of monotonicity featuring prominently in, e.g., the context of rectangular operators and Rockafellar's characterization of subgradients of convex functions is that of cyclic monotonicity:

Definition 2.12 (cyclic monotonicity, see e.g. [2, Definition 22.13])**.** An operator $A : X \to 2^X$ is n-cyclically monotone for $n \geq 2$ if for any $x_1, \ldots, x_{n+1} \in X$ and any $u_1, \ldots, u_n \in X$: if $u_i \in Ax_i$ and $x_{n+1} = x_1$, then

$$\sum_{i=1}^{n} \langle x_{i+1} - x_i, u_i \rangle \leq 0.$$

An operator A is cyclically monotone if it is n-cyclically monotone for any $n \geq 2$.

Obviously, A is 2-cyclically monotone if and only if A is monotone. Further, cyclically monotone operators behave nice with respect to maximality in the sense of the following proposition.

Proposition 2.13 (see e.g. [2])**.** *An operator A is maximally monotone and cyclically monotone if and only if A is maximally cyclically monotone.*

Correspondingly, the previous approach to formally treating maximal and non-maximal monotone operators immediately extends to the cyclic case in the sense that we can treat (maximally) cyclically monotone operators by just adding the cyclic monotonicity assumption to the previous system $\mathcal{T}^\omega$ (or $\mathcal{T}^\omega_p$, respectively).
Concretely, for any fixed $n \in \mathbb{N}$ the condition of A being n-cyclically monotone can be immediately expressed in the language of $\mathcal{T}^\omega$ by the purely universal axiom

$$(\mathrm{c(n)}) \qquad \forall x_1^X, \ldots, x_n^X, u_1^X, \ldots, u_n^X \left(\bigwedge_{i=1}^{n} (u_i \in Ax_i) \to \sum_{i=1}^{n-1} \langle x_{i+1} -_X x_i, u_i \rangle_X + \langle x_1 -_X x_n, u_n \rangle_X \leq_{\mathbb{R}} 0 \right)$$

and cyclic monotonicity is a straightforward extension of that by setting

$$(\mathrm{c}) \quad \forall x^{\mathbb{N}\to X}, u^{\mathbb{N}\to X}, n^{\mathbb{N}} \left(n \geq_{\mathbb{N}} 2 \wedge \forall i \leq_{\mathbb{N}} n (u_i \in Ax_i) \to \sum_{i=1}^{n-1} \langle x_{i+1} -_X x_i, u_i \rangle_X + \langle x_1 -_X x_n, u_n \rangle_X \leq_{\mathbb{R}} 0 \right)$$

where the iterated summation is definable by the recursor R_0 contained in the language of $\mathcal{A}^\omega[X, \langle \cdot, \cdot \rangle]$ (R_0 instead of R_1 is sufficient by using rational approximations of the reals being added, see [15] for details).

Without indulging in the details of the proof, we note that by an immediate extension of the proof given in [31, Theorem 3.11], one can obtain the following analogous characterization of maximally cyclically monotone operators.

Theorem 2.14. *Over $\mathcal{T}^\omega_c$, extensionality of A is equivalent to a suitable formulation of maximal cyclic monotonicity, where $\mathcal{T}^\omega_c$ is the extension of $\mathcal{T}^\omega$ by the axiom (c).*

As mentioned before, we will discuss a particular use of cyclic monotone operators later on in the context of rectangular operators but we want to hint on a potential use of them here already. Consider the following classical result of Rockafellar (see also [2, Theorem 22.18]):

Theorem 2.15 ([32]). *A set-valued operator A on a Hilbert space X is maximally cyclically monotone if and only if $A = \partial f$ for some proper lower semicontinuous convex function $f : X \to (-\infty, +\infty]$, where ∂f denotes the subdifferential of f.*

In that way, the above systems may be used in combination with the corresponding bound extraction theorems to extract quantitative information from proofs involving subgradients of convex functions which only use properties of the subgradient seen as maximal cyclically monotone operator.

2.6. **Rectangular operators.** The central notion in the context of the Brezis-Haraux theorem is that of rectangular operators (sometimes also called 3^* monotone operators).

Definition 2.16 (Fitzpatrick function, see e.g. [2]). For a monotone operator A, the Fitzpatrick function $F_A : X \times X \to [-\infty, +\infty]$ of A is defined by

$$(x, u) \mapsto \langle x, u\rangle - \inf_{(y,v)\in \mathrm{gra}A} \langle x - y, u - v\rangle.$$

Definition 2.17 (Rectangular operator, see e.g. [2]). A monotone operator A is called rectangular if for any $x \in \mathrm{dom}A$ and any $u \in \mathrm{ran}A$:

$$\sup_{(y,v)\in \mathrm{gra}A} \langle x - y, v - u\rangle < \infty.$$

Alternatively, A is called rectangular if $\mathrm{dom}A \times \mathrm{ran}A \subseteq \mathrm{dom}F_A$ where $(x, u) \in \mathrm{dom}F_A$ means $F_A(x, u) < \infty$ (see [2, Definition 25.10]).

We now introduce a uniform strengthening of the property of being rectangular together with a modulus function:

Definition 2.18 (Uniformly rectangular operator). A monotone operator $A : X \to 2^X$ is called uniformly rectangular with modulus $\alpha : \mathbb{N} \to \mathbb{N}$ if

$$(*) \qquad \forall n \in \mathbb{N}, (x, \hat{x}), (\check{u}, u), (y, v) \in \mathrm{gra}A \left(\|x\|, \|\hat{x}\|, \|u\|, \|\check{u}\| < n \to \langle x - y, v - u\rangle \leq \alpha(n)\right).$$

Note that $(*)$ is purely universal and implies that A is rectangular where a bound for

$$\sup_{(y,v)\in \mathrm{gra}A} \langle x - y, v - u\rangle < \infty$$

is given via $\alpha(n)$ in terms of a norm upper bound for x and u as well as for $\hat{x}$ and $\check{u}$ witnessing that $x \in \mathrm{dom}A$ and $u \in \mathrm{ran}A$. In that way, $(*)$ is a uniform quantitative version of stating that A is rectangular via

$$\forall x \in \mathrm{dom}A\, \forall u \in \mathrm{ran}A\, ((x, u) \in \mathrm{dom}F_A)$$

with 'uniform' here referring to the fact that the bound $\alpha(n)$ depends only on norm upper bounds on the points involved but not on the points themselves.

Remark 2.19. If A is majorizable and A^* is a majorant for A, then $x \in \mathrm{dom}A$ with $\|x\| \leq n$ implies that there exists an $\hat{x} \in Ax$ with $\|\hat{x}\| \leq A^*(n)$ and if A^* is a uniform majorant for A, then for $u \in \mathrm{ran}A$, if $\check{u}$ exists with $u \in A\check{u}$ and $\|\check{u}\| \leq n$, then this implies that $\|u\| \leq A^*(n)$.
So it suffices to assume that $\|x\|, \|\check{u}\| \leq n$ to conclude that $\alpha(\max(n, A^*(n)))$ is an upper bound for

$$\sup_{(y,v)\in \mathrm{gra}A} \langle x - y, v - u\rangle$$

in that case.

The following is an immediate quantitative version of Proposition 25.12 from [2] where the latter proof can be carried out in $\mathcal{T}^\omega_{i,p,c(3)} := \mathcal{T}^\omega_{i,p} + (c(3))$.

Proposition 2.20. *If A is 3-cyclically monotone, then A is uniformly rectangular with modulus $\alpha(n) = 4n^2$.*

Proof. By the proof of [2, Proposition 25.12], one has that for all $(x, \hat{x}), (\check{u}, u) \in \mathrm{gra}A$:

$$\begin{aligned} F_A(x, u) &\leq |\langle \check{u}, u\rangle| + |\langle \check{u} - x, \hat{x}\rangle| \\ &\leq \|\check{u}\| \|u\| + \|\check{u}\| \|\hat{x}\| + \|x\| \|\hat{x}\| \\ &\leq 3n^2 \end{aligned}$$

if $\|x\|, \|\hat{x}\|, \|\check{u}\|, \|u\| \leq n$. Thus, we have

$$\sup_{(y,v)\in \mathrm{gra}A} \langle x-y, v-u\rangle \leq 3n^2 - \langle x,u\rangle \leq 3n^2 + \|x\|\,\|u\| \leq 4n^2.$$

□

2.7. **The bound extraction theorem.** The following theorem is an adaptation and extension due to the second author of metamathematical bound extraction theorems from [10, 12, 17, 18] to the additional constants and axioms for monotone operators and their resolvents discussed above.

These bound extraction results allow for the inclusion of non-constructive principles Δ in the sense of [18] (see also [12]), i.e. of sentences of the form

$$\forall \underline{a}^{\underline{\delta}} \exists \underline{b} \preceq_{\underline{\sigma}} \underline{r}\underline{a} \forall \underline{c}^{\underline{\gamma}} B_{qf}(\underline{a},\underline{b},\underline{c})$$

where B_{qf} is quantifier-free, $\underline{r}$ is a tuple of closed terms of suitable types and all types in $\underline{\delta},\underline{\sigma},\underline{\gamma}$ are admissible, i.e. they are of the form

$$\sigma_1 \to \ldots \to \sigma_k \to X \text{ or } \sigma_1 \to \ldots \to \sigma_k \to \mathbb{N},$$

including $\mathbb{N}, X$ (with $\sigma_1 \to \ldots \to \sigma_k \to \tau$ to be read as $\sigma_1 \to (\sigma_2 \to (\ldots \to \tau)\ldots))$, where each σ_i is of the form $\mathbb{N} \to \ldots \to \mathbb{N} \to \mathbb{N}$ or $\mathbb{N} \to \ldots \to \mathbb{N} \to X$ (also including $\mathbb{N}, X$). Here $\preceq$ is defined pointwise with $x \preceq_X y := \|x\|_X \leq_{\mathbb{R}} \|y\|_X$ for the base type X.
As shown in [13], e.g. the binary ('weak') König's lemma WKL can be written in the form of a sentence Δ and thus can be added to these systems to obtain mathematically strong systems even without the use of DC.

A formula is called a $\forall$-formula (respectively $\exists$-formula) if it has the form $\forall \underline{a}\, F_{qf}(\underline{a})$ (respectively $\exists \underline{a}\, F_0(\underline{a})$), where F_{qf} is quantifier-free and $\underline{a}$ are variables of admissible types.

In the next theorem $z^* \gtrsim z$ denotes the strong majorization relation from the definition of the model $\mathcal{M}^{\omega,X}$ as defined for the finite types over $\mathbb{N}$ in [5] and extended to the finite types over $\mathbb{N}, X$ in [10]. For a finite type τ over $\mathbb{N}, X$, the type $\widehat{\tau}$ is the result of replacing everywhere X by $\mathbb{N}$.

Theorem 2.21 ([31])**.** *Let τ be admissible, δ be of the form $\mathbb{N} \to \ldots \to \mathbb{N} \to \mathbb{N}$ and s be a closed term of $\mathcal{T}^\omega$ of type $\delta \to \sigma$ for admissible σ. Let $B_\forall(x,y,z,u)/C_\exists(x,y,z,v)$ be $\forall$-/$\exists$-formulas of $\mathcal{T}^\omega$ with only $x,y,z,u/x,y,z,v$ free. Let $\mathcal{T}^\omega_-$ be $\mathcal{T}^\omega$ without* DC[2]*. If*

$$\mathcal{T}^\omega_- + \Delta \vdash \forall x^\delta \forall y \preceq_\sigma s(x) \forall z^\tau \left(\forall u^{\mathbb{N}} B_\forall(x,y,z,u) \to \exists v^{\mathbb{N}} C_\exists(x,y,z,v)\right),$$

then one can extract a primitive recursive (in the sense of Gödel's T) $\Phi : S_\delta \times S_{\widehat{\tau}} \times \mathbb{N} \to \mathbb{N}$ such that for all $x \in S_\delta$, $z \in S_\tau$, $z^* \in S_{\widehat{\tau}}$ and all $n \in \mathbb{N}$, if $z^* \gtrsim z$ and $n \geq_{\mathbb{R}} \left\|J_1^A(0)\right\|_X$, then

$$\mathcal{S}^{\omega,X} \models \forall y \preceq_\sigma s(x)\,(\forall u \leq_{\mathbb{N}} \Phi(x,z^*,n)\, B_\forall(x,y,z,u) \to \exists v \leq_{\mathbb{N}} \Phi(x,z^*,n)\, C_\exists(x,y,z,v))$$

holds for all (real) Hilbert spaces $(X,\langle\cdot,\cdot\rangle)$ with χ_A interpreted by the characteristic function of a maximally monotone operator A and J^{χ_A} by corresponding resolvents J^A_γ for $\gamma > 0$ whenever $\mathcal{S}^{\omega,X} \models \Delta$.

Remark 2.22.
(i) If the proof also uses DC, then $\Phi : S_\delta \times S_{\widehat{\tau}} \times \mathbb{N} \rightharpoonup \mathbb{N}$ is a partial functional which is total and (bar-recursively) computable on $S_\delta \times M_{\widehat{\tau}} \times \mathbb{N}$.
If $\widehat{\tau}$ is of the form $\mathbb{N} \to \ldots \to \mathbb{N} \to \mathbb{N}$ in that case, then Φ is again a total computable functional.
(ii) We may have tuples instead of single variables x,y,z,u,v and a finite conjunction instead of a single premise $\forall u^{\mathbb{N}} B_\forall(x,y,z,u)$.
(iii) Instead of $\mathcal{T}^\omega$, we may use $\mathcal{T}^\omega_p$, where the conclusion is then drawn over the appropriate operators such that, in particular, $\bigcap_{\gamma>0} \mathrm{dom} J^A_\gamma \neq \emptyset$. In this case we have to interpret c_X by an element of $\bigcap_{\gamma>0} \mathrm{dom} J^A_\gamma$ and need to replace $n \geq \|J^A_1(0)\|$ by $n \geq \|c_X - J^A_1(c_X)\|, \|c_X\|$.
(iv) If we assume that the resolvents of A are total, then the conclusion even holds in all inner product spaces (in Hilbert spaces this is a consequence of A being maximally monotone by Minty's theorem).

[2]In the absence of DC, one can actually allow τ and σ as well as the types of the quantified variables in $B_\forall, C_\exists$ and in the sentences contained in Δ to be arbitrary. In that case, $\gtrsim$ is defined as the interpretation of the syntactic majorizability relation in $\mathcal{S}^{\omega,X}$.

Proposition 2.23. *Instead of $\mathcal{T}^\omega$ or $\mathcal{T}^\omega_p$ we may use variants of those obtained by adding the formalized versions of cyclic monotonicity or being uniformly rectangular with modulus α or adding the axioms $(i),(ii)$ for defining $A+B$ with a bounding function ξ or also the assumption that $\mathrm{dom}A \subseteq \mathrm{dom}B$ with a witnessing modulus β where then the extracted bound will additionally depend on α or ξ or β (or any combination thereof).*

Proof. The proposition is a consequence of the fact that the axioms expressing that A is cyclically monotone and uniformly rectangular with modulus α are purely universal and so, in particular, of the form Δ, and involve only admissible types. Furthermore, the axiom (i) in defining $A+B$ is purely universal while axiom (ii) as well as the axiom (β) on β is equivalent to a statement of the form Δ. □

The statement that $A : X \to 2^X$ is rectangular can be written as

$$\forall (x,\hat{x}),(\check{u},u) \in \mathrm{gra}A\,\exists m \in \mathbb{N}\,\forall (y,v) \in \mathrm{gra}A\,(\langle x-y, v-u\rangle < m) \tag{2.1}$$

and hence formally (disregarding that we actually have tuples of variables and implicitly using some rational approximation to the scalar product to avoid the existential quantifier hidden in '<' between reals[3]) in the logical form

$$\forall x^X\,\exists m^{\mathbb{N}}\,\forall y^X F_{qf}(x,m,y), \tag{2.2}$$

where F_{qf} is quantifier free. Using classical logic and QF-AC (which is stated for tuples, see [18]) the latter is equivalent in $\mathcal{T}^\omega$ to its Herbrand normal form

$$\forall x^X, \Gamma^{\mathbb{N}\to X}\,\exists m^{\mathbb{N}}\,F_{qf}(x,m,\Gamma m) \tag{2.3}$$

which is implied by

$$\forall x^X,\,\gamma^{\mathbb{N}\to\mathbb{N}}\,\exists m^{\mathbb{N}}\,\forall y^X\,(\|y\|_X <_{\mathbb{R}} \gamma(m) \to F_{qf}(x,m,y)) \tag{2.4}$$

and in turn by

$$\exists \varphi^{X\to(\mathbb{N}^{\mathbb{N}}\to\mathbb{N})}\,\forall x^X, \gamma^{\mathbb{N}\to\mathbb{N}}, y^X\,\left(\|y\|_X <_{\mathbb{R}} \gamma(\varphi(x,\gamma)) \to F_{qf}(x,\varphi(x,\gamma),y)\right). \tag{2.5}$$

Note that noneffectively, (2.2) - (2.5) are all equivalent.

Now in a proof using that A is rectangular as an implicative assumption, we can instead use (2.5) (given that $\mathcal{T}^\omega$ is based on classical logic and contains QF-AC), pull $\exists\varphi$ as a universal quantifier out of the implication and apply Theorem 2.21 which then produces a bound which no longer depends on α but only on a strong majorant φ^* for some φ satisfying (2.5), i.e. a self-majorizing functional φ^* such that

$$\forall n^{\mathbb{N}}, \gamma^{\mathbb{N}\to\mathbb{N}}, x^X\,\left(\|x\|_X \leq_{\mathbb{R}} n \to \exists m \leq_{\mathbb{N}} \varphi^*(n,\gamma)\,\forall y^X\,(\|y\|_X <_{\mathbb{R}} \gamma(m) \to F_{qf}(x,m,y))\right). \tag{2.6}$$

This motivates the following:

Definition 2.24. We say that A is weakly uniformly rectangular with modulus φ (short: φ wur A) if

$$\begin{cases} \forall n \in \mathbb{N}\,\forall \gamma : \mathbb{N}\to\mathbb{N}\,\forall (x,\hat{x}),(\check{u},u) \in \mathrm{gra}A\,(\|x\|, \|\hat{x}\|, \|u\|, \|\check{u}\| \leq n \\ \qquad \to \exists m \leq \varphi(n,\gamma)\,\forall (y,v) \in \mathrm{gra}A\,(\|y\|, \|v\| \leq \gamma(m) \to \langle x-y, v-u\rangle < m)). \end{cases}$$

Theorem 2.25. *Under the assumptions of Theorem 2.21 we have the following: If*

$$\mathcal{T}^\omega_- + \Delta \vdash \forall x^\delta \forall y \preceq_\sigma s(x) \forall z^\tau \left(A \text{ rectangular } \to \exists v^{\mathbb{N}} C_\exists(x,y,z,v)\right),$$

then one can extract a primitive recursive functional $\Phi : S_\delta \times S_{\hat{\tau}} \times S_{\mathbb{N}\to\mathbb{N}^{\mathbb{N}}\to\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for all $x \in S_\delta$, $z \in S_\tau$, $z^* \in S_{\hat{\tau}}, \varphi \in S_{\mathbb{N}\to\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}$ and all $n \in \mathbb{N}$, if $z^* \gtrsim z$ and $n \geq_{\mathbb{R}} \left\|J^A_1(0)\right\|_X$, then

$$\mathcal{S}^{\omega,X} \models \forall y \preceq_\sigma s(x)\,(\varphi \text{ wur } A \to \exists v \leq_{\mathbb{N}} \Phi(x,z^*,\varphi,n)\,C_\exists(x,y,z,v))$$

holds for all (real) Hilbert spaces $(X, \langle\cdot,\cdot\rangle)$ with χ_A interpreted by the characteristic function of a maximally monotone operator A and J^{χ_A} by corresponding resolvents J^A_γ for $\gamma > 0$ whenever $\mathcal{S}^{\omega,X} \models \Delta$.
The theorem also extends to the systems formulated in Proposition 2.23 and Remark 2.22 applies here as well (where then the bound depends on a (strong) majorant φ^* of φ instead of the latter).

[3]In practice we do not use such a rational approximation but the fact that we could have equivalently used '≤' instead of '<' and so can always choose things in such a way that the additional quantifier does not matter.

Proof. It remains to show that we can use any modulus φ for A being *wur* instead of a majorant of such a modulus. This due to the fact that if DC is not used in the proof we do not have to go through the model $\mathcal{M}^{\omega,X}$ since we do not need bar recursion and can use throughout the proof the ordinary Howard-majorizability relation instead of Bezem's strong majorizability (see [5, 18]).[4] But **any** modulus φ for A being weakly uniformly rectangular is in fact a majorant (though in general not a strong one!) of some such modulus, namely of φ^- which selects, given n, γ, the least m satisfying the claim in the definition of A being weakly uniformly rectangular instead of $\varphi(n,\gamma)$. □

Whereas a modulus α can only be guaranteed to be extractable from (semi-)constructive proofs for A being rectangular (see Corollary 2.30 below), such a modulus φ for A being weakly uniformly rectangular can be extracted whenever the latter fact is provable in $\mathcal{T}^\omega + \Delta$ (or one of its variants considered above), i.e. also from noneffective proofs, as we will show now: if $\mathcal{T}^\omega + \Delta$ proves that A is rectangular it, in particular, proves the reformulation corresponding to (2.4), i.e.

$$(+)\qquad \begin{cases} \forall n \in \mathbb{N}\,\forall \gamma : \mathbb{N} \to \mathbb{N}\,\forall (x,\hat{x}), (\check{u}, u) \in \mathrm{gra}A\,(\|x\|, \|\hat{x}\|, \|u\|, \|\check{u}\| \le n \\ \qquad\qquad \to \exists m \in \mathbb{N}\,\forall (y,v) \in \mathrm{gra}A\,(\|y\|, \|v\| \le \gamma(m) \to \langle x-y, v-u\rangle < m)). \end{cases}$$

By treating the 'bounded' formula

$$(b)\qquad \forall (y,v) \in \mathrm{gra}\,A(\|y\|, \|v\| \le \gamma(m) \to \langle x-y, v-u\rangle < m)$$

as purely existential, we can extract by Theorem 2.21 a bound φ such that

$$\begin{cases} \forall n \in \mathbb{N}\,\forall \gamma : \mathbb{N} \to \mathbb{N}\,\forall (x,\hat{x}), (\check{u}, u) \in \mathrm{gra}A\,(\|x\|, \|\hat{x}\|, \|u\|, \|\check{u}\| \le n \\ \qquad\qquad \to \exists m \le \varphi(n,\gamma)\,\forall (y,v) \in \mathrm{gra}A\,(\|y\|, \|v\| \le \gamma(m) \to \langle x-y, v-u\rangle < m)). \end{cases}$$

It follows from the construction of φ as provided by the bound extraction theorem that it is a strong majorant.

Remark 2.26. To justify the treatment of (b) as a purely existential formula, we introduce (similar to [12]) 'ε-terms' χ_1, χ_2 with the purely universal axiom

$$(\chi)\quad \begin{cases} \forall \underbrace{x^X, u^X, m^{\mathbb{N}}, \gamma^{\mathbb{N}\to\mathbb{N}}}_{:=a}\,\forall y^X, v^X\,(\|\chi_1 a\|_X, \|\chi_2 a\|_X \le_{\mathbb{R}} \gamma(m) \wedge [(y,v) \in \mathrm{gra}A \wedge \|y\|_X, \|v\|_X <_{\mathbb{R}} \gamma(m) \wedge \\ \langle x -_X y, v -_X u\rangle_X >_{\mathbb{R}} m \to (\chi_1 a, \chi_2 a) \in \mathrm{gra}A \wedge \langle x -_X \chi_1 a, \chi_2 a -_X u\rangle_X \ge_{\mathbb{R}} m]). \end{cases}$$

The interpretation of $(\chi_1 a, \chi_2 a)$ in $\mathcal{M}^{\omega,X}$ is to be a pair of points $(y,v) \in \mathrm{gra}A$ with $\|y\|, \|v\| \le \gamma(m)$ and $\langle x-y, v-u\rangle \ge m$ if existent and $(0,0)$ otherwise. Both χ_1 and χ_2 are trivially majorizable via γ.
Now, if $\mathcal{T}^\omega + \Delta$ proves $(+)$, then $\mathcal{T}^\omega + \Delta + (\chi)$ proves

$$\forall n, \gamma\,\forall (x,\hat{x}), (\check{u}, u) \in \mathrm{gra}A\,(\|x\|, \|\hat{x}\|, \|u\|, \|\check{u}\| \le n \to \exists m\,((\chi_1 a, \chi_2 a) \in \mathrm{gra}A \to \langle x - \chi_1 a, \chi_2 a - u\rangle < m))$$

where $\langle x - \chi_1 a, \chi_2 a - u\rangle < m$ is Σ^0_1. We can now extract a bound $\varphi(n,\gamma)$ on '$\exists m$'. With (χ), this implies

$$\forall (y,v) \in \mathrm{gra}A(\|y\|, \|v\| < \gamma(m) \to \langle x-y, v-u\rangle \le m).$$

Indeed, if $\langle x-y, v-u\rangle > m$ for some $(y,v) \in \mathrm{gra}A$ with $\|y\|, \|v\| < \gamma(m)$, then - by (χ) -

$$(\chi_1 a, \chi_2 a) \in \mathrm{gra}A \wedge \langle x - \chi_1 a, \chi_2 a - u\rangle \ge m$$

which is a contradiction.

In total, we have shown:

Corollary 2.27. If $\mathcal{T}^\omega_{(-)} + \Delta$ proves that A is rectangular, then from the proof one can extract a bar-recursively (primitive recursively, respectively) computable modulus φ for A being weakly uniformly rectangular which is a strong majorant.

Now, regarding the semi-constructive variants $\mathcal{T}^\omega_i, \mathcal{T}^\omega_{i,p}$, etc., we can obtain a similar bound extraction result by following the methods developed in [9]. Those bound extraction results, however, make it possible to allow a much larger class of ideal principles to be used in the proof and also a more general class of theorems to be analyzed.

[4] This is due to the fact that we use the monotone functional interpretation of the first author [14] in proving the bound extraction theorem and not the bounded functional interpretation due to [8] which crucially uses strong majorizability throughout.

Concretely, the restrictions which are lifted are (without indulging in the precise details, see [9] for a more detailed discussion), that for one, instead of requiring the formula to be of the form $\forall\exists A_{qf}$ with A_{qf} quantifier-free which is necessary in the classical setting to recover from the negative translation using Markov's principle, we can now allow for arbitrary formulas A.
For another, while the interpretation of the negative translation of dependent choice requires the use of bar-recursion which forces the detour through the model $\mathcal{M}^{\omega,X}$ of strongly majorizable functionals in the classical setting, this is no longer necessary in this context. In that vein, the full axiom of choice can be allowed and the type restrictions on the quantifiers to essentially admissible types can be lifted.
Further, we can allow the full extensionality principle instead of restricting the system to the quantifier-free rule of extensionality.

Hence, as mentioned before, the systems $\mathcal{T}^\omega_i, \mathcal{T}^\omega_{i,p}$ and their extensions are based on E-HA$^\omega[X,\langle\cdot,\cdot\rangle]$+AC (denoted by $\mathcal{A}^\omega_i[X,\langle\cdot,\cdot\rangle]$ in [9]) and with the axioms of A and its resolvent J^A_γ added. While the method used for deriving the above bound extraction result for the classical systems rests on a combination of Gödel's functional interpretation together with a negative translations (due to Kuroda) and majorization, the following bound extraction result is established through the use of a monotone variant (due to the first author [16]) of Kreisel's modified realizability interpretation [25, 26].

In this context we can now as in [9] add an even larger class of highly non-constructive principles $\Gamma_\neg$ without affecting the bound extraction results. These principles in particular include comprehension for negated formulas $\mathrm{CA}_\neg$ which is the union of the principles

$$(\mathrm{CA}^{\underline{\rho}}_\neg) \qquad \exists\Phi \leq_{\underline{\rho}\to\mathbb{N}} \lambda\underline{x}^{\underline{\rho}}.1^{\mathbb{N}}\forall\underline{y}^{\underline{\rho}}\,(\Phi(\underline{y}) =_{\mathbb{N}} 0 \leftrightarrow \neg A(\underline{y})).$$

Concretely, we can show the following result by adapting [9, Theorem 4.11] (where now $\gtrsim$ denotes majorization - not necessarily strong - interpreted in the model $\mathcal{S}^{\omega,X}$):

Theorem 2.28. *Let δ be of the form $\mathbb{N}\to\ldots\to\mathbb{N}\to\mathbb{N}$ and σ,τ be arbitrary, s be a closed term of suitable type. Let $\Gamma_\neg$ be a set of sentences of the form $\forall\underline{u}^{\underline{\zeta}}(C(\underline{u})\to\exists\underline{v}\preceq_{\underline{\beta}}\underline{t}\underline{u}\neg D(\underline{u},\underline{v}))$ with $\underline{\zeta},\underline{\beta}$ and C,D arbitrary types and formulas respectively and $\underline{t}$ be closed terms.*
Let $B(x,y,z)/A(x,y,z,u)$ be arbitrary formulas of $\mathcal{T}^\omega_i$ with only $x,y,z/x,y,z,u$ free. If

$$\mathcal{T}^\omega_i + \mathrm{CA}_\neg + \Gamma_\neg \vdash \forall x^\delta\,\forall y\preceq_\sigma(x)\,\forall z^\tau\,(\neg B(x,y,z)\to\exists u^0 A(x,y,z,u)),$$

one can extract a $\Phi:\mathcal{S}_\delta\times\mathcal{S}_{\hat{\tau}}\times\mathbb{N}\to\mathbb{N}$ with is primitive recursive in the sense of Gödel's T such that for any $x\in\mathcal{S}_\delta$, any $y\in\mathcal{S}_\sigma$ with $y\preceq_\sigma s(x)$, any $z\in\mathcal{S}_\tau$ and $z^*\in\mathcal{S}_{\hat{\tau}}$ with $z^*\gtrsim z$ and any $n\in\mathbb{N}$ with $n\geq_{\mathbb{R}}\left\|J^A_1(0)\right\|_X$, we have that

$$\mathcal{S}^{\omega,X}\models\exists u\leq\Phi(x,z^*,n)\,(\neg B(x,y,z)\to A(x,y,z,u))$$

holds for all (real) Hilbert spaces $(X,\langle\cdot,\cdot\rangle)$ with χ_A interpreted by the characteristic function of a maximally monotone operator A and J^{χ_A} by corresponding resolvents J^A_γ for $\gamma>0$ whenever $\mathcal{S}^{\omega,X}\models\Gamma_\neg$.

Remark 2.29. Similar to before, instead of $\mathcal{T}^\omega_i$, we may use $\mathcal{T}^\omega_{i,p}$ or any modification of those obtained by adding the formalized versions of cyclic monotonicity or being uniformly rectangular with modulus α where the conclusion is then drawn over the appropriate operators such that, in particular, $\bigcap_{\gamma>0}\mathrm{dom}J^A_\gamma\neq\emptyset$ in the case of the partial systems where c_X then has to be interpreted by an element of $\bigcap_{\gamma>0}\mathrm{dom}J^A_\gamma$ and we need to replace $n\geq\|J^A_1(0)\|$ by $n\geq\|c_X-J^A_1(c_X)\|,\|c_X\|$.

Corollary 2.30. If $\mathcal{T}^\omega_i+\mathrm{CA}_\neg+\Gamma_\neg$ proves that A is rectangular, then by Theorem 2.28 one can extract from the proof a primitive recursively computable (in the extended sense of Gödel's T) modulus α for A being uniformly rectangular.

Proof. Suppose that

$$\mathcal{T}^\omega_i+\mathrm{CA}_\neg+\Gamma_\neg\vdash\forall x^X,\hat{x}^X,u^X,\check{u}^X\,\left((x,\hat{x}),(\check{u},u)\in\mathrm{gra}A\to\exists m^{\mathbb{N}}\,\forall(y,v)\in\mathrm{gra}A\,(\langle x-y,v-u\rangle<m)\right).$$

Then by Theorem 2.28 (note that $(x,\hat{x})\in\mathrm{gra}A$ is quantifier-free and hence equivalent to its double negation, i.e. a negated premise) we extract a primitive recursive (in the sense of Gödel's T) Φ such that for all n with $n\gtrsim_X x,\hat{x},u,\check{u}$, i.e. $n\geq\|x\|,\|\hat{x}\|,\|u\|,\|\check{u}\|$, it is true in $\mathcal{S}^{\omega,X}$ that

$$(x,\hat{x}),(\check{u},u)\in\mathrm{gra}A\to\exists m\leq\Phi(n)\,\forall(y,v)\in\mathrm{gra}A\,(\langle x-y,v-u\rangle<m)),$$

i.e. $\alpha:=\Phi$ is a modulus for A being uniformly rectangular. ☐

This covers the situation of Proposition 2.20 as well as the concrete proofs for being rectangular in the applications given in [20] and [34].

Note further that the above results can be immediately extended in the following ways:

(1) We may add further abstract metric and normed spaces which are treated simultaneously (see the discussion in [18] in Section 17.6) or further constants for all previously discussed types of operators and their resolvents which in particular may mix partial and non-partial resolvents.
(2) We may add additional constants (where corresponding defining axioms guarantee majorizability) which are of admissible types (see the discussion in [18], Section 17.5) and corresponding $\Delta/\Gamma_{\neg}$-formulas as axioms.

3. The Brezis-Haraux theorem and its formalization in logical systems

3.1. Quantitative forms of the Brezis-Haraux theorem. We now show that the (part of the) Brezis-Haraux theorem [6] used in Bauschke's solution to the zero displacement conjecture [1] as well as in its generalizations to compositions of firmly nonexpansive mappings in [3] and of averaged mappings in [4] can be formulated in an appropriate formal system to which logical bound extraction metatheorems apply. These then also guarantee a quantitative version of this theorem as given below (which - in more special forms - is used in analyzing quantitatively the specific situations mentioned above in [20] and [34]).

The theorem states that if $A, B : X \to 2^X$ are monotone (X a real Hilbert space), $A+B$ is maximally monotone, $\mathrm{dom}A \subseteq \mathrm{dom}B$ and B is rectangular, then $\overline{\mathrm{ran}(A+B)} = \overline{\mathrm{ran}A + \mathrm{ran}B}$. Since $\overline{\mathrm{ran}(A+B)} \subseteq \overline{\mathrm{ran}A + \mathrm{ran}B}$ is trivial, we only have to consider

$$\overline{\mathrm{ran}A + \mathrm{ran}B} \subseteq \overline{\mathrm{ran}(A+B)}. \tag{3.1}$$

Clearly $x \in \overline{\mathrm{ran}A + \mathrm{ran}B}$ if and only if

$$\forall n \in \mathbb{N} \exists p_n \in \mathrm{dom}A, q_n \in \mathrm{dom}B, \tilde{p}_n \in Ap_n, \tilde{q}_n \in Bq_n \left(\|x - (\tilde{p}_n + \tilde{q}_n)\| \leq \frac{1}{n+1} \right) \tag{3.2}$$

and $x \in \overline{\mathrm{ran}(A+B)}$ if and only if

$$\forall n \in \mathbb{N} \exists u_n \in \mathrm{dom}A \subseteq \mathrm{dom}B, \tilde{u}_n \in Au_n + Bu_n \left(\|x - \tilde{u}_n\| \leq \frac{1}{n+1} \right). \tag{3.3}$$

By (a suitable instantiation of) Theorem 2.25 one can extract from a given proof of the Brezis-Haraux theorem (3.1) an effective transformation of a majorant $K : \mathbb{N} \to \mathbb{N}$ for $(p_n), (q_n), (\tilde{p}_n), (\tilde{q}_n)$, i.e.

$$\forall n \in \mathbb{N} \left(\|p_n\|, \|q_n\|, \|\tilde{p}_n\|, \|\tilde{q}_n\| \leq K(n) \right),$$

a modulus φ for B being weakly uniformly rectangular, a modulus β for $\mathrm{dom}A \subseteq \mathrm{dom}B$, a bound ξ witnessing the axiom (ii) in the representation of $A+B$, a bound $n \geq \|J_1^{A+B}(0)\|$ and a norm bound on x[5] into a majorant $\tilde{K} : \mathbb{N} \to \mathbb{N}$ for (u_n), i.e. $\forall n \in \mathbb{N} \left(\|u_n\| \leq \tilde{K}(n) \right)$. Note that given a norm bound on x, a majorant for $(\tilde{u}_n)$ follows trivially from $\forall n \in \mathbb{N} \left(\|x - \tilde{u}_n\| \leq \frac{1}{n+1} \right)$.

We first study this under the stronger assumption that we even have a modulus α for B being uniformly rectangular as this simplifies the bound and is the situation given in the applications so far. Moreover, it turns out that under this assumption the bound does not depend on ξ from the representation of $A+B$ and also not on $n \geq \|J_1^{A+B}(0)\|$.

For convenience, we use $\varepsilon > 0$ instead of $\frac{1}{n+1}$ and $K : (0, \infty) \to \mathbb{N} \setminus \{0\}$. Analyzing the proof of (3.1) from [6] one obtains the following (compare also [20, 34]):

Proposition 3.1. *Let $A, B : X \to 2^X$ be monotone, $A+B$ be maximally monotone,* $\mathrm{dom}A \subseteq \mathrm{dom}B$ *with modulus β, B uniformly rectangular with modulus α and let $\varepsilon \in (0,1)$ and $K(\varepsilon) \in \mathbb{N} \setminus \{0\}$. Let $x \in X$ be such that*

$$\exists p \in \mathrm{dom}A, q \in \mathrm{dom}B, \tilde{p} \in Ap, \tilde{q} \in Bq \left(\|p\|, \|q\|, \|\tilde{p}\|, \|\tilde{q}\| \leq K(\varepsilon) \wedge \|x - (\tilde{p} + \tilde{q})\| \leq \frac{3}{4}\varepsilon \right). \tag{3.4}$$

[5] This can actually also be obtained from K by $\|x\| \leq \|x - (\tilde{p}_0 + \tilde{q}_0)\| + \|\tilde{p}_0\| + \|\tilde{q}_0\| \leq 1 + 2K(0)$.

Then

$$\exists u \in \mathrm{dom}A \subseteq \mathrm{dom}B \, \exists \tilde{u} \in Au + Bu \left(\|u\| \leq \frac{4(K(\varepsilon)^2 + 2L)}{\varepsilon} \wedge \|x - \tilde{u}\| \leq \varepsilon \right) \tag{3.5}$$

where $L := \alpha(\max\{K(\varepsilon), \beta(K(\varepsilon))\})$.

Proof. Let $\varepsilon, x, K(\varepsilon), \alpha, \beta, p, q, \tilde{p}, \tilde{q}$ be as in the proposition and assume that (3.4) holds. Since A is monotone, we get

$$\forall (z, h_1) \in \mathrm{gra}A \left(\langle h_1 - \tilde{p}, z - p \rangle \geq 0 \right). \tag{3.6}$$

The fact that B is uniformly rectangular with modulus α implies

$$\forall (w, \tilde{w}) \in \mathrm{gra}B \left(\langle \tilde{w} - \tilde{q}, w - p \rangle \geq -L \right) \tag{3.7}$$

since $p \in \mathrm{dom}A \subseteq \mathrm{dom}B$, $q \in \mathrm{dom}B$ with $(p, \tilde{p}) \in \mathrm{gra}A, (q, \tilde{q}) \in \mathrm{gra}B$ and since $(p, \tilde{v}) \in \mathrm{gra}B$ for some $\tilde{v}$ with $\|\tilde{v}\| \leq \beta(K(\varepsilon))$ where $\|p\|, \|\tilde{p}\|, \|q\|, \|\tilde{q}\| \leq K(\varepsilon)$.

By (3.6) and (3.7), we get for $f := \tilde{p} + \tilde{q}$:

$$\forall (w, h) \in \mathrm{gra}(A + B) \left(\langle h - f, w - p \rangle \geq -L \right). \tag{3.8}$$

Indeed, let $h = h_1 + h_2 \in Aw + Bw$ with $h_1 \in Aw$ and $h_2 \in Bw$. By (3.6), we have $\langle h_1 - \tilde{p}, w - p \rangle \geq 0$ and so

$$\langle h - \tilde{p} - \tilde{q}, w - p \rangle = \langle h - h_2 - \tilde{p}, w - p \rangle + \langle h_2 - \tilde{q}, w - p \rangle \geq -L$$

using (3.7). By (3.8), we get

$$\forall (w, h) \in \mathrm{gra}(A + B) \left(\langle f - h, w - p \rangle \leq L \right). \tag{3.9}$$

Since $A + B$ is maximally monotone, Minty's theorem implies that for all $\tilde{\varepsilon} > 0$: $\mathrm{ran}(\tilde{\varepsilon}I + (A + B)) = X$. Hence

$$\forall \tilde{\varepsilon} > 0 \exists u_{\tilde{\varepsilon}} \in X \left(f \in \tilde{\varepsilon} u_{\tilde{\varepsilon}} + (A + B)(u_{\tilde{\varepsilon}}) \right),$$

where $f = \tilde{p} + \tilde{q}$ as before. In particular, $u_{\tilde{\varepsilon}} \in \mathrm{dom}(A + B)$ with $(u_{\tilde{\varepsilon}}, f - \tilde{\varepsilon} u_{\tilde{\varepsilon}}) \in \mathrm{gra}(A + B)$. (3.9) applied to $w := u_{\tilde{\varepsilon}}$, $h := f - \tilde{\varepsilon} u_{\tilde{\varepsilon}}$ yields

$$\langle \tilde{\varepsilon} u_{\tilde{\varepsilon}}, u_{\tilde{\varepsilon}} - p \rangle \leq L$$

and so

$$\frac{1}{2} \tilde{\varepsilon} \|u_{\tilde{\varepsilon}}\|^2 \leq \frac{1}{2} \tilde{\varepsilon} \|p\|^2 + L \leq \frac{1}{2} \tilde{\varepsilon} K(\varepsilon)^2 + L.$$

Hence, for $\tilde{\varepsilon} \in (0, 1)$:

$$\sqrt{\tilde{\varepsilon}} \|u_{\tilde{\varepsilon}}\| \leq \sqrt{\tilde{\varepsilon} K(\varepsilon)^2 + 2L} \leq \sqrt{K(\varepsilon)^2 + 2L}.$$

Now take $\tilde{\varepsilon} := (\varepsilon/4)^2 / (K(\varepsilon)^2 + 2L)$. Then $\tilde{\varepsilon} \|u_{\tilde{\varepsilon}}\| \leq \varepsilon/4$ and so

$$\|x - (f - \tilde{\varepsilon} u_{\tilde{\varepsilon}})\| \leq \|x - f\| + \|\tilde{\varepsilon} u_{\tilde{\varepsilon}}\| \leq \frac{3}{4}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon$$

using (3.4). Hence the proposition is satisfied with $u := u_{\tilde{\varepsilon}}$ and $\tilde{u} := f - \tilde{\varepsilon} u_{\tilde{\varepsilon}}$, where

$$\|u\| \leq \frac{\sqrt{K(\varepsilon)^2 + 2L}}{\sqrt{\tilde{\varepsilon}}} = \frac{4(K(\varepsilon)^2 + 2L)}{\varepsilon}.$$

□

As predicted by Theorem 2.25, an inspection of the proof of Proposition 3.1 shows that it is sufficient to have a modulus φ for A being weakly uniformly rectangular instead of α:

Theorem 3.2. *Let $A, B : X \to 2^X$ be monotone, $A + B$ be maximally monotone and represented with bounding function ξ satisfying axiom (ii), $\mathrm{dom}A \subseteq \mathrm{dom}B$ with modulus β, B weakly uniformly rectangular with modulus φ, $n > \|J_1^{A+B}(0)\|$ and let $\varepsilon \in (0, 1)$ and $K(\varepsilon) \in \mathbb{N} \setminus \{0\}$. Let $x \in X$ be such that*

$$\exists p \in \mathrm{dom}A, q \in \mathrm{dom}B, \tilde{p} \in Ap, \tilde{q} \in Bq \left(\|p\|, \|q\|, \|\tilde{p}\|, \|\tilde{q}\| \leq K(\varepsilon) \wedge \|x - (\tilde{p} + \tilde{q})\| \leq \frac{3}{4}\varepsilon \right). \tag{3.4}$$

Then

$$\exists u \in \mathrm{dom}A \subseteq \mathrm{dom}B \, \exists \tilde{u} \in Au + Bu \left(\|u\| \leq \frac{4(K(\varepsilon)^2 + 2L^*)}{\varepsilon} \wedge \|x - \tilde{u}\| \leq \varepsilon \right) \tag{3.5*}$$

where $L^* := \varphi(\max\{K(\varepsilon), \beta(K(\varepsilon))\}, \gamma_\varepsilon)$ *with*

$$\gamma_\varepsilon(m) = \max\left\{ \lceil \tilde{\varepsilon}^{-1} 2K(\varepsilon) + (2 + \tilde{\varepsilon}^{-1})n \rceil, \xi\left(\lceil \tilde{\varepsilon}^{-1} 2K(\varepsilon) + (2 + \tilde{\varepsilon}^{-1})n \rceil \right) \right\},$$

where $\tilde{\varepsilon} := \frac{(\varepsilon/4)^2}{K(\varepsilon)^2+2m}$.

Proof. We use the notations from the proof of Proposition 3.1. In that proof, (3.9) is applied to $w := u_{\tilde{\varepsilon}}, h := f - \tilde{\varepsilon}u_{\tilde{\varepsilon}}$ and so the uniform rectangularity (3.7) is used for $w := u_{\tilde{\varepsilon}}$ and some $\tilde{w} := h_2 \in Bu_{\tilde{\varepsilon}}$ such that an $h_1 \in Au_{\tilde{\varepsilon}}$ exists with $f - \tilde{\varepsilon}u_{\tilde{\varepsilon}} = h_1 + h_2$.
In order to replace the use of uniform rectangularity by its weak version we need, given m (alias L) as a parameter, to construct a norm bound $\gamma_\varepsilon(m) \geq \|u_{\tilde{\varepsilon}}\|, \|h_2\|$, where

$$\tilde{\varepsilon} := \frac{(\varepsilon/4)^2}{K(\varepsilon)^2 + 2m}.$$

Firstly, note that by definition, $u_{\tilde{\varepsilon}}$ can be expressed via the resolvent of $A + B$ as

$$u_{\tilde{\varepsilon}} = J^{A+B}_{\tilde{\varepsilon}^{-1}}(\tilde{\varepsilon}^{-1}f).$$

In a similar vein to [31], where a majorant for the resolvent was constructed in the context of general logical metatheorems, an upper bound for the resolvent can be obtained (in an upper bound for the argument), using that the resolvent is nonexpansive and satisfies the so-called resolvent equation, via

$$\begin{aligned}\left\|J^{A+B}_{\tilde{\varepsilon}^{-1}}(\tilde{\varepsilon}^{-1}f)\right\| &\leq \tilde{\varepsilon}^{-1}\|f\| + (2+\tilde{\varepsilon}^{-1})\left\|J^{A+B}_1(0)\right\| \\ &\leq \tilde{\varepsilon}^{-1}2K(\varepsilon) + (2+\tilde{\varepsilon}^{-1})\left\|J^{A+B}_1(0)\right\| \\ &< \tilde{\varepsilon}^{-1}2K(\varepsilon) + (2+\tilde{\varepsilon}^{-1})n.\end{aligned}$$

The dependence on $n > \left\|J^{A+B}_1(0)\right\|$ is similarly explained by the bound extraction results given in Theorem 2.25.

Now, to use (3.7), we further need $\gamma_\varepsilon(m)$ to be an upper bound on some $h_2 \in Bu_{\tilde{\varepsilon}}$ where $f - \tilde{\varepsilon}u_{\tilde{\varepsilon}} = h_1 + h_2$ with $h_1 \in Au_{\tilde{\varepsilon}}$. For this we invoke the modulus ξ from the axiomatization of $A + B$ in the sense of Section 2.4 which, however, needs as argument an upper bound for both $\|u_{\tilde{\varepsilon}}\|$ and $\|f - \tilde{\varepsilon}u_{\tilde{\varepsilon}}\|$. We now show that our upper bound for the former also serves as an upper bound for the latter: By the above we have that

$$\|h_1 + h_2\| \leq \|f\| + \tilde{\varepsilon}\|u_{\tilde{\varepsilon}}\| < 2K(\varepsilon) + 2K(\varepsilon) + 3n = 4K(\varepsilon) + 3n \leq \tilde{\varepsilon}^{-1}2K(\varepsilon) + (2+\tilde{\varepsilon}^{-1})n$$

where we used that $\varepsilon \leq 1$ and consequently $\tilde{\varepsilon}^{-1} \geq 2$. Using now the bounding function ξ satisfying axiom (ii) on $A + B$ we get

$$\|h_2\| \leq \xi\left(\left\lceil\tilde{\varepsilon}^{-1}2K(\varepsilon) + (2+\tilde{\varepsilon}^{-1})n\right\rceil\right)$$

for some h_2 satisfying the requirements above.

A simultaneous upper bound on the norms of $u_{\tilde{\varepsilon}}$ and the respective h_2 can then be given via[6]

$$\gamma_\varepsilon(m) = \max\left\{\left\lceil\tilde{\varepsilon}^{-1}2K(\varepsilon) + (2+\tilde{\varepsilon}^{-1})n\right\rceil, \xi\left(\left\lceil\tilde{\varepsilon}^{-1}2K(\varepsilon) + (2+\tilde{\varepsilon}^{-1})n\right\rceil\right)\right\}.$$

Now, we use that φ is a modulus for B being weakly uniformly rectangular to construct an

$$m \leq \varphi(\max\{K(\varepsilon), \beta(K(\varepsilon))\}, \gamma_\varepsilon)$$

satisfying the conclusion in the defining condition for this property. This m can serve as the original $L := m$ in the proof of Proposition 3.1 and we obtain for $L^* := \varphi(\max\{K(\varepsilon), \beta(K(\varepsilon))\}, \gamma_\varepsilon) \geq L$ that

$$\|u\| \leq \frac{4(K(\varepsilon)^2 + 2L^*)}{\varepsilon}.$$

□

Remark 3.3. Note that the Brezis-Haraux theorem is formulated with the assumption that $A + B$ is maximally monotone. By the characterization of maximality laid out in Theorem 4, this would imply a dependence on some quantitative version of extensionality, like a modulus of uniform continuity for A as discussed in [31], in the quantitative results formulated above. However, an inspection of the proof yields that this assumption is made only in order to use the totality of the resolvents of $A + B$ and it can consequently be formulated in the systems discussed in the previous sections without the use of the extensionality axiom. This explains the absence of a dependence on such a quantitative assumption. For the same reason, the completeness of X (needed only to infer the totality of J^{A+B}_γ via Minty's theorem) is not needed in formalizing the proof.

[6]If B is uniformly majorizable with majorant B^*,, then we can simply take $\xi := B^*$ in the definition of γ_ε.

3.2. **Applications of the Brezis-Haraux theorem and their logical analysis.** In this section we indicate how recent proof minings of arguments which are based on the Brezis-Haraux theorem can be logically explained in terms of our notion of modulus for being uniformly rectangular and our general quantitative Brézis-Haraux theorem (Proposition 3.1):

In [20][7], the first author analyzed Bauschke's [1] solution of the zero displacement conjecture and its extension to compositions of N-many arbitrary firmly nonexpansive mappings $T_1, \ldots, T_N$ in a Hilbert space X from [3] and extracted polynomial rates of asymptotic regularity. Both proofs use the theory of strongly nonexpansive mappings which nicely fits into the existing framework of logical metatheorems as has been shown in [19]. On X^N the maximally monotone operator

$$A(x) := A_1(x_1) \times \cdots \times A_N(x_N), \text{ where } A_i := T_i^{-1} - Id,$$

is defined. In that context, we need to provide a treatment of the space X^N relative to X which can, for example, be achieved as follows: one can treat the product space X^N of the space X for a variable N by combining a treatment of finite sequences in X via elements of type $\mathbb{N} \to X$ with the approach explored by Günzel [11] for the treatment of infinite product spaces X_∞. Concretely, we identify elements in X^N with elements of $X^\mathbb{N}$ of the form $\overline{x, N}$ for $x \in X^\mathbb{N}$ with

$$\overline{x, N}(i) = \begin{cases} x(i) & \text{if } i < N, \\ 0_X & \text{otherwise.} \end{cases}$$

Note that this 'truncation' can be defined via a closed term in our language. The operations $\|\cdot\|_{X^N}, +_{X^N}, -_{X^N}, \cdot_{X^N}$ on objects of type $\mathbb{N} \to X$ and the constant 0_{X^N} are introduced as abbreviations by

(1) $\|x\|_{X^N} := \left(\sum_{i=1}^N \|x(i)\|_X^2\right)^{1/2}$,
(2) $x +_{X^N} y := \lambda i^\mathbb{N}.(\overline{x, N}(i) +_X \overline{y, N}(i))$,
(3) $-_{X^N} x := \lambda i^\mathbb{N}.(-_X(\overline{x, N}(i)))$,
(4) $\alpha \cdot_{X^N} x := \lambda i^\mathbb{N}.(\alpha \cdot_X (\overline{x, N}(i)))$,
(5) $0_{X^N} := \lambda i^\mathbb{N}.(0_X)$.

In particular, $\|\cdot\|_{X^N}$ can be defined via the recursors of the underlying language. Note that the abbreviations (1) - (4) depend on N as a free variable. Further, we introduce equality on X^N by the abbreviation

$$x^{\mathbb{N}\to X} =_{X^N} y^{\mathbb{N}\to X} := \|x -_{X^N} y\|_{X^N} =_\mathbb{R} 0.$$

To characterize the spaces X^N as inner product space, we add the parallelogram law (similar to [10, 17])

$$\forall N^\mathbb{N}, x^{\mathbb{N}\to X}, y^{\mathbb{N}\to X} \left(\|x +_{X^N} y\|_{X^N}^2 + \|x -_{X^N} y\|_{X^N}^2 =_\mathbb{R} 2\left(\|x\|_{X^N}^2 + \|y\|_{X^N}^2\right)\right)$$

where the inner product can then be defined as

$$\langle x^{\mathbb{N}\to X}, y^{\mathbb{N}\to X}\rangle_{X^N} := \frac{1}{4}\left(\|x +_{X^N} y\|_{X^N}^2 - \|x -_{X^N} y\|_{X^N}^2\right).$$

It is straightforward to see that in that case we have provably

$$\langle x, y\rangle_{X^N} =_\mathbb{R} \sum_{i=1}^N \langle x(i), y(i)\rangle_X$$

for any N.

The main benefit of this kind of treatment is that by treating N as a variable, the bound extraction theorems surveyed before actually **guarantee** that the extracted rates are primitive recursive (in the extended sense of Gödel's T) also in N. By a 'pointwise' treatment of the spaces X^N, e.g. in the way described in [11], the above extraction could be carried out 'locally' for any N but there would be no a priori complexity information on those rates seen as functions of N which are provided by this approach.

The second main component of the proofs is the Brezis-Haraux theorem. Based on the treatment of the latter above we are now in the position to explain the concrete proof minings from [20] and [34] in terms of logical

[7]Corrections to [20]: P.94, l.-13: '$A_1x_1 \times \ldots \times A_Nx_N$' instead of '$(A_1x_1, \ldots, A_Nx_N)$', P.95, last line: drop '$\forall\varepsilon \in (0,1)$', P.97, l.7: in the definition of α_K replace '$K(\varepsilon/4)$' by '$K(\varepsilon/4N^2)$' (3 times).

metatheorems: in the concrete application of Proposition 3.1 in the proof Theorem 4 in [20] via [20, Lemma 8], only a norm upper bound $K(\varepsilon)$ on $\tilde{T}p_\varepsilon$ (playing the role of p in Proposition 3.1) is imposed where

$$B := M := Id - R \text{ with } R(x_1, x_2, \ldots, x_N) := (x_N, x_1, \ldots, x_{N-1})$$

is an explicitly defined single-valued total Lipschitz-2 (and hence uniformly majorizable) operator on X^N and q being a point with $\|q\| \leq \varepsilon/4 \leq 1$ so that $\|\tilde{q} := M(q)\| \leq \varepsilon/2 \leq 1$ since $\varepsilon \leq 1$. Moreover, $\tilde{p}$ is taken in this application to be $q \in A(\tilde{T}p_\varepsilon)$ with $A = \tilde{T}^{-1} - I$, where $\tilde{T}(x) := (T_1(x_1), \ldots, T_N(x_N))$. Lemma 7 in [20] establishes that M is uniformly rectangular and provides a corresponding modulus. Note that it in fact corresponds exactly to our definition of 'uniformly rectangular' since M has - as discussed above a simple uniform majorant M^* so that Remark 2.19 applies where x, f in [20, Lemma 7] play the role of $x, \check{u}$ (that f is assumed to be bounded by 1 rather than some general bound n is only due to the fact that the lemma is used in this form but the proof easily extends to a general bound). In line with Remark 2.11, [20] did not need any extra modulus β witnessing that $\mathrm{dom}A \subseteq \mathrm{dom}B$ since the majorant M^* for $M(= B)$ does the job. The (quantitative) Brezis-Haraux theorem is applicable since, for $B := M$, the operator $A + B$ is maximally monotone by [2, Corollary 25.5(i)] which we 'hardwire' into our formal system by following the approach for maximal operators described in Section 2.1, in combination with Section 2.4: For the treatment of $A + M$, we add constants χ_{A+B}, $J^{\chi_{A+B}}$ (which take N as an additional input) with the axioms expressing monotonicity of $A + B$ as well as that $J^{\chi_{A+B}}$ codes the resolvents which fulfill $\mathrm{ran}(Id + \gamma(A + B)) = X$ as discussed in Section 2.1, here now formulated by additionally quantifying over N. As mentioned before, the function M on X^N can be represented by a closed term in the underlying language, e.g. via

$$M_N x := x -_{X^N} \lambda i^{\mathbb{N}}.x(\pi_N(i))$$

where x is of type $\mathbb{N} \to X$ and

$$\pi_N(i) := \begin{cases} N - 1 & \text{if } i = 0, \\ i - 1 & \text{otherwise,} \end{cases}$$

implements the right shift in the indices from the definition of M. To fit the format of the representation of set-valued operators chosen in our context, we add another constant χ_B (with additional input N) with the accompanying axioms

$$\begin{cases} \forall N^{\mathbb{N}}, x^{\mathbb{N}\to X}((\chi_B N)(x, M_N x) =_{\mathbb{N}} 0), \\ \forall N^{\mathbb{N}}, x^{\mathbb{N}\to X}, y^{\mathbb{N}\to X}((\chi_B N)(x, y) =_{\mathbb{N}} 0 \to M_N x =_{X^N} y), \end{cases}$$

which express that $\chi_B N$ (intentionally) represents the graph of M_N on X^N pointwise for every N. We can then proceed as described in Section 2.4 and use the corresponding axioms (i), (ii) with the uniform majorant M^* of M (which can similarly be given by a closed term) in place of ξ to express that χ_{A+B} represents the graph of $A + M$.

In [34], an extension of the result in [3] to compositions of averaged mappings due to [4] is proof-theoretically analyzed, building partly upon [20], and a rate of asymptotic regularity is extracted. [34] uses a Proposition 2.1 which 'expresses quantitatively the fact that cocoercive operators are rectangular'. Here the operator A in question is a total single valued β-cocoercive operator which therefore, in particular is $L := 1/\beta$-Lipschitzian (see [34]). The bounding assumptions made are that norm upper bounds on (in our notation) $x, \check{u}, u = A\check{u}$ are given. Using the Lipschitz continuity this also gives the missing upper bound on $\hat{x} := Ax$ since

$$\|Ax\| \leq \|Ax - A\check{u}\| + \|A\check{u}\| \leq L\|x - \check{u}\| + \|u\| \leq L(\|x\| + \|\check{u}\|) + \|u\|.$$

So [34, Proposition 2.1] states exactly that A has a modulus for being uniformly rectangular in our sense. The proof of [34, Theorem 2.2] then uses the special case of our Proposition 3.1 for the situation at hand (just as [20] had done in its respective setting discussed above).

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