A quantitative analysis of the "Lion-Man" game

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Abstract

In this paper we focus on a discrete lion and man game with an ε-capture criterion. We prove that in uniformly convex bounded domains the lion always wins and, using ideas stemming from proof mining, we extract a uniform rate of convergence for the successive distances between the lion and the man. As a byproduct of our analysis, we study the relation among different convexity properties in the setting of geodesic spaces.

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1 Introduction

The lion and man problem, which goes back to R. Rado, is one of the most challenging pursuit-evasion games and can be described as follows: a lion and a man move in a circular arena with equal maximum speeds. If the arena is viewed as a closed disc, the positions of the lion and of the man are regarded as two points (considering the initial position of the lion at the center of the disc) and the lion moves so that the center of the disc, its position and the one of the man are collinear in this order, can the lion catch the man? A detailed discussion of the solution to this problem can be found in [14, 27, 29]. Very similar problems have appeared under different names in the literature (e.g. the robot and the rabbit [20] or the cop and the robber [1]).

The analysis of the lion and man game is closely tied to the geometric structure of the domain where the game is played. This fact, as well as the potential applications in different fields such as robotics [7], biology [9] and random processes [10, 11], have given rise to several variants of this game, both continuous [8] and discrete (the discrete version is attributed to D. Gales, see [33] for more details). Such games involve one or more evaders in a fixed domain being hunted by one or more pursuers who win the game if certain appropriate capture criteria are satisfied. Such criteria may be physical capture (the pursuers move to the location of the evaders) or ε-capture [4] (the pursuers get within a distance less than ε to the evaders).

Here we focus on a discrete-time equal-speed game with an ε-capture criterion that was considered in [2, 3]. The domain A of our game is a convex subset of a uniquely geodesic space. Initially, the lion and the man are located at two points in A, L_0 and M_0, respectively. One fixes a positive upper bound $D > 0$ on the distance the lion and the man may jump. After n steps, the man moves from the point M_n to any point $M_{n+1} \in A$ which is within distance D. The lion moves from the point L_n to the point L_{n+1} along the geodesic from L_n to M_n such that its distance to L_n equals $\min\{D, d(L_n, M_n)\}$. We say that the lion wins if $\lim_{n\to\infty} d(L_{n+1}, M_n) = 0$. When we refer in the sequel to the Lion-Man game, we will always mean the game we have just described. In [3], it was stated that in CAT(0) spaces the lion always wins if and only if the domain is compact. Nonetheless, this characterization of compactness proved to be false as [6] contains an example of an unbounded CAT(0) space where the lion always wins. Further advances in this problem were made in [26] where a characterization of compactness of the domain in terms of the success of the lion was obtained in complete, locally compact, uniquely geodesic spaces that satisfy a betweenness property. In addition, the success of the lion in the Lion-Man game was also linked in this setting to the fixed point property for continuous mappings. However, none of these results provides any information on the speed of convergence towards 0 of the sequence $d(L_{n+1}, M_n)$.

Our aim in this paper is twofold: on the one hand to weaken the topological and geometric hypotheses that ensure the success of the lion, and on the other hand to give a rate of convergence for the sequence $d(L_{n+1}, M_n)$ that only depends on some geometric properties of the domain. The ideas that led to our results have their roots in proof mining. By "proof mining" we mean the logical analysis, using proof-theoretic tools, of mathematical proofs with the aim of extracting relevant information hidden in the proofs. This new information can be both of quantitative nature, such as algorithms and effective bounds, as well as of qualitative nature, such as uniformities in the bounds or weakening of the premises. A comprehensive reference for proof mining is the book [22]. The first step towards obtaining the desired rate of convergence is to introduce quantitative uniform versions of the convexity properties used in [26] as main ingredients in the study of the Lion-Man game. More precisely, the properties in question refer to the uniqueness of geodesics and a betweenness relation.

The organization of the paper is as follows. In Section 2 we analyze the geometric background that we later rely on to prove our main result. Although the existence of unique geodesics between any two given points in a geodesic space is a widely known and well-understood condition, here we consider a quantitative uniform version thereof (see Definition 2.4) and study its connection with other convexity properties. In Theorem 2.6 we prove that uniformly convex geodesic spaces admitting a monotone modulus of uniform convexity η are uniformly uniquely geodesic and one can define a modulus of uniform uniqueness in terms η. Actually, in normed vector spaces, uniform convexity is equivalent to uniform uniqueness of geodesics, but in general these two concepts are different. This distinction in nonlinear settings is an interesting feature of uniform uniqueness which could motivate the further study of its relevance. Along with the uniqueness of geodesics, a betweenness property also plays an important role in the study of the Lion-Man game. This property (see Definition 2.9) holds e.g. in all strictly convex normed spaces, but also in a wide class of geodesic spaces as we will point out. Betweenness relations have already been considered in very early works such as [21, 15]. We introduce a quantitative uniform variant of the betweenness property (see Definition 2.10) which will be central to the quantitative analysis of the Lion-Man game. In Theorem 2.13 we show that the uniform betweenness property holds in uniformly uniquely geodesic spaces whose distance function satisfies a convexity condition, and that a modulus of uniform uniqueness generates a modulus of uniform betweenness. In Section 3 we prove our main result contained in Theorem 3.2 and Corollary 3.3, which states that under the assumption of boundedness, uniform uniqueness, and uniform betweenness for the domain where the Lion-Man game is played, the lion always wins. The result applies in particular for all uniformly convex normed spaces, CAT(κ) spaces (of sufficiently small diameter for $\kappa > 0$), or compact uniquely geodesic spaces satisfying the betweenness property. Consequently, we notably weaken and unify previously known geometric conditions that were imposed on the domain in order to guarantee the success of the lion. Moreover, Corollary 3.3 provides a rate of convergence for the sequence $d(L_{n+1}, M_n)$ towards 0, and hence gives an explicit bound on the number of steps to be taken for an ε-capture. This rate of convergence is expressed in terms of the moduli of uniform uniqueness and of uniform betweenness, and can be explicitly computed e.g. in L_p spaces over measurable spaces with $1 < p < \infty$ or CAT(κ) spaces. The last section contains a general discussion on the proof mining techniques used to develop our quantitative analysis.

2 Convexity properties in geodesic spaces

This section discusses several geometric properties of geodesic spaces with emphasis on convexity notions that play an essential role in our main result. We start with a brief account of some basic definitions on geodesic spaces and refer to [12] for a more detailed treatment.

Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we denote the *closed ball* centered at x with radius r by $\overline{B}(x, r)$. If A is a nonempty and bounded subset of X, the *diameter* of A is

$$\operatorname{diam}(A) = \sup\{d(a, a') : a, a' \in A\},$$

the *separation* of A is

$$\operatorname{sep}(A) = \inf\{d(a, a') : a, a' \in A, a \neq a'\},$$

and the *distance* of a point $x \in X$ to A is

$$\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$

Let $x, y \in X$. A *geodesic* joining x to y is a mapping $\gamma : [0, l] \subseteq \mathbb{R} \to X$ such that $\gamma(0) = x$, $\gamma(l) = y$ and

$$d(\gamma(s), \gamma(s')) = |s - s'| \quad \text{for all } s, s' \in [0, l].$$

It follows that $l = d(x, y)$. We say that a geodesic γ *starts* at x if $\gamma(0) = x$. If every two points in X are joined by a (unique) geodesic, then X is called a *(uniquely) geodesic space*. The image $\gamma([0, l])$ of a geodesic γ is called a *geodesic segment* with endpoints x and y. A point $z \in X$ belongs to a geodesic segment with endpoints x and y if and only if there exists $t \in [0, 1]$ such that

$$d(z, x) = td(x, y) \quad \text{and} \quad d(z, y) = (1 - t)d(x, y).$$

In this case, if γ is the geodesic in question, then $z = \gamma(tl)$. When $t = 1/2$, we call such a point z a *midpoint* of x and y and also denote it by $m(x, y)$. In a geodesic space, two given points x and y may be joined by more than one geodesics and thus may have more than one midpoint. If there is a unique geodesic segment with endpoints x and y, we denote it by $[x, y]$ and in this case for all $t \in [0, 1]$ there exists only one point $z \in X$, denoted by $(1-t)x+ty$, satisfying $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$. In particular, x and y have a unique midpoint $m(x, y) = (1/2)x + (1/2)y$.

Definition 2.1. Let (X, d) be a geodesic space. We say that X is *strictly convex* if for all $z, x, y \in X$ with $x \neq y$ and all midpoints $m(x, y)$ of x and y we have

$$d(z, m(x, y)) < \max\{d(z, x), d(z, y)\}.$$

Strictly convex geodesic spaces are uniquely geodesic. Indeed, let γ_1 and γ_2 be two geodesics joining x to y. Denote $u_s = \gamma_1(s)$ and $v_s = \gamma_2(s)$, where $s \in [0, d(x, y)]$. If $u_s \neq v_s$, then taking any midpoint $m(u_s, v_s)$ it follows that

$$\begin{aligned} d(x, y) &\leq d(x, m(u_s, v_s)) + d(y, m(u_s, v_s)) \\ &< \max\{d(x, u_s), d(x, v_s)\} + \max\{d(y, u_s), d(y, v_s)\} \\ &= s + d(x, y) - s = d(x, y), \end{aligned}$$

a contradiction. Hence $u_s = v_s$ for any $s \in [0, d(x,y)]$, which shows that $\gamma_1 = \gamma_2$. This also shows that in Definition 2.1 one can equivalently consider some midpoint of x and y instead of all midpoints as this is enough to prove the uniqueness of geodesics.

Any normed vector space is a geodesic space. For this class of spaces, strict convexity is actually equivalent to the existence of unique geodesics between any two points. However, in general this equivalence fails to hold as the following example shows.

Example 2.2. The 2-*dimensional sphere* $\mathbb{S}^2$ is the set $\{u \in \mathbb{R}^3 : (u \mid u) = 1\}$, where $(\cdot \mid \cdot)$ is the Euclidean scalar product. Endowed with the distance $d : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ that assigns to each $(x,y) \in \mathbb{S}^2 \times \mathbb{S}^2$ the unique number $d(x,y) \in [0,\pi]$ such that $\cos d(x,y) = (x \mid y)$, $\mathbb{S}^2$ is a geodesic space called the spherical space. Any octant of $\mathbb{S}^2$ is a uniquely geodesic space that is not strictly convex.

Uniform convexity is a strengthening of strict convexity and was first introduced in the linear case in [13] and in a nonlinear setting in [18, 17]. Since then it was used in various forms in metric spaces and we consider here the following variant (see also [25]).

Definition 2.3. A geodesic space (X,d) is *uniformly convex* if for all $\varepsilon \in (0,2]$ and $r > 0$ there exists $\delta \in (0,1]$ such that for all $z,x,y \in X$ and all midpoints $m(x,y)$ we have

$$\left.\begin{array}{l} d(z,x) \leq r \\ d(z,y) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d(z, m(x,y)) \leq (1-\delta) r.$$

A mapping $\eta : (0,2] \times (0,\infty) \to (0,1]$ providing for given $r > 0$ and $\varepsilon \in (0,2]$ such a $\delta = \eta(\varepsilon, r)$ is called *a modulus of uniform convexity*. A modulus of uniform convexity is said to be *monotone* if it is nonincreasing in the second argument.

Every uniformly convex geodesic space is strictly convex, hence uniquely geodesic. Again, uniform convexity is in fact equivalent to the condition obtained by considering the above implication for some midpoint of x and y instead of all midpoints. Besides, one can show that every compact strictly convex geodesic space is uniformly convex.

In normed vector spaces that are uniformly convex in the sense of Definition 2.3, by rescaling balls, one can always find moduli of uniform convexity that do not depend on the second argument, namely on the radii r. In fact, one usually considers the notion of *the modulus of convexity* of a normed vector space X defined as the function $\delta : [0,2] \to [0,1]$ given by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\},$$

or equivalently,

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = 1, \|y\| = 1, \|x-y\| \geq \varepsilon\right\}.$$

Note that δ is nondecreasing on $[0,2]$ and continuous on $[0,2)$. The normed vector space X is uniformly convex (in the sense of Definition 2.3 and equivalently in the sense of [13]) if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. In this case δ is the largest possible modulus of uniform convexity one can define for X.

For $1 < p < \infty$, an L_p space over a measurable space is uniformly convex and, if δ is its modulus of convexity and $\eta : (0,2] \to (0,1]$ is defined by

$$\eta(\varepsilon) = \begin{cases} \dfrac{p-1}{8}\varepsilon^2, & \text{if } 1 < p \leq 2, \\ \dfrac{1}{p2^p}\varepsilon^p, & \text{if } 2 < p < \infty, \end{cases} \tag{2.1}$$

then $\delta(\varepsilon) \geq \eta(\varepsilon)$ for all $\varepsilon \in (0, 2]$. Hence, η is a modulus of uniform convexity for L_p.

A related notion is *the characteristic of convexity* of a normed vector space defined as the number

$$\varepsilon_0 = \sup\{\varepsilon \in [0, 2] : \delta(\varepsilon) = 0\}.$$

Then X is uniformly convex if and only if $\varepsilon_0 = 0$. In addition, δ is strictly increasing on $[\varepsilon_0, 2]$. These concepts and proofs of the properties mentioned above can be found e.g. in [16, Chapter 5].

A particular notion of uniform convexity, called p-uniformly convexity, was introduced by Ball, Carlen and Lieb [5] in the linear case and, more recently, in the setting of geodesic spaces by Naor and Silberman [28] in the following way: given $1 < p < \infty$, a geodesic space (X, d) is *p-uniformly convex* if there exists a parameter $c > 0$ such that for all $x, y, z \in X$, all $t \in [0, 1]$ and all geodesics γ joining x to y,

$$d(z, \gamma(td(x, y)))^p \leq (1 - t)\, d(z, x)^p + t\, d(z, y)^p - \frac{c}{2}\, t(1 - t)\, d(x, y)^p. \tag{2.2}$$

Thus, a geodesic space that is p-uniformly convex geodesic space with parameter $c > 0$ is uniformly convex (in the sense of Definition 2.3) and admits a modulus of uniform convexity that does not depend on the second argument

$$\eta(\varepsilon) = \frac{c}{8p}\varepsilon^p. \tag{2.3}$$

Estimations on c depending on the value of p can be found in [24]. Every L_p space over a measurable space is p-uniformly convex if $p > 2$ and 2-uniformly convex if $p \in (1, 2]$. As for geodesic spaces, every CAT(0) space is 2-uniformly convex with parameter $c = 2$ and, in this case, (2.2) provides a characterization of CAT(0) spaces. For $\kappa > 0$, any CAT(κ) space X with $\mathrm{diam}(X) < \pi/(2\sqrt{\kappa})$ is 2-uniformly convex with parameter $c = (\pi - 2\sqrt{\kappa}\,\varepsilon)\, \tan(\sqrt{\kappa}\,\varepsilon)$ for any $0 < \varepsilon \leq \pi/(2\sqrt{\kappa}) - \mathrm{diam}(X)$, see [31]. We remark at this point that CAT(κ) spaces are defined in terms of comparisons with the model planes i.e. the complete simply connected 2-dimensional Riemannian manifolds of constant sectional curvature κ. More precisely, in CAT(κ) spaces, geodesic triangles (which consist of three points and three geodesic segments joining them) are "thin" when compared to triangles with the same side lengths in the model planes. Note also that a normed real vector space which is CAT(κ) for some $\kappa \in \mathbb{R}$ is pre-Hilbert. A comprehensive exposition of CAT(κ) spaces can be found in [12].

Unless otherwise stated, in what follows we always assume that (X, d) is a uniquely geodesic space. As we pointed out in the introduction, for our main result we need a quantitative uniform version of the property that there exists exactly one geodesic joining two points in X, and we define it below.

Definition 2.4. We say that X is *uniformly uniquely geodesic* if for all $\varepsilon, b > 0$ there exists $\varphi > 0$ such that for all $x, y, z \in X$ and all $r_1, r_2 \in (0, b]$ we have

$$\left.\begin{array}{l} d(z, x) \leq r_1 \\ d(z, y) \leq r_2 + \varphi \\ d(x, y) \geq r_1 + r_2 \end{array}\right\} \quad \Rightarrow \quad \mathrm{dist}(z, [x, y]) < \varepsilon.$$

A mapping $\Phi : (0, \infty) \times (0, \infty) \to (0, \infty)$ providing for given $\varepsilon, b > 0$ such a $\varphi = \Phi(\varepsilon, b)$ is called *a modulus of uniform uniqueness.*

A somehow related property for CAT(0) spaces can be found in [22, Lemma 17.20] and [12, Chapter II, Lemma 9.15].

Proposition 2.5. Compact uniquely geodesic spaces are uniformly uniquely geodesic.

Proof. Although this fact is almost straightforward, we include its proof because uniform uniqueness of geodesics plays an essential role in this work. We argue by contradiction. Suppose that (X, d) is compact and uniquely geodesic, but not uniformly uniquely geodesic. Then there exist $\varepsilon, b > 0$ such that for all $n \in \mathbb{N}$ we can find points $x_n, y_n, z_n \in X$ and numbers $r_1^n, r_2^n \in (0, b]$ satisfying

$$d(z_n, x_n) \leq r_1^n, \quad d(z_n, y_n) \leq r_2^n + \frac{1}{n}, \quad d(x_n, y_n) \geq r_1^n + r_2^n,$$

and

$$\operatorname{dist}(z_n, [x_n, y_n]) \geq \varepsilon. \tag{2.4}$$

By compactness, we may assume that there exist $x, y, z \in X$ and $r_1, r_2 \in [0, b]$ such that $x_n \to x$, $y_n \to y$, $z_n \to z$, $r_1^n \to r_1$ and $r_2^n \to r_2$. Then

$$r_1 + r_2 \leq d(x, y) \leq d(z, x) + d(z, y) \leq r_1 + r_2,$$

from where $z \in [x, y]$. Note that if γ_n is the geodesic starting at x_n whose image is $[x_n, y_n]$, then (γ_n) converges uniformly to the geodesic starting at x whose image is $[x, y]$ (see, e.g., [12, Chapter I, Lemma 3.12]). This contradicts (2.4). □

In normed vector spaces that are uniformly uniquely geodesic, by rescaling balls, it is enough to define $\Phi(\cdot, 1)$ in order to obtain a modulus of uniform uniqueness: one can take $\Phi(\varepsilon, b) = b\Phi(\varepsilon/b, 1)$ for all $\varepsilon, b > 0$.

We show next that uniform convexity with a monotone modulus of uniform convexity η implies uniform uniqueness of geodesics and one can define a modulus of uniform uniqueness in terms of η.

Theorem 2.6. *Let (X, d) be a uniformly convex geodesic space that admits a monotone modulus of uniform convexity η. The X is uniformly uniquely geodesic and $\Phi : (0, \infty) \times (0, \infty) \to (0, \infty)$ defined by*

$$\Phi(\varepsilon, b) = \varepsilon \eta \left(\frac{\varepsilon}{b + \varepsilon}, b + \varepsilon \right)$$

is a modulus of uniform uniqueness for X.

In addition, if η can be written as $\eta(\varepsilon, r) = \varepsilon \tilde{\eta}(\varepsilon, r)$, where $\tilde{\eta}$ is nondecreasing in ε, then one can take

$$\Phi(\varepsilon, b) = \varepsilon \tilde{\eta} \left(\frac{\varepsilon}{b + \varepsilon}, b + \varepsilon \right).$$

Proof. Let $\varepsilon, b > 0$ and denote $\varphi = \Phi(\varepsilon, b)$. Note that $\varphi \leq \varepsilon$.

Take $x, y, z \in X$, $r_1, r_2 \in (0, b]$ with

$$d(z, x) \leq r_1, \quad d(z, y) \leq r_2 + \varphi, \quad \text{and} \quad d(x, y) \geq r_1 + r_2.$$

We need to show that $\operatorname{dist}(z, [x, y]) < \varepsilon$. Suppose, on the contrary, that $\operatorname{dist}(z, [x, y]) \geq \varepsilon$. In this case

$$\varepsilon \leq \operatorname{dist}(z, [x, y]) \leq d(z, y) \leq r_2 + \varphi,$$

so

$$r_2 + \varphi \geq \varepsilon. \tag{2.5}$$

Let $z' \in [x, y]$ such that $d(x, z') = r_1$. As $d(x, z) \leq r_1$ and

$$d(z, z') \geq \operatorname{dist}(z, [x, y]) \geq \varepsilon = \frac{\varepsilon}{r_1} r_1,$$

by uniform convexity,

$$d\left(x, m(z,z')\right) \leq \left(1 - \eta\left(\frac{\varepsilon}{r_1}, r_1\right)\right) r_1. \tag{2.6}$$

Since

$$r_1 + d(z', y) = d(x,y) \leq d(z,x) + d(z,y) \leq r_1 + r_2 + \varphi,$$

it follows that $d(z', y) \leq r_2 + \varphi$. Moreover, $d(y,z) \leq r_2 + \varphi$ and

$$d(z,z') \geq \varepsilon \geq \frac{\varepsilon}{b+\varepsilon}(r_2+\varphi).$$

Again, by uniform convexity,

$$d\left(y, m(z,z')\right) \leq \left(1 - \eta\left(\frac{\varepsilon}{b+\varepsilon}, r_2+\varphi\right)\right)(r_2+\varphi). \tag{2.7}$$

Hence,

$$\begin{aligned} r_1 + r_2 &\leq d(x,y) \leq d(x, m(z,z')) + d(y, m(z,z')) \\ &\leq \left(1 - \eta\left(\frac{\varepsilon}{r_1}, r_1\right)\right) r_1 + \left(1 - \eta\left(\frac{\varepsilon}{b+\varepsilon}, r_2+\varphi\right)\right)(r_2+\varphi) \quad \text{by (2.6) and (2.7)} \\ &\leq r_1 + r_2 - r_1\eta\left(\frac{\varepsilon}{r_1}, r_1\right) + \varphi - \varepsilon\eta\left(\frac{\varepsilon}{b+\varepsilon}, r_2+\varphi\right) \quad \text{by (2.5).} \end{aligned}$$

Using the monotonicity of η we obtain

$$0 < r_1\eta\left(\frac{\varepsilon}{r_1}, r_1\right) \leq \varphi - \varepsilon\eta\left(\frac{\varepsilon}{b+\varepsilon}, b+\varepsilon\right),$$

a contradiction.

Suppose now $\eta(\varepsilon, r) = \varepsilon\tilde{\eta}(\varepsilon, r)$ with $\tilde{\eta}$ nondecreasing in ε. As $d(z', y) \leq r_2 + \varphi$, $d(y,z) \leq r_2 + \varphi$ and

$$d(z,z') \geq \varepsilon = \frac{\varepsilon}{r_2+\varphi}(r_2+\varphi),$$

by uniform convexity and the monotonicity of η we have

$$\begin{aligned} d\left(y, m(z,z')\right) &\leq \left(1 - \eta\left(\frac{\varepsilon}{r_2+\varphi}, r_2+\varphi\right)\right)(r_2+\varphi) \\ &\leq \left(1 - \eta\left(\frac{\varepsilon}{r_2+\varphi}, b+\varepsilon\right)\right)(r_2+\varphi) \\ &= \left(1 - \frac{\varepsilon}{r_2+\varphi}\tilde{\eta}\left(\frac{\varepsilon}{r_2+\varphi}, b+\varepsilon\right)\right)(r_2+\varphi). \end{aligned}$$

Using the monotonicity of $\tilde{\eta}$ we obtain

$$d\left(y, m(z,z')\right) \leq \left(1 - \frac{\varepsilon}{r_2+\varphi}\tilde{\eta}\left(\frac{\varepsilon}{b+\varepsilon}, b+\varepsilon\right)\right)(r_2+\varphi). \tag{2.8}$$

The same reasoning as before applying now (2.8) instead of (2.7) finishes the proof. $\square$

In particular, using (2.1), for $1 < p < \infty$, L_p spaces over measurable spaces admit a modulus of uniform uniqueness $\Phi : (0,\infty) \times (0,\infty) \to (0,\infty)$ defined by

$$\Phi(\varepsilon, b) = \begin{cases} \dfrac{p-1}{8} \dfrac{\varepsilon^2}{(b+\varepsilon)}, & \text{if } 1 < p \leq 2, \\ \dfrac{1}{p2^p} \dfrac{\varepsilon^p}{(b+\varepsilon)^{p-1}}, & \text{if } 2 < p < \infty. \end{cases} \tag{2.9}$$

If X is p-uniformly convex with parameter c, then, according to (2.3),

$$\Phi(\varepsilon, b) = \frac{c}{8p} \frac{\varepsilon^p}{(b+\varepsilon)^{p-1}}, \tag{2.10}$$

acts as a modulus of uniform uniqueness for X.

Revisiting Example 2.2 we can immediately notice that there exist uniformly uniquely geodesic spaces that are not uniformly convex. Indeed, any octant of $\mathbb{S}^2$ is a compact uniquely geodesic space, thus, by Proposition 2.5, it is uniformly uniquely geodesic. However, it is not strictly convex, and hence not uniformly convex.

On the other hand, recall that in normed vector spaces, strict convexity is equivalent to uniqueness of geodesics. This equivalence still holds when passing to the uniform versions of these properties. Namely, Theorems 2.6 and 2.8 show that uniform uniqueness of geodesics is equivalent to uniform convexity, and respective moduli can be expressed in terms of each other. Before proving Theorem 2.8, we recall the following property of the modulus of convexity. Its proof can be found e.g. in [16, p. 56], but since it is short, for completeness we include it below.

Lemma 2.7 (Goebel and Kirk [16])**.** Let $(X, \|\cdot\|)$ be a normed vector space with modulus of convexity δ and characteristic of convexity ε_0. If $\varepsilon_0 < 2$, then

$$\delta(2(1-\delta(\varepsilon))) \leq 1 - \frac{\varepsilon}{2},$$

for all $\varepsilon \in (\varepsilon_0, 2]$.

Proof. Let $\varepsilon \in (\varepsilon_0, 2]$. Clearly, if $\delta(\varepsilon) = 1$ (which can only happen for $\varepsilon = 2$), then the inequality holds. Moreover, $\delta(\varepsilon) > 0$ and so we can assume that $\delta(\varepsilon) \in (0,1)$. Let $\tau \in (0, 1-\delta(\varepsilon))$ and take $x, y \in X$ with

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon, \quad \text{and} \quad \left\| \frac{x+y}{2} \right\| \geq 1 - \delta(\varepsilon) - \tau.$$

As δ is nondecreasing, we get $\delta(\|x+y\|) \geq \delta(2(1-\delta(\varepsilon)-\tau))$. Furthermore,

$$\frac{\varepsilon}{2} \leq \frac{\|x-y\|}{2} = \frac{\|x+(-y)\|}{2} \leq 1 - \delta(\|x-(-y)\|) = 1 - \delta(\|x+y\|).$$

Hence,

$$\delta(2(1-\delta(\varepsilon)-\tau)) \leq 1 - \frac{\varepsilon}{2}$$

and we only need to let $\tau \searrow 0$ to obtain the desired inequality. □

Theorem 2.8. *Let $(X, \|\cdot\|)$ be a normed vector space that is uniformly uniquely geodesic with a modulus of uniform uniqueness Φ satisfying $\Phi < 1$. Then X is uniformly convex and its modulus of convexity δ can be estimated by*

$$\delta(\varepsilon) \geq \frac{1}{2}\Phi\left(\frac{\varepsilon}{3}, 1\right),$$

for all $\varepsilon \in (0,2]$. In particular, $\eta : (0,2] \to (0,1]$ defined by

$$\eta(\varepsilon) = \frac{1}{2}\Phi\left(\frac{\varepsilon}{3}, 1\right)$$

is a modulus of uniform convexity for X.

Proof. Let ε_0 be the characteristic of convexity of X and denote for simplicity $\varphi : (0,\infty) \to (0,1)$, $\varphi(\varepsilon) = \Phi(\varepsilon, 1)$.

We show first that $\varepsilon_0 < 2$. If $\varepsilon_0 = 2$, then $\delta(2 - \varphi(1/2)) = 0$, so there exist $x, y \in X$ with

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq 2 - \varphi(1/2), \quad \text{and} \quad \left\|\frac{x+y}{2}\right\| > \frac{1}{2}.$$

Writing $\|y\| = 1 - \varphi(1/2) + \varphi(1/2)$, by uniform uniqueness,

$$\left\|\frac{x+y}{2}\right\| = \operatorname{dist}(0, [x,y]) < \frac{1}{2},$$

a contradiction.

Since δ is continuous on $[0,2)$, we have $\delta(\varepsilon_0) = 0$. Suppose that $\varepsilon_0 > 0$. Then we can take $\varepsilon \in (\varepsilon_0, 2)$ such that $\delta(\varepsilon) < \varphi(\varepsilon_0/2)/2$. Applying Lemma 2.7, there exist $x, y \in X$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq 2(1 - \delta(\varepsilon)), \quad \text{and} \quad \left\|\frac{x+y}{2}\right\| > \frac{\varepsilon_0}{2}.$$

Because $\|y\| = 1 - \varphi(\varepsilon_0/2) + \varphi(\varepsilon_0/2)$ and $\|x - y\| \geq 2 - \varphi(\varepsilon_0/2)$, we get

$$\left\|\frac{x+y}{2}\right\| < \frac{\varepsilon_0}{2},$$

another contradiction. Therefore, $\varepsilon_0 = 0$, so X is uniformly convex.

Assume now that $\delta(\varepsilon) < \varphi(\varepsilon/3)/2$ for some $\varepsilon \in (0,2]$. By Lemma 2.7, there exist $x, y \in X$ satisfying

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq 2(1 - \delta(\varepsilon)), \quad \text{and} \quad \left\|\frac{x+y}{2}\right\| > \frac{\varepsilon}{3}.$$

Arguing as before one shows that the assumption is false, hence the desired estimate holds. □

We describe next another convexity condition that refers to betweenness and plays in [26] an important role in the study of the Lion-Man game and whose quantitative uniform version will be essential in the proof of our main result. Postulates for betweenness relations and their relevance with other convexity conditions already appeared in very early works such as [21, 15]. Here we consider the following relation.

Definition 2.9. A subset A of a uniquely geodesic space X satisfies the *betweenness property* if for every four pairwise distinct points $x, y, z, w \in A$,

$$\left.\begin{array}{l} y \in [x,z] \\ z \in [y,w] \end{array}\right\} \Rightarrow \quad y, z \in [x,w].$$

This property was further studied in connection to the geometry of geodesic spaces in [32, 30, 26]. Proposition 3.4 in [30] shows in particular that the betweenness property holds in every uniquely geodesic space X satisfying the following convexity condition

$$d(z,(1-t)x+ty) \le (1-t)d(z,x)+td(z,y), \tag{2.11}$$

for all $x, y, z \in X$ and all $t \in [0,1]$. In other words, given any $z \in X$, the function $d(z,\cdot)$ is convex. This condition holds e.g. in any strictly convex normed space, any geodesic space that is nonpositively curved in the sense of Busemann or in any CAT(κ) space (of diameter smaller than $\pi/(2\sqrt{\kappa})$ if $\kappa > 0$).

We introduce now a quantitative uniform variant of this property.

Definition 2.10. We say that X satisfies the *uniform betweenness property* if for all $\varepsilon, a, b > 0$ there exists $\theta > 0$ such that for all $x, y, z, w \in X$ we have

$$\left.\begin{array}{l} \mathrm{sep}\{x,y,z,w\} \ge a \\ \mathrm{diam}\{x,y,z,w\} \le b \\ \mathrm{dist}(y,[x,z]) < \theta \\ \mathrm{dist}(z,[y,w]) < \theta \end{array}\right\} \quad \Rightarrow \quad \max\left\{\mathrm{dist}(y,[x,w]),\mathrm{dist}(z,[x,w])\right\} < \varepsilon.$$

A mapping $\Theta : (0,\infty)\times(0,\infty)\times(0,\infty)\to(0,\infty)$ providing for given $\varepsilon, a, b > 0$ such a $\theta = \Theta(\varepsilon, a, b)$ is called *a modulus of uniform betweenness.*

In the presence of compactness, betweenness coincides with uniform betweenness. The proof of this fact is similar to the one of Proposition 2.5.

Proposition 2.11. Compact uniquely geodesic spaces with the betweenness property satisfy the uniform betweenness property as well.

The remainder of this subsection is devoted to showing that uniformly uniquely geodesic spaces where (2.11) holds also satisfy the uniform betweenness property. In addition, given a modulus of uniform uniqueness, one can convert it into a modulus of uniform betweenness. First we prove a rather technical property needed to this end and also in the proof of the main result.

Lemma 2.12. Let (X, d) be a uniformly uniquely geodesic space with a modulus of uniform uniqueness Φ satisfying $\Phi(\varepsilon, b) < \varepsilon$ for all $\varepsilon, b > 0$. Let $\varepsilon, b > 0$, $x, y, z \in X$ with

$$\max\{d(x,y),d(x,z)\} \le b \quad \text{and} \quad d(y,z) \le \frac{1}{2}\Phi(\varepsilon,b).$$

Then

$$\max\{\mathrm{dist}(y_t,[x,z]),\mathrm{dist}(z_t,[x,y])\} < \varepsilon,$$

for all $t \in [0,1]$, where $y_t = (1-t)x+ty$ and $z_t = (1-t)x+tz$.

Proof. By symmetry, it is enough to prove that $\mathrm{dist}(y_t,[x,z]) < \varepsilon$. First note that for $t \in \{0,1\}$ the conclusion is immediate. Thus, we may assume $t \in (0,1)$. By the triangle inequality,

$$d(y_t,z) \le d(y_t,y)+d(y,z) = (1-t)d(x,y)+d(y,z).$$

If $(1-t)d(x,y) \le d(y,z)$, then

$$d(y_t,z) \le 2d(y,z) \le \Phi(\varepsilon,b) < \varepsilon,$$

hence $\mathrm{dist}(y_t, [x, z]) < \varepsilon$. Otherwise, $(1-t)d(x, y) > d(y, z)$ and we denote

$$r_1 = td(x, y) \quad \text{and} \quad r_2 = (1-t)d(x, y) - d(y, z).$$

Clearly, $r_1, r_2 \in (0, b]$. As $d(y_t, x) = td(x, y) = r_1$,

$$d(y_t, z) \le 2d(y, z) + r_2 \le \Phi(\varepsilon, b) + r_2$$

and

$$d(x, z) \ge d(x, y) - d(y, z) = r_1 + r_2,$$

by uniform uniqueness, $\mathrm{dist}(y_t, [x, z]) < \varepsilon$. $\square$

Theorem 2.13. *Let (X, d) be a uniformly uniquely geodesic space with a modulus of uniform uniqueness Φ satisfying $\Phi(\varepsilon, b) < \varepsilon$ for all $\varepsilon, b > 0$. Additionally, suppose that (2.11) holds for all $x, y, z \in X$ and all $t \in [0, 1]$. Then X satisfies the uniform betweenness property and the mapping $\Theta : (0, \infty) \times (0, \infty) \times (0, \infty) \to (0, \infty)$ defined by*

$$\Theta(\varepsilon, a, b) = \frac{1}{2} \min \left\{ \varepsilon, \Phi\left(\frac{a}{6b} \Phi\left(\frac{\Phi(\varepsilon/2, b)}{2}, b \right), b + \frac{\varepsilon}{2} \right) \right\}$$

for $\varepsilon \le a$ and by $\Theta(\varepsilon, a, b) = \Theta(a, a, b)$ for $\varepsilon > a$ is modulus of uniform betweenness for X.

Proof. Let $\varepsilon, a, b > 0$. Note that it is sufficient to consider the case $\varepsilon \le a$. Denoting $\theta = \Theta(\varepsilon, a, b)$, we have

$$\theta \le \frac{\varepsilon}{2} \le \frac{a}{2}. \tag{2.12}$$

Let $x, y, z, w \in X$ such that

$$\mathrm{sep}\{x, y, z, w\} \ge a, \quad \mathrm{diam}\{x, y, z, w\} \le b, \quad \mathrm{dist}(y, [x, z]) < \theta, \quad \text{and} \quad \mathrm{dist}(z, [y, w]) < \theta.$$

Denote

$$\tau = \frac{a}{6b} \Phi\left(\frac{\Phi(\varepsilon/2, b)}{2}, b \right).$$

Then

$$\theta < \frac{\tau}{2} \tag{2.13}$$

and

$$\tau < \frac{a}{6b} \frac{\Phi(\varepsilon/2, b)}{2} < \frac{a\varepsilon}{24b} \le \frac{a^2}{24b} \tag{2.14}$$

Let $z' \in [y, w]$ and $y^* \in [x, z]$ with $d(z', z) < \theta$ and $d(y^*, y) < \theta$. Observe now that

$$d(z', z) < \theta \le \frac{\Phi(\tau, b + \varepsilon/2)}{2}.$$

At the same time, $d(z, x) \le b$ and

$$d(z', x) \le d(z, x) + d(z', z) \le b + \theta \le b + \frac{\varepsilon}{2}.$$

Using Lemma 2.12 we deduce $\mathrm{dist}(y^*, [x, z']) < \tau$, hence there exists $y' \in [x, z']$ such that $d(y^*, y') < \tau$. Therefore,

$$d(y, y') \le d(y, y^*) + d(y^*, y') < \theta + \tau < \frac{3\tau}{2} \quad \text{by (2.13)}.$$

By the triangle inequality,

$$d(x,y) \leq d(x,y') + \frac{3\tau}{2} \quad \text{and} \quad d(y,z') \leq d(z',y') + \frac{3\tau}{2}.$$

Adding these two inequalities we obtain

$$d(x,y) + d(y,z') \leq d(x,z') + 3\tau. \tag{2.15}$$

As $z' \in [y,w]$, there exists $t \in [0,1]$ such that $z' = (1-t)y + tw$. Since

$$tb \geq td(y,w) = d(y,z') \geq d(y,z) - d(z,z') > a - \theta \geq \frac{a}{2} \quad \text{by (2.12)},$$

we get $t \geq a/(2b)$. Then

$$\begin{aligned} d(x,y) + td(y,w) &= d(x,y) + d(y,z') \\ &\leq d(x,z') + 3\tau \quad \text{by (2.15)} \\ &\leq (1-t)d(x,y) + td(x,w) + 3\tau \quad \text{by (2.11)}. \end{aligned}$$

Thus,

$$d(x,y) + d(y,w) \leq d(x,w) + \frac{3\tau}{t} \leq d(x,w) + \frac{6b}{a}\tau. \tag{2.16}$$

Denote $r_1 = d(x,y)$ and

$$r_2 = d(y,w) - \frac{6b}{a}\tau \geq a - \frac{6b}{a}\tau > \frac{3a}{4} > 0 \quad \text{by (2.14)}.$$

Thus, $r_1, r_2 \in (0,b]$, $d(x,w) \geq r_1 + r_2$ by (2.16), and

$$d(y,w) = r_2 + \frac{6b}{a}\tau = r_2 + \Phi\left(\frac{\Phi(\varepsilon/2,b)}{2}, b\right).$$

By uniform uniqueness,

$$\operatorname{dist}(y,[x,w]) < \frac{\Phi(\varepsilon/2,b)}{2} < \varepsilon.$$

This means that there exists $y^{**} \in [x,w]$ such that $d(y,y^{**}) < \Phi(\varepsilon/2,b)/2$. Because $z' \in [y,w]$, using again Lemma 2.12 we obtain $\operatorname{dist}(z',[y^{**},w]) < \varepsilon/2$, so $\operatorname{dist}(z',[x,w]) < \varepsilon/2$. Therefore,

$$\operatorname{dist}(z,[x,w]) \leq d(z,z') + \operatorname{dist}(z',[x,w]) < \theta + \frac{\varepsilon}{2} \leq \varepsilon \quad \text{by (2.12)}.$$

$\square$

3 A rate of convergence for the Lion-Man game

The main goal of this section is to analyze the Lion-Man game from the quantitative viewpoint using the convexity notions and results given in Section 2. We recall first the exact definition of the game.

Let (X,d) be a uniquely geodesic space and $A \subseteq X$ nonempty and convex. Take $D > 0$ and suppose that $L_0, M_0 \in A$ are the starting points of the lion and the man, respectively. At step $n+1$, $n \in \mathbb{N}$, the lion moves from the point L_n to the point $L_{n+1} \in [L_n, M_n]$ such that $d(L_n, L_{n+1}) = \min\{D, d(L_n, M_n)\}$. The man moves from the point M_n to any point $M_{n+1} \in A$ as long as $d(M_n, M_{n+1}) \leq D$. We say that lion wins if the sequence $(d(L_{n+1}, M_n))$ converges to 0. Otherwise the man wins.

Denote $D_n = d(L_n, M_n)$, $n \in \mathbb{N}$. Note first that if $D_n \geq D$, then

$$D_{n+1} \leq d(L_{n+1}, M_n) + d(M_n, M_{n+1}) = D_n - D + d(M_n, M_{n+1}) \leq D_n. \tag{3.17}$$

Thus, if $D_n \geq D$ for all $n \in \mathbb{N}$, then (D_n) is nonincreasing.

We can distinguish two mutually exclusive situations when the lion wins:

(1) there exists $n_0 \in \mathbb{N}$ such that $D_{n_0} < D$.

(2) $D_n \geq D$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} D_n = D$.

The main result of this paper provides a rate for D being an approximate upper bound on (D_n) (and as a consequence, a rate of convergence for the sequence $(d(L_{n+1}, M_n))$ towards 0) under appropriate regularity conditions imposed on the geodesic space that ensure the success of the lion. These conditions refer to uniform uniqueness of geodesics and uniform betweenness. We fix below the precise setting and notation we consider.

Let $A \subseteq X$ be a nonempty, convex and bounded set of diameter $b \geq D$ where the Lion-Man game is played. Take $N \in \mathbb{N}$ such that

$$b + 1 < ND. \tag{3.18}$$

Suppose that X is uniformly uniquely geodesic with a modulus of uniform uniqueness Φ. Moreover, assume that X satisfies the uniform betweenness property with a modulus of uniform betweenness Θ and that the moduli Φ and Θ satisfy the condition

$$\Phi(\varepsilon, b) < \varepsilon \quad \text{and} \quad \Theta(\varepsilon, a, b) < \varepsilon \quad \text{for all } \varepsilon, a > 0. \tag{3.19}$$

Additionally, denote

$$\Delta(\varepsilon) = \Theta(\varepsilon, \varepsilon, b) \quad \text{and} \quad \Psi(\varepsilon) = \frac{\Phi(\Delta(\varepsilon), b)}{2} \quad \text{for all } \varepsilon > 0.$$

Then

$$\Psi(\varepsilon) < \Delta(\varepsilon) < \varepsilon, \tag{3.20}$$

for all $\varepsilon > 0$.

Before giving our main result, we recall the following property of bounded nonincreasing real sequences which follows from Proposition 2.27 and Remark 2.29 in [22].

Lemma 3.1 (Kohlenbach [22])**.** Let $b > 0$ and (a_n) be a nonincreasing sequence in $[0, b]$. Then

$$\forall \tau > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists I \leq \tilde{g}^{(\lceil \frac{b}{\tau} \rceil)}(0)\, \forall n, m \in [0, g(I)]\ \left(|a_{I+n} - a_{I+m}| \leq \tau\right),$$

where $\tilde{g} = \mathrm{Id} + g$.

Theorem 3.2. *For all $\alpha > 0$ and all $n \geq \Omega_{D,b,N,\Phi,\Theta}(\alpha)$, $D_n < D + \alpha$, where*

$$\Omega_{D,b,N,\Phi,\Theta}(\alpha) = N + \Gamma_{b,\Phi,\Theta}(\varepsilon), \quad \Gamma_{b,\Phi,\Theta}(\varepsilon) = N \left\lceil \frac{b}{\Phi(\Psi^N(\varepsilon), b)} \right\rceil$$

and

$$0 < \varepsilon \leq \min\left\{\frac{1}{3N}, \frac{D}{4}, \frac{\alpha}{3}\right\}. \tag{3.21}$$

Proof. Let $\alpha > 0$. We use the notation introduced above. For simplicity, denote

$$\omega = \Omega_{D,b,N,\Phi,\Theta}(\alpha).$$

Suppose first that there exists $n_0 \in \mathbb{N}$ such that $D_{n_0} < D$. If $n_0 \le \omega$, then for all $n \ge n_0$, $D_n \le D < D + \alpha$ and the conclusion holds.

So we only need to consider the following two cases:

(i) there exists $n_0 > \omega$ with $D_{n_0} < D$ and $D_n \ge D$ for all $n \le n_0 - 1$.

(ii) $D_n \ge D$ for all $n \in \mathbb{N}$.

Observe that in case (i), applying (3.17), $D_{n+1} \le D_n$ for all $n \le n_0 - 1$. Then it is enough to show that there exists $n \le \omega$ such that $D_n < D + \alpha$. Indeed, if $k \in [\omega, n_0 - 1]$, then $D_k \le D_\omega \le D_n < D + \alpha$. Otherwise, if $k \ge n_0$, $D_k \le D < D + \alpha$.

For case (ii), as (D_n) is nonincreasing, again we only need to show that there exists $n \le \omega$ such that $D_n < D + \alpha$. Consequently, in the following we treat both cases at once.

Consider the sequence (E_n) defined by

$$E_n = \begin{cases} D_n, & \text{if } n \le \omega, \\ D, & \text{otherwise.} \end{cases}$$

This is a nonincreasing sequence in $[0, b]$ and we can apply Lemma 3.1 taking $\tau = \Phi(\Psi^N(\varepsilon), b)$ and the function g constantly equal to N. Thus, there exists $I \le \Gamma_{b,\Phi,\Theta}(\varepsilon)$ such that for all $n, m \in [0, N]$,

$$|D_{I+n} - D_{I+m}| \le \Phi(\Psi^N(\varepsilon), b). \tag{3.22}$$

Note that for all $n \in [0, N]$, $I + n \le \omega$, so $E_{I+n} = D_{I+n}$. Assume that for all $n \in [0, N]$, $D_{I+n} \ge D + \alpha$. Denoting $\gamma = D_{I+N} - D \ge \alpha$, we have

$$\begin{aligned} D + \gamma = D_{I+N} &\le D_{I+n} \\ &\le D_{I+N} + \Phi(\Psi^N(\varepsilon), b) \quad \text{by (3.22)} \\ &= D + \gamma + \Phi(\Psi^N(\varepsilon), b), \end{aligned}$$

hence

$$D + \gamma \le D_{I+n} \le D + \gamma + \Phi(\Psi^N(\varepsilon), b), \tag{3.23}$$

for all $n \in [0, N]$.

Denote now $l_n = L_{I+n}$ and $m_n = M_{I+n}$ for $n \in [0, N]$. Then $d(l_n, m_n) = D_{I+n}$.

Claim. $\operatorname{dist}(l_n, [l_0, m_n]) < \Psi^{N-n}(\varepsilon)$ and $d(l_0, l_n) \ge n(D - 3\varepsilon)$ for all $n \in [0, N]$.

Proof of Claim. For $n = 0$, the two inequalities are obviously true. Suppose that they hold for $n = k \le N - 1$. We prove that they also hold for $n = k + 1$.

Let $p \in [l_0, m_k]$ such that

$$d(l_k, p) < \Psi^{N-k}(\varepsilon) = \frac{1}{2}\Phi(\Delta(\Psi^{N-k-1}(\varepsilon)), b).$$

Because $l_{k+1} \in [l_k, m_k]$, by Lemma 2.12, $\operatorname{dist}(l_{k+1}, [p, m_k]) < \Delta(\Psi^{N-k-1}(\varepsilon))$, hence

$$\operatorname{dist}(l_{k+1}, [l_0, m_k]) < \Delta(\Psi^{N-k-1}(\varepsilon)) \tag{3.24}$$

and, by (3.20), $d(l_k, p) < \varepsilon$. Then

$$d(l_0, p) \geq d(l_0, l_k) - d(l_k, p) > k(D - 3\varepsilon) - \varepsilon. \tag{3.25}$$

At the same time, as $D + \gamma \leq d(l_k, m_k)$ by (3.23),

$$d(p, m_k) \geq d(l_k, m_k) - d(l_k, p) > D + \gamma - \varepsilon. \tag{3.26}$$

Adding (3.25) and (3.26),

$$d(l_0, m_k) = d(l_0, p) + d(p, m_k) > (k+1)D + \gamma - (3k+2)\varepsilon. \tag{3.27}$$

Hence,

$$\begin{aligned} d(l_0, l_{k+1}) &\geq d(l_0, m_k) - d(l_{k+1}, m_k) = d(l_0, m_k) - D_{I+k} + D \\ &\geq d(l_0, m_k) - \gamma - \Phi(\Psi^N(\varepsilon), b) \quad \text{by (3.23)} \\ &\geq d(l_0, m_k) - \gamma - \varepsilon \quad \text{by (3.19) and (3.20)} \\ &\geq (k+1)(D - 3\varepsilon) \quad \text{by (3.27).} \end{aligned}$$

Because $d(m_k, m_{k+1}) \leq D$, $d(m_k, l_{k+1}) \leq \gamma + \Phi(\Psi^N(\varepsilon), b)$, $d(m_{k+1}, l_{k+1}) \geq D + \gamma$ by (3.23), and $D, \gamma \in (0, b]$, using the uniform uniqueness and (3.20) we get

$$\operatorname{dist}(m_k, [m_{k+1}, l_{k+1}]) < \Psi^N(\varepsilon) < \Psi^{N-k}(\varepsilon) < \Delta(\Psi^{N-k-1}(\varepsilon)). \tag{3.28}$$

We verify next that $\operatorname{sep}\{l_0, l_{k+1}, m_k, m_{k+1}\} \geq \varepsilon$. Using (3.21), (3.27) and (3.23) it is easy to see that

1. $d(l_0, l_{k+1}) \geq (k+1)(D - 3\varepsilon) \geq \varepsilon$.
2. $d(l_0, m_k) \geq (k+1)D + \gamma - (3k+2)\varepsilon \geq (k+1)(D - 3\varepsilon) + \alpha \geq 4\varepsilon$.
3. $d(l_0, m_{k+1}) \geq d(l_0, m_k) - d(m_k, m_{k+1}) \geq k(D - 3\varepsilon) + \gamma - 2\varepsilon \geq \varepsilon$.
4. $d(l_{k+1}, m_k) = D_{I+k} - D \geq \gamma \geq 3\varepsilon$.
5. $d(l_{k+1}, m_{k+1}) = D_{I+k+1} \geq D + \gamma \geq 7\varepsilon$.
6. $d(m_k, m_{k+1}) \geq d(l_{k+1}, m_{k+1}) - d(l_{k+1}, m_k) \geq D + \gamma - \gamma - \varepsilon \geq 3\varepsilon$.

As $\operatorname{diam}\{l_0, l_{k+1}, m_k, m_{k+1}\} \leq b$, taking into account (3.24) and (3.28), the uniform betweenness property yields $\operatorname{dist}(l_{k+1}, [l_0, m_{k+1}]) < \Psi^{N-k-1}(\varepsilon)$. This finishes the proof of the claim. □

Consequently,

$$\begin{aligned} d(l_0, l_N) &\geq N(D - 3\varepsilon) \\ &> b + 1 - 3N\varepsilon \quad \text{by (3.18)} \\ &\geq b \quad \text{by (3.21),} \end{aligned}$$

a contradiction to the fact that b is the diameter of A. This shows that there exists $n \leq I + N \leq \omega$ such that $D_n < D + \alpha$. □

As an immediate consequence we obtain the next result.

Corollary 3.3. For all $\alpha > 0$ and all $n \geq \Omega_{D,b,N,\Phi,\Theta}(\alpha)$, $d(L_{n+1}, M_n) < \alpha$, where $\Omega_{D,b,N,\Phi,\Theta}$ is defined above.

Proof. Let $n \geq \Omega_{D,b,N,\Phi,\Theta}(\alpha)$. According to Theorem 3.2,

$$d(L_{n+1}, M_n) = \max\{0, D_n - D\} < \alpha.$$

□

Remark 3.4. As the domain of the game is the set A, it is sufficient for the conditions referring to uniform uniqueness and uniform betweenness to hold only on A.

Based on the discussion from Section 2, it is immediate that Theorem 3.2 applies in particular for L_p spaces over measurable spaces with $1 < p < \infty$ and CAT(κ) spaces with diameter smaller than $\pi/(2\sqrt{\kappa})$ if $\kappa > 0$. For these classes of spaces, using (2.1), (2.3) and Theorem 2.13, we have explicit moduli Φ and Θ satisfying (3.19) and therefore we can compute the exact expression of the rate of convergence provided by Theorem 3.2. Disregarding the quantitative aspect, in the setting of CAT(κ) spaces, this recovers corresponding results from [3].

Observe that Theorem 3.2 also guarantees the success of the lion when the Lion-Man game is played in a bounded and convex subset of a uniformly convex normed space. Moreover, by Propositions 2.5 and 2.11, we obtain as a consequence one of the implications proved in [26, Theorem 4.2], namely that the lion always wins the Lion-Man game played in a compact uniquely geodesic space that satisfies the betweenness property.

4 Comments on the use of logic in arriving at the quantitative analysis ('proof mining')

The point of departure for the investigation in this paper has been the noneffective proof for the convergence $\lim_{n\to\infty} D_n = D$, when $D_n \geq D$ for all $n \in \mathbb{N}$, for *compact* uniquely geodesic spaces satisfying the betweenness property as given in [26] (see Theorem 4.2). Since the sequence (D_n) *decreases* to D this statement is of the logical form

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N} \left(D_n < D + \frac{1}{2^n} \right) \in \forall\exists.$$

General logical metatheorems due to the first author (see, e.g., [22]) guarantee in such situations the extractability of an explicit and effective rate of convergence which only depends on general metric bounds, a modulus of total boundedness (as a quantitative form of the compactness assumption), and moduli providing quantitative forms of 'uniformized' versions of being 'uniquely geodesic' and satisfying the 'betweenness property'. Technically speaking, these moduli serve to produce a solution for the monotone Gödel functional interpretation (see [22]) of the respective properties which in this uniformized form become (essentially) purely universal assumptions in these moduli which can be taken as number-theoretic functions (although, for convenience, we used their ε/δ-variants). Hence, the moduli can w.l.o.g. be assumed to be self-majorizing (in the technical sense of [22]) which would not be the case if these moduli would not be uniform by depending on points in X.

In the case of uniqueness of geodesics, the process of uniformization and subsequent witnessing by a modulus $\Phi : \mathbb{N}^2 \to \mathbb{N}$ yields

$$\forall k, b \in \mathbb{N}\, \forall x, y, z \in X\, \forall r_1, r_2 \in [0, b] \left(\left. \begin{array}{l} d(z, x) < r_1 \\ d(z, y) < r_2 + \Phi(k, b) \\ d(x, y) > r_1 + r_2 \end{array} \right\} \Rightarrow \operatorname{dist}(z, [x, y]) \leq \frac{1}{2^k} \right),$$

which is (using that $<,>$ resp. $\leq$ on $\mathbb{R}$ are purely existential resp. purely universal relations; see [22, Chapter 4]) equivalent (essentially) to a purely universal sentence. We omit here various details on how to represent $[x, y]$ by a suitable operator W and also that there is a bounded quantifier hidden in

$$\text{dist}(z, [x, y]) \leq \frac{1}{2^k}$$

which should be written as

$$\exists t \in [0, 1] \left(d(z, W(t, x, y)) \leq \frac{1}{2^k} \right),$$

so that the sentence actually has the form of an axiom Δ as in [19] which is as good as being purely universal in this context. For convenience, we use in our official definition of the modulus the non-strict forms $\leq, \geq$ in the premise and $<$ instead of $\leq$ in the conclusion, which - modulo a simple shift in the modulus - of course is inessential. A similar discussion applies to the concepts of moduli of uniform convexity and the uniform betweenness property.

The actual extraction of the rate of convergence in these moduli from the convergence proof of [26, Theorem 4.2] made this proof completely constructive by avoiding altogether the (even nested) sequential compactness argument used in the original proof and hence the need to assume compactness in the first place *once* the uniqueness of geodesics and the betweenness property are written in their uniform variants (to which they are equivalent in the presence of compactness). As a consequence, the actual rate of convergence extracted only uses moduli for the uniform uniqueness of geodesics and the uniform betweenness property, but no modulus of total boundedness. The phenomenon that 'compactness' disappears from the proof in the process of its logical analysis is a feature of this particular proof being analyzed (see [23] for situations where this is not the case).

The uniqueness of geodesics follows from the strict convexity of a geodesic space. Again, the general methods from [22] guarantee that from a proof of this fact it must be possible to extract a procedure on how to translate a modulus of *uniform* convexity into a modulus of being *uniformly* uniquely geodesic (Theorem 2.6). In a normed vector space X, the uniqueness of geodesics implies conversely the strict convexity of X. In line with general proof mining facts, this explains why it must be possible to translate a modulus for being uniformly uniquely geodesic into a modulus of uniform convexity of X in the linear case (Theorem 2.8).

From the proof of the fact that the property of being uniquely geodesic together with the convexity condition (2.11) implies the betweenness property, the methods of proof mining guarantee that any given modulus for being *uniformly* uniquely geodesic can be transformed into a modulus for the uniform betweenness property (Theorem 2.13). This crucially uses that the convexity property can be written in purely universal form (and so only contributes to the verification of the new modulus but not to its construction) when uniqueness of geodesics is already assumed as then (2.11) follows from stipulating this only for $W(t, x, y)$ instead of corresponding points on any geodesic (with W satisfying the clauses $(i), (ii)$ of Definition 17.9 in [22]).

As usual with case studies in proof mining, when the actual extraction of the data in question is carried out it also comes with an ordinary analytical proof of their correctness which does not refer to any results from logic which, however, were instrumental for finding these data.

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