HIGHER ORDER REVERSE MATHEMATICS

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§1. Introduction. Reverse mathematics as developed by H. Friedman, S. Simpson and others (see [17] for a comprehensive treatment) focuses on the language of second order arithmetic 'because that language is the weakest one that is rich enough to express and develop the bulk of core mathematics' ([17], p.viii).
However, as we have argued in [15], already the treatment of continuous functions $f : X \to Y$ between Polish spaces X, Y not only requires a quite complicated encoding. Even more importantly, the restricted language makes it necessary (already for $X = \mathbb{N}^{\mathbb{N}}, Y = \mathbb{N}$) to use a constructively slightly enriched definition of continuous functions whose equivalence with the usual definition cannot be proved e.g. in the finite type extension $\mathbf{E\text{-}PA}^{\omega}$+QF-AC^{1,0} of (a variant with function variables instead of set variables of) the second order system **RCA** (i.e. $\mathbf{RCA}_0$ plus full induction, where $\mathbf{RCA}_0$ is the well-known base system used in reverse mathematics, see [17]). Here QF-AC^{1,0} denotes the schema of quantifier-free choice from functions to numbers. In fact, the encoding of continuous functions used in reverse mathematics amounts (for the spaces mentioned above) to the representation of such functions via an associate in the sense of Kleene and Kreisel. This representation, however, entails implicitly a (continuous) modulus of pointwise continuity which cannot be shown (in the finite type extension of **RCA** mentioned above) to exist for a general continuous functional $\varphi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. Of course, in the presence of arithmetical comprehension the difference between the encoding of continuous functionals and their direct treatment disappears. For functions $f : 2^{\mathbb{N}} \to \mathbb{N}$, already the binary König's lemma WKL suffices for this but it is open whether this holds e.g. in $\mathbf{E\text{-}PA}^{\omega}$+QF-AC^{1,0} (see [15] for all this).

Thus already for those parts of analysis which only deal with continuous functions, there are reasons to extend the context of reverse mathematics to the language of arithmetic in all finite types. This need becomes even more

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urgent if one considers principles involving non-continuous functions since whereas one *can* reason and quantify about continuous functions in systems based on the language of $\mathbf{RCA}_0$ (though only using the constructively enriched representation mentioned above), one cannot even talk about single discontinuous functions $f : \mathbb{R} \to \mathbb{R}$ as objects (of course it is possible to formulate $\forall\exists$-dependencies '$\forall x \in \mathbb{R} \exists! y \in \mathbb{R}\, A(x,y)$' such that the function $f : \mathbb{R} \to \mathbb{R}$ which is uniquely determined by this property is discontinuous. However, the existence of this function as an object cannot even been stated in the language of second order arithmetic).

In systems formulated in the language of functionals of all finite types, however, one can represent arbitrary (and hence in particular continuous) functions between Polish spaces in a rather direct way: the language contains variables for arbitrary functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and via the so-called standard representation of elements of Polish spaces X, Y by number theoretic functions, arbitrary functions $f : X \to Y$ are directly given as functionals $\Phi_f^{1\to 1} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ which happen to be extensional w.r.t. $=_X$ and $=_Y$, where $g^1 =_X h^1$ iff g, h represent the same element of X (similarly for Y).

The availability of variables for arbitrary (and not just continuous) functions within the language allows for an extension of reverse mathematics. In this paper we indicate that there is in fact an interesting kind of reverse mathematics for such principles which naturally takes place over a conservative finite type extension of $\mathbf{RCA}_0$ as base system.[1] As a natural candidate we propose the system $\mathbf{RCA}_0^\omega := \mathbf{E}\text{-}\mathbf{PRA}^\omega + \text{QF-AC}^{1,0}$, where $\mathbf{E}\text{-}\mathbf{PRA}^\omega$ is Feferman's ([4],[1]) restriction of $\mathbf{E}\text{-}\mathbf{PA}^\omega$ with quantifier-free induction and predicative primitive recursion only.[2]
We will show that $\mathbf{RCA}_0^\omega$ is conservative over $\mathbf{RCA}_0$ so that for principles which can be formalized already in $\mathbf{RCA}_0$ nothing is lost by using $\mathbf{RCA}_0^\omega$ as the base system.

In this paper we show that the principles which relative to $\mathbf{RCA}_0^\omega$ are equivalent to

$$(\exists^2) :\equiv \exists \varphi^2 \forall f^1 \big(\varphi(f) =_0 0 \leftrightarrow \exists x^0 (fx =_0 0)\big)$$

[1]Here (and also two sentences below) we again identify the official formulation of $\mathbf{RCA}_0$ (from [17]) with its (inessential) variant with function variables instead of set variables. As soon as we have defined that variant precisely in the next section we will call it $\mathbf{RCA}_0^2$ and reserve the name $\mathbf{RCA}_0$ for the official version. Note that Friedman's original systems proposed in [6] also had function variables.

[2]It is an easy exercise to show that $\mathbf{RCA}_0^\omega$ proves the second order axiom of Σ_1^0-induction on which $\mathbf{RCA}_0$ is based upon. 'Predicative' here means that we have only primitive recursion in the type 0 (but with parameters of arbitrary types). So for pure types this corresponds to the primitive recursive functionals in the sense of Kleene's ([9]) schemata S1-S8.

form a rich and very robust class. We conjecture that one gets further interesting and robust classes by considering other functional existence principles than $(\exists^2)$, like the existence of the Suslin operator ([1],[4])

$$(\text{Suslin}): \exists S^2 \forall f^1 \big(S(f) =_0 0 \leftrightarrow \exists g \forall x \big(f(\overline{g}x) =_0 0\big)\big)$$

This indicates that there is an interesting extension of the currently existing kind of reverse mathematics to higher order statements.

§2. Description of the theory $\mathbf{RCA}_0^\omega$. The set $\mathbf{T}$ of all finite types is defined inductively by

$$(i)\ 0 \in \mathbf{T} \text{ and } (ii)\ \rho, \tau \in \mathbf{T} \Rightarrow \rho \to \tau \in \mathbf{T}.$$

Terms which denote a natural number have type 0. Elements of type $\rho \to \tau$ are functions which map objects of type ρ to objects of type τ.
The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by

$$(i)\ 0 \in \mathbf{P} \text{ and } (ii)\ n \in \mathbf{P} \Rightarrow n+1 := n \to 0 \in \mathbf{P}.$$

Brackets whose occurrences are uniquely determined are often omitted. Also $\rho_1 \to \ldots \to \rho_k \to \tau$ stands for $\rho_1 \to (\rho_2 \ldots \to (\rho_k \to \tau)\ldots)$. For arbitrary types $\rho \in \mathbf{T}$ the degree of ρ (for short deg(ρ)) is defined by $\deg(0) := 0$ and $\deg(\rho \to \tau) := \max(\deg(\tau), \deg(\rho)+1)$.

The theory $\mathbf{E\text{-}PRA}^\omega$ is based on many–sorted classical logic formulated in the language of all finite types plus the combinators $\Pi_{\rho,\tau}, \Sigma_{\delta,\rho,\tau}$ which allow the definition of λ–abstraction.
Furthermore we include the axioms of extensionality

$$(E):\ \forall x^\rho, y^\rho, z^{\rho\to\tau}(x =_\rho y \to zx =_\tau zy)$$

for all finite types ($x =_\rho y$ is defined as $\forall z_1^{\rho_1}, \ldots, z_k^{\rho_k}(xz_1\ldots z_k =_0 yz_1 \ldots z_k)$ where $\rho = \rho_1 \to \ldots \to \rho_k \to 0$).
In addition to the defining axioms for the combinators, the Kleene recursor constant R_0, the equality axioms for type-0 equality and the successor axioms we have the schema of quantifier-free induction

$$\text{QF-IA: } A_0(0) \wedge \forall x\big(A_0(x) \to A_0(x')\big) \to \forall x A_0(x),$$

where A_0 is quantifier-free.
This finishes the description of $\mathbf{E\text{-}PRA}^\omega$. The theory $\mathbf{E\text{-}PA}^\omega$ is the extension of $\mathbf{E\text{-}PRA}^\omega$ obtained by the addition of the schema of full induction and all (impredicative) primitive recursive functionals in the sense of [7].

The schema of quantifier-free choice for the types ρ, τ is given by

$$\begin{aligned} \text{QF-AC}^{\rho,\tau}\ &:\ \forall x^\rho \exists y^\tau A_0(x,y) \to \exists Y^{\rho\to\tau} \forall x^\rho A_0(x, Yx), \\ \text{QF-AC}\ &:=\ \bigcup_{\rho,\tau\in\mathbf{T}} \{\text{QF-AC}^{\rho,\tau}\}, \end{aligned}$$

where A_0 is quantifier-free.

The theory $\mathbf{RCA}_0^\omega$ is defined as

$$\mathbf{RCA}_0^\omega :=\textbf{E-PRA}^\omega+\text{QF-AC}^{1,0}.$$

In deviating slightly from the 'official' definition of $\mathbf{RCA}_0$ with set variables we define a version with function variables as follows

$$\mathbf{RCA}_0^2 :=\textbf{E-PRA}^2+\text{QF-AC}^{0,0},$$

where $\textbf{E-PRA}^2$ denotes the second order fragment of $\textbf{E-PRA}^\omega$ (see [13] for details).

The base system $\mathbf{RCA}_0$ used in reverse mathematics can easily be seen as a subsystem of $\mathbf{RCA}_0^2$ by identifying sets with their characteristic functions. The axiom schemata of Σ^0_1-induction and Δ^0_1-comprehension from $\mathbf{RCA}_0$ are then easily derivable in $\mathbf{RCA}_0^2$ by QF-AC^{0,0} and – in the case of Σ^0_1-IA – the primitive recursive functional

$$\Phi_{it}(0,y,f) :=_0 y,\ \Phi_{it}(x+1,y,f) :=_0 f(x,\Phi_{it}(x,y,f)).$$

Conversely, $\mathbf{RCA}_0^2$ can be viewed as an inessential extension of $\mathbf{RCA}_0$ by identifying functions with their graphs. The only 'extension' provided by $\mathbf{RCA}_0^2$ is the existence of primitive recursive type-2-functionals (in the sense of Kleene) which allow to define a new function $g := \Phi(f)$ primitive recursively in a function f. However, this can be simulated in $\mathbf{RCA}_0$ in the form $\forall f \exists g A_\Phi(f,g)$, where $A_\Phi(f,g)$ expresses in terms of recursion equations that $g = \Phi(f)$.

Notation: For $\rho = \rho_1 \to \ldots \to \rho_k \to 0$, we define $1^\rho := \lambda x_1^{\rho_1}\ldots x_k^{\rho_k}.1^0$, where $1^0 := S0$.

In the following we will need the definition of the binary ('weak') König's lemma as given in [19]:

Definition 2.1 (Troelstra(74)).
WKL:$\equiv \forall f^1\big(T^\infty(f) \to \exists b \leq_1 \lambda k.1 \forall x^0 (f(\bar{b}x) =_0 0)\big)$,
where

$$T^\infty(f) :\equiv \begin{cases} \forall n^0, m^0(f(n*m) =_0 0 \to fn =_0 0) \\ \wedge \forall n^0, x^0(f(n*\langle x\rangle) =_0 0 \to x \leq_0 1) \\ \wedge \forall x^0 \exists n^0(lth\ n =_0 x \wedge fn =_0 0) \end{cases}$$

(i.e. $T^\infty(f)$ asserts that f represents an infinite 0,1–tree).

§3. First steps towards reverse mathematics in higher types.
In this section we show that various analytical principles are equivalent to $(\exists^2)$ (over our base system $\mathbf{RCA}_0^\omega$).
The fact that the class of these principles is rather rich and robust is mainly due to the following facts

1. a great deal of non-continuous analysis can be done already in $\mathbf{RCA}_0^\omega+(\exists^2)$
2. if a principle A implies the existence of a discontinuous function, then one can use an argument known as Grilliot's trick (see [8]) to derive the existence of $(\exists^2)$.

We first show that nothing is lost by working relative to the base system $\mathbf{RCA}_0^\omega$ instead of $\mathbf{RCA}_0^2$:

Proposition 3.1. *$\mathbf{RCA}_0^\omega$ is a conservative extension of $\mathbf{RCA}_0^2$.*

Proof: Locally, one can show in $\mathbf{RCA}_0^2$ that the type structure ECF of all extensional hereditarily continuous functionals (see [18] for the technical definition) forms a model of $\mathbf{RCA}_0^\omega$, i.e.

$$(1)\ \mathbf{RCA}_0^\omega \vdash A \Rightarrow\ \mathbf{RCA}_0^2 \vdash [A]_{\mathrm{ECF}}.$$

Together with the fact that

$$(2)\ \mathbf{RCA}_0^2 \vdash \forall f^1(\Phi(f) =_0 [\Phi]_{\mathrm{ECF}}(f))$$

for all ordinary primitive recursive functionals Φ^2 of type 2 (i.e. the functionals definable in $\mathbf{RCA}_0^2$) this yields the conservation result.
(1) is proved similarly to (and in fact easier than) the corresponding result for $\mathbf{E\text{-}HA}^\omega$+QF-AC from [18](2.6.20). In particular, no induction beyond Σ_1^0-IA is needed. $\dashv$

As a corollary of proposition 3.1 we get that the finite type extensions $\mathbf{RCA}_0^\omega$+WKL and $\mathbf{RCA}_0^\omega + \Pi_\infty^0$-CA etc. of the second order systems $\mathbf{WKL}_0$ and $\mathbf{ACA}_0$ used in reverse mathematics are conservative over their second order part.
However, we are now in the position to state conservation results which could not even been expressed with second order systems:

Theorem 3.2 ([4]). *$\mathbf{RCA}_0^\omega+(\exists^2)$ is conservative over first order Peano arithmetic* **PA**.

Definition 3.3. 1. *For k^0, f^1 we define*

$$(\lim_{n\to\infty} f(n) =_0 k) :\equiv \exists n \forall m > n(f(m) =_0 k).$$

2. *For $g^1, g_{(\cdot)}^{0\to1}$ we define*

$$(\lim_{n\to\infty} g_n =_1 g) :\equiv \forall k \exists n \forall m > n\big(\overline{g_m}(k) =_0 \overline{g}(k)\big).$$

3. *A functional Φ^2 is everywhere sequentially continuous if*

$$\forall g^1, \forall g_{(\cdot)}^{0\to1}\big(\lim_{n\to\infty} g_n =_1 g \to \lim_{n\to\infty} \Phi(g_n) =_0 \Phi(g)\big).$$

4. *A functional* $\Phi^{1\to1}$ *is everywhere sequentially continuous if*

$$\forall g^1, \forall g^{0\to1}_{(\cdot)}\big(\lim_{n\to\infty} g_n =_1 g \to \lim_{n\to\infty}\Phi(g_n) =_1 \Phi(g)\big).$$

LEMMA 3.4. $\mathbf{RCA}_0^\omega$ *proves that the existence of a not everywhere sequentially continuous functional* $\Phi^{1\to1}$ *implies the existence of a not everywhere sequentially continuous functional* Ψ^2.

Proof: Define $\tilde{\Phi}(f,k) := \overline{\Phi(f)}(k)$.[3] If $\lambda f.\tilde{\Phi}(f,k)$ is everywhere sequentially continuous for all fixed k then Φ is everywhere sequentially continuous. So if Φ is not everywhere sequentially continuous there must exist a k such that $\Psi := \lambda f.\tilde{\Phi}(f,k)$ is not everywhere sequentially continuous. ⊣

DEFINITION 3.5. *A functional* $\Phi^{1\to1}$ *is called everywhere* ε-δ*-continuous if*

$$\forall g^1, k^0 \exists n^0 \forall h^1(\overline{g}n =_0 \overline{h}n \to (\overline{\Phi g})k =_0 (\overline{\Phi h})k).$$

PROPOSITION 3.6. 1. $\mathbf{RCA}_0^\omega$ *proves that* $\Phi^{1\to1}$ *is everywhere sequentially continuous iff* Φ *is everywhere* ε-δ*-continuous.*

2. $\mathbf{RCA}_0^\omega$+QF-AC^{0,1} *proves*

$$\forall \Phi^{1\to1}, g^1(\Phi \textit{ sequentially continuous in } g \leftrightarrow \Phi\ \varepsilon\text{-}\delta\textit{-continuous in } g).$$

Proof: 1) We reason in $\mathbf{RCA}_0^\omega$. It is trivial that ε-δ-continuity implies sequential continuity. For the converse, let $\Phi^{1\to1}$ be everywhere sequentially continuous and assume that there exists a point g^1 at which Φ is not ε-δ-continuous, i.e. for some k^0

$$\forall n^0 \exists h^1(\overline{g}n =_0 \overline{h}n \wedge (\overline{\Phi g})k \neq (\overline{\Phi h})k).$$

By the global sequential continuity of Φ this implies

$$\forall n^0 \exists i^0(\overline{g}n =_0 (\overline{\lambda j.(i)_j})n \wedge (\overline{\Phi g})k \neq (\overline{\Phi(\lambda j.(i)_j)})k),$$

where we refer again to the sequence coding as e.g. carried out in [18], i.e. $\lambda j.(i)_j$ denotes the function which continues the finite sequence encoded by i with 0's. By QF-AC^{0,0} this yields

$$\exists \alpha^1 \forall n^0(\overline{g}n =_0 (\overline{\lambda j.(\alpha n)_j})n \wedge (\overline{\Phi g})k \neq (\overline{\Phi(\lambda j.(\alpha n)_j)})k).$$

Hence with $g_n := \lambda j.(\alpha n)_j$ we have

$$\forall n(\overline{g}n = \overline{g_n}n \wedge (\overline{\Phi g})k \neq (\overline{\Phi g_n})k),$$

i.e. Φ is not sequential continuous in g in contradiction to its assumed global sequential continuity.
2) is proved similarly noting that the only use of *global* sequential continuity of Φ made was to replace '$\exists h$' by '$\exists i$'. In the presence of QF-AC^{0,1}, however, we can keep '$\exists h$' and form directly the required sequence g_n. ⊣

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[3]Here $\overline{\alpha}k$ denotes the sequence code of $(\alpha(0), \dots, \alpha(k-1))$ (see e.g. [18]).

The following result essentially is the observation that a recursion theoretic argument known as 'Grilliot's trick' can be carried out in $\mathbf{RCA}_0^\omega$:

PROPOSITION 3.7. *Relative to* $\mathbf{RCA}_0^\omega$ *the following principles are equivalent:*

1. $(\exists^2)$,
2. *there exists a functional* Φ^2 *which is not everywhere sequentially continuous.*
3. *there exists a functional* $\Phi^{1\to 1}$ *which is not everywhere sequentially continuous.*
4. *there exists a functional* $\Phi^{1\to 1}$ *which is not everywhere* ε-δ*-continuous.*

Proof: 1. $\to$ 2. and 2. $\to$ 3. are obvious. 3. $\to$ 2. follows from lemma 3.4. The equivalence of 3. and 4. follows from proposition 3.6. So it remains to show that 2. $\to$ 1.
2. implies the existence of $\Phi^2, g_{(\cdot)}^{0\to 1}, g^1$ such that

$$\forall n \exists m \big(\overline{g_m}(n) = \overline{g}(n) \wedge \Phi(g_m) \neq \Phi(g)\big).$$

With QF-AC^{0,0} this yields

$$\exists h^1 \forall n \big(\overline{g_{h(n)}}(n) = \overline{g}(n) \wedge \Phi(g_{h(n)}) \neq \Phi(g)\big).$$

So with $\tilde{g}_n(k) := g_{h(n+1)}(k)$ and $\tilde{\Phi}(f) := \begin{cases} 1, & \text{if } \Phi(f) \neq \Phi(g) \\ 0, & \text{otherwise} \end{cases}$ we get

$$\forall n \forall i \leq n (\tilde{g}_n(i) = g(i)) \wedge \forall n, m \big(\tilde{\Phi}(\tilde{g}_n) = \tilde{\Phi}(\tilde{g}_m) \neq \tilde{\Phi}(g)\big).$$

We are now in the position to apply Grilliot's trick as in the proof of prop.3.4 in [14]. For completeness we repeat that short argument here:
In $\mathbf{RCA}_0^\omega$ we can define a functional $\xi(f^1, \tilde{g}_{(\cdot)}^{0\to 1}, i^0)$ such that

$$\xi(f, \tilde{g}_{(\cdot)}, i) = \begin{cases} \tilde{g}_j(i), & \text{for the least } j < i \text{ such that } f(j) > 0 \text{ if it exists} \\ \tilde{g}_i(i), & \text{otherwise.} \end{cases}$$

Using $\forall j \forall i \leq j (\tilde{g}_j(i) = \tilde{g}_i(i))$ and $\forall i (\tilde{g}_i(i) = g(i))$ one gets

$$(1)\ \exists j \big(f(j) > 0\big) \to \xi(f, \tilde{g}_{(\cdot)}) =_1 \tilde{g}_j \text{ for the least such } j$$

and

$$(2)\ \forall j \big(f(j) = 0\big) \to \xi(f, \tilde{g}_{(\cdot)}) =_1 g.$$

Hence by the extensionality axiom for type-2-functionals we obtain

$$\forall j \big(f(j) = 0\big) \leftrightarrow \tilde{\Phi}(\xi(f, \tilde{g}_{(\cdot)})) = \tilde{\Phi}(g).$$

Thus $\varphi := \lambda f^1.\overline{sg}(|\tilde{\Phi}(\xi(\overline{sg} \circ f, \tilde{g}_{(\cdot)})) - \tilde{\Phi}(g)|)$, where $\overline{sg}(x) := 0$ for $x \neq 0$ and $\overline{sg}(x) := 1$ otherwise, satisfies $(\exists^2)$. $\dashv$

DEFINITION 3.8.

1. $(\mu^2) :\equiv \exists \mu^2 \forall f^1 \big(\exists x^0 (fx = 0) \to f(\mu f) = 0\big)$ (see [4]),

2. *The uniform weak König's lemma* UWKL *is the principle*

$$\mathrm{UWKL} :\equiv \exists \Phi^{1\to 1} \forall f^1 \big(T^\infty(f) \to \forall x^0 \big(f((\overline{\Phi f})x) = 0\big)\big) \quad ([14]).$$

PROPOSITION 3.9 ([14]). *Relative to* $\mathbf{RCA}_0^\omega$ *the following principles are pairwise equivalent:*

(i) $(\exists^2)$,
(ii) (μ^2),
(iii) UWKL.

REMARK 3.10. *In addition to* WKL *and* UWKL *one can also consider an intermediate 'weak' uniform version of* WKL *which asserts for every given sequence* $(f_n)_{n\in\mathbb{N}}$ *of infinite binary trees the existence of a sequence* $(b_n)_{n\in\mathbb{N}}$ *of infinite paths* b_n *of* f_n. *This version however is implied already by* WKL *(relative to* $\mathbf{RCA}_0^\omega$*).*

We now sketch the representation of real numbers and functions $f : \mathbb{R} \to \mathbb{R}$ but only to the very limited extent needed here (for more details see [2],[10] and [12]. A systematic treatment of a general theory of representations can be found in [21]). Rational numbers are represented as codes $j(n,m)$ of pairs (n,m) of natural numbers n,m. $j(n,m)$ represents

the rational number $\frac{\frac{n}{2}}{m+1}$, if n is even,
and the negative rational $-\frac{\frac{n+1}{2}}{m+1}$, if n is odd.

Here j is the surjective pairing function $j(x,y) := \frac{1}{2}\big((x+y)^2+3x+y\big)$. On the codes of $\mathbb{Q}$, i.e. on $\mathbb{N}$, we have an equivalence relation by

$$n_1 =_{\mathbb{Q}} n_2 :\equiv \frac{\frac{j_1 n_1}{2}}{j_2 n_1 + 1} = \frac{\frac{j_1 n_2}{2}}{j_2 n_2 + 1} \text{ if } j_1 n_1, j_1 n_2 \text{ both are even}$$

and analogously in the remaining cases, where $\frac{a}{b} = \frac{c}{d}$ is defined to hold iff $ad =_0 cb$ (for $bd > 0$).

On $\mathbb{N}$ one easily defines functions $|\cdot|_{\mathbb{Q}}, +_{\mathbb{Q}}, -_{\mathbb{Q}}, \cdot_{\mathbb{Q}}\ :_{\mathbb{Q}}, \max_{\mathbb{Q}}, \min_{\mathbb{Q}} \in \mathbf{RCA}_0^\omega$ and (quantifier–free) relations $<_{\mathbb{Q}}, \leq_{\mathbb{Q}}$ which represent the corresponding functions and relations on $\mathbb{Q}$. We sometimes omit the index $\mathbb{Q}$ if this does not cause any confusion. We write $\langle q \rangle$ to denote the canonical code of $q \in \mathbb{Q}$.

We next want to represent real numbers as Cauchy sequences of rational number with rate of convergence 2^{-n}. Using the encoding of rational numbers by natural numbers, such a Cauchy sequence is given by a function f^1 satisfying

$$(*)\ \forall n \forall m, \tilde{m}\big(m, \tilde{m} \geq n \to |f(m) -_{\mathbb{Q}} f(\tilde{m})| <_{\mathbb{Q}} \langle 2^{-n} \rangle\big).$$

$(*)$ is implied by

$$(**)\ \forall n\big(|f(n) -_{\mathbb{Q}} f(n+1)| <_{\mathbb{Q}} \langle 2^{-n-1} \rangle\big)$$

and conversely for any f satisfying $(*)$, $\tilde{f}(n) := f(n+1)$ satisfies $(**)$. That is why we can use the more convenient condition $(**)$ on our representing sequences instead of $(*)$. To achieve that **any** function f^1 can be viewed as a representative of (a uniquely determined) real number we use the construction

$$\widehat{f}(n) := \begin{cases} f(n), \text{ if } \forall k < n(|f(k) -_{\mathbb{Q}} f(k+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-k-1}\rangle), \\ f(k) \text{ for the least } k < n \text{ s.t. } |f(k) -_{\mathbb{Q}} f(k+1)| \geq_{\mathbb{Q}} \langle 2^{-k-1}\rangle, \\ \qquad\qquad \text{otherwise.} \end{cases}$$

$\widehat{f}$ always satisfies $(**)$ and if already f satisfies $(**)$ then $f =_1 \widehat{f}$. So in particular $\widehat{\widehat{f}} =_1 \widehat{f}$.
On the representatives of reals, i.e. on the number theoretic functions f_1^1, f_2^1, we can define an equivalence relation $=_{\mathbb{R}}$

$$f_1 =_{\mathbb{R}} f_2 :\equiv \forall n\big(\widehat{f}_1(n+1) -_{\mathbb{Q}} \widehat{f}_2(n+1)| <_{\mathbb{Q}} \langle 2^{-n}\rangle\big),$$

which holds iff f_1 and f_2 represent the same real number. Similarly one defines relations $\leq_{\mathbb{R}}$ and $<_{\mathbb{R}}$. Note that $=_{\mathbb{R}}, \leq_{\mathbb{R}} \in \Pi^0_1$ while $<_{\mathbb{R}} \in \Sigma^0_1$. The usual arithmetical operations $+_{\mathbb{R}}, -_{\mathbb{R}}$ etc. can easily be defined as functionals (definable in $\mathbf{RCA}_0^\omega$) on the representation of the real numbers. Functions $F : \mathbb{R} \to \mathbb{R}$ are represented as functionals $\Phi^{1\to 1}$ which satisfy

$$\forall f_1, f_2\big(f_1 =_{\mathbb{R}} f_2 \to \Phi(f_1) =_{\mathbb{R}} \Phi(f_2)\big).$$

In a similar but technically somewhat more involved way one can also represent more general Polish spaces X, Y by $\mathbb{N}^{\mathbb{N}}$ and functions $F : X \to Y$ as functionals $\Phi^{1\to 1}$ respecting the corresponding equivalence relations $=_X$ and $=_Y$ (for details see e.g. [10]).

Lemma 3.11. $\mathbf{RCA}_0^\omega$ *proves*

1. $\forall f_1, f_2, n\big(\overline{f}_1(n+2) =_0 \overline{f}_2(n+2) \to |f_1 -_{\mathbb{R}} f_2| <_{\mathbb{R}} \langle 2^{-n}\rangle\big)$.
2. $\forall f^1, f_{(\cdot)}^{0\to 1}\big(\forall n(|f_n -_{\mathbb{R}} f| <_{\mathbb{R}} \langle 2^{-n-2}\rangle) \to$
$\qquad \exists \tilde{f}^1, \tilde{f}_{(\cdot)}^{0\to 1}(\tilde{f} =_{\mathbb{R}} f \wedge \forall n(\tilde{f}_n =_{\mathbb{R}} f_n \wedge \overline{\tilde{f}_n}(n) =_0 \overline{\tilde{f}}(n)))\big)$.

Proof: 1. follows from

$$\overline{f}_1(n+2) =_0 \overline{f}_2(n+2) \to \overline{\widehat{f}_1}(n+2) =_0 \overline{\widehat{f}_2}(n+2)$$

and

$$|f -_{\mathbb{R}} \lambda k.\widehat{f}(n+1)| <_{\mathbb{R}} \langle 2^{-n-1}\rangle.$$

2. Define $\tilde{f} := \lambda k.\widehat{f}(k+3)$, $\tilde{f}_n(k) := \begin{cases} \widehat{f}_n(k+3), \text{ for } k \geq n \\ \widehat{f}(k+3), \text{ for } k < n. \end{cases}$

It is clear that

$$\tilde{f} =_{\mathbb{R}} f \wedge \forall n\big(\overline{\tilde{f}_n}(n) =_0 \overline{\tilde{f}}(n)\big).$$

It remains to show that $\forall n(\tilde{f}_n =_{\mathbb{R}} f_n)$. This easily follows from the fact that $\tilde{f}_n$ satisfies $(**)$ (so that $\widehat{\tilde{f}}_n =_1 \tilde{f}_n$ for all n). Thus we have to show the latter. The only problematic case is $|\tilde{f}_n(n-1) -_{\mathbb{Q}} \tilde{f}_n(n)| < \langle 2^{-n}\rangle$ (for $n \geq 1$) which we establish as follows (using the assumption $\forall n(|f_n -_{\mathbb{R}} f| <_{\mathbb{R}} \langle 2^{-n-2}\rangle)$):

$$\begin{aligned}
&|\tilde{f}_n(n-1) -_{\mathbb{Q}} \tilde{f}_n(n)| =_{\mathbb{Q}} |\widehat{f}(n+2) -_{\mathbb{Q}} \widehat{f}_n(n+3)| \\
&\leq_{\mathbb{Q}} |\widehat{f}(n+2) -_{\mathbb{Q}} \widehat{f}_n(n+2)| + |\widehat{f}_n(n+2) -_{\mathbb{Q}} \widehat{f}_n(n+3)| \\
&\leq_{\mathbb{R}} |\lambda k.\widehat{f}(n+2) -_{\mathbb{R}} f| + |f -_{\mathbb{R}} f_n| + |f_n -_{\mathbb{R}} \lambda k.\widehat{f}_n(n+2)| \\
&\qquad + |\widehat{f}_n(n+2) -_{\mathbb{Q}} \widehat{f}_n(n+3)| \\
&<_{\mathbb{R}} \langle 2^{-n-2} + 2^{-n-2} + 2^{-n-2} + 2^{-n-3}\rangle <_{\mathbb{Q}} \langle 2^{-n}\rangle.
\end{aligned}$$

$\dashv$

PROPOSITION 3.12. *The following principles are pairwise equivalent relative to* $\mathbf{RCA}_0^\omega$:

1. $(\exists^2)$,
2. *the function* $F : \mathbb{R} \to \mathbb{R}$ *determined by*

$$F(x) := \begin{cases} 0, & \text{for } x \leq_{\mathbb{R}} 0 \\ 1, & \text{for } x >_{\mathbb{R}} 0 \end{cases}$$

   exists,
3. *there exists a function* $F : \mathbb{R} \to \mathbb{R}$ *which is not everywhere sequentially continuous,*
4. *there exists a function* $F : \mathbb{R} \to \mathbb{R}$ *which is not everywhere* ε-δ-*continuous.*

Proof: 1. $\to$ 2. and 2. $\to$ 3. are obvious. The equivalence of 3. and 4. follows similarly to proposition 3.6.1 using that $<_{\mathbb{R}} \in \Sigma^0_1$.
It remains to show that 3. $\to$ 1. Let $F : \mathbb{R} \to \mathbb{R}, x \in \mathbb{R}$ and (x_n) be a sequence in $\mathbb{R}$ such that

$$x_n \to x \wedge \neg(F(x_n) \to F(x)),$$

where '$\to$' indicates convergence in the sense of $\mathbb{R}$. Then

$$\exists l \forall k \exists n \big(|x_n -_{\mathbb{R}} x| <_{\mathbb{R}} \langle 2^{-k}\rangle \wedge |F(x_n) -_{\mathbb{R}} F(x)| >_{\mathbb{R}} \langle 2^{-l}\rangle\big).$$

Since $<_{\mathbb{R}} \in \Sigma^0_1$, we can apply QF-AC^{0,0} to obtain

$$\exists l \exists g \forall k \big(|x_{g(k)} -_{\mathbb{R}} x| <_{\mathbb{R}} \langle 2^{-k-3}\rangle \wedge |F(x_{g(k)}) -_{\mathbb{R}} F(x)| >_{\mathbb{R}} \langle 2^{-l}\rangle\big).$$

Lemma 3.11.2 applied to $f_k := x_{g(k)}$ yields $\tilde{f}_k, \tilde{f}$ with

$$\tilde{f} =_{\mathbb{R}} x \wedge \forall k \big(\tilde{f}_k =_{\mathbb{R}} x_{g(k)} \wedge \overline{\tilde{f}_k}(k) =_0 \overline{\tilde{f}}(k)\big).$$

F is given by some functional $\Phi^{1\to 1}$. Using the extensionality of Φ w.r.t. $=_{\mathbb{R}}$ we get

$$\forall k \big(|\Phi(\tilde{f}_k) -_{\mathbb{R}} \Phi(\tilde{f})| >_{\mathbb{R}} \langle 2^{-l}\rangle\big)$$

and hence by lemma 3.11.1

$$\forall k\big(\overline{\Phi(\tilde{f}_k)}(l+2) \neq \overline{\Phi(\tilde{f})}(l+2)\big).$$

So put together we have shown that

$$\exists l \forall k\big(\overline{\tilde{f}}_k(k) =_0 \overline{\tilde{f}}(k) \wedge \overline{\Phi(\tilde{f}_k)}(l+2) \neq \overline{\Phi(\tilde{f})}(l+2)\big).$$

Hence $\Phi^{1\to 1}$ is not everywhere sequentially continuous (in the sense of definition 3.3.4). By proposition 3.7 this implies $(\exists^2)$. ⊣

Remark 3.13. *Whereas the equivalence of global sequential continuity and global ε-δ-continuity of $f : \mathbb{R} \to \mathbb{R}$ is provable in $\mathbf{RCA}_0^\omega$ along the lines of proposition 3.6.1 (see also* [20]*(7.2.9)), the use of QF-AC^{0,1} is unavoidable (but also sufficient even for general Polish spaces X, Y, see* [15]*) to prove the pointwise ('local') equivalence. In fact, as shown in* [5]*, the pointwise equivalence of sequential and ε-δ-continuity for $f : \mathbb{R} \to \mathbb{R}$ is independent from ZF.*

Notation: $\overline{C}$ denotes the space of all functions $f \in C[0,1]$ with $f(0) \leq 0 \wedge f(1) \geq 0$.

We now consider uniform versions of the following principles:

1. the intermediate value theorem:
$\forall f \in \overline{C} \exists x \in [0,1](f(x) =_{\mathbb{R}} 0)$,
2. the attainment of the maximum principle:
$\forall f \in C([0,1]^d)\exists x \in [0,1]^d \forall y \in [0,1]^d\big(f(x) \geq_{\mathbb{R}} f(y)\big)$,
3. Brouwer's fixed point theorem:
$\forall f \in C([0,1]^d, [0,1]^d)\exists x \in [0,1]^d(f(x) =_{\mathbb{R}^d} x)$.

These principles differ in strength: whereas 2. and 3. imply (already for continuous functions as defined in reverse mathematics) WKL, 1.can be proved in $\mathbf{RCA}_0^\omega$ (see [17]). In fact, there is also some difference between 2. and 3., since 2. implies even in the case $d = 1$ WKL whereas 3. is provable in RCA_0 for $d = 1$ but implies WKL for $d \geq 2$ (see [16]).
In contrast to this, the uniform versions of 1.-3. are all equivalent to $(\exists^2)$ (independently of whether e.g. $f \in C[0,1]$ is given as a type-2 functional, with a code in the sense of reverse mathematics, or even with a modulus of uniform continuity).

Proposition 3.14. *The following principles are pairwise equivalent relative to $\mathbf{RCA}_0^\omega$:*

1. $(\exists^2)$,
2. $\exists F : \overline{C} \to [0,1] \forall f \in \overline{C}\big(f(F(f)) =_{\mathbb{R}} 0\big)$,
3. *the restriction of 2. to Lipschitz*[4] *continuous functions with $\lambda = 1$,*
4. $\exists F : C([0,1]^d) \to [0,1]^d \forall f \in C([0,1]^d)\forall y \in [0,1]^d\big(f(F(f)) \geq_{\mathbb{R}} f(y)\big)$,

[4] E.g. with respect to the Euclidean norm.

5. *the restriction of 4. to Lipschitz continuous functions with* $\lambda = 1$,
6. $\exists F : C([0,1]^d, [0,1]^d) \to [0,1]^d \forall f \in C([0,1]^d, [0,1]^d)\big(f(F(f)) =_{\mathbb{R}^d} F(f)\big)$.
7. *the restriction of 6. to Lipschitz continuous functions with* $\lambda = 1$.

Proof: It is a routine verification, that 2.-7. can be proved within $\mathbf{RCA}_0^\omega + (\exists^2)$ by inspecting the proofs of the non-uniform versions of these theorems. This holds true even if $f \in C[0,1]$ (and similarly $f \in C([0,1]^d)$ and $f \in C([0,1]^d, [0,1]^d)$) is given just as a functional of type $1 \to 1$ which happens to be ε-δ-continuous w.r.t. the usual topologies of $[0,1]^d$ and $\mathbb{R}$, but without any witness information for this continuity. We sketch this for 4. and $d = 1$: Let r_n be a suitable enumeration of all rational numbers in $[0,1]$ and define[5]

$$g(n) := \begin{cases} \mu i \le 2^n - 1[\forall k^0, l^0 \exists j^0 \big(r_j \in [\frac{i}{2^n}, \frac{i+1}{2^n}] \wedge f(r_j) \ge_{\mathbb{R}} f(r_k) - 2^{-l}\big)], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if existent} \\ 0^0, \text{ otherwise.} \end{cases}$$

Note that g is (Kleene-)primitive recursively definable in $(\exists^2)$ and (a functional representing) f since the property '$[\ldots]$' is arithmetical. In $\mathbf{RCA}_0^\omega + (\exists^2)$ one easily shows that the case 'otherwise' cannot occur. Moreover, using the continuity of f it follows that $(g(n)/2^n)_{n\in\mathbb{N}}$ is a Cauchy sequence with rate of convergence 2^{-n} which converges to the least $x \in [0,1]$ such that $f(x) = \sup_{y\in[0,1]} f(y)$.

We now prove that any of 2.-7. implies $(\exists^2)$. It is clear that it suffices to consider the case of Lipschitz continuous functions. We show this now for 3. (for 5. and 7. the proofs are very similar): Let $f_0 : [0,1] \to \mathbb{R}$ be the constant-0-function $f_0(x) := 0$. C is the space of all Lipschitz continuous functions $f \in C[0,1]$ with Lipschitz constant $\lambda = 1$ satisfying $f(0) \le 0, f(1) \ge 0$, and $F : C \to [0,1]$ is a function that satisfies 3.

Case 1: $F(f_0) \in [0, \frac{1}{2}]$. For $y \in [0,1]$, define $f_y : [0,1] \to \mathbb{R}$ by $f_y(x) := yx - y$. $f_y \in C$ and $f_y(0) \le 0, f_y(1) \ge 0$ for all $y \in [0,1]$. Moreover,

$$\forall y \in (0,1] \forall x \in [0,1]\big(f_y(x) =_{\mathbb{R}} 0 \leftrightarrow x =_{\mathbb{R}} 1\big).$$

Hence

$$\forall y \in (0,1]\big(F(f_y) =_{\mathbb{R}} 1\big).$$

Define $g : [0,1] \to \mathbb{R}$ by $g(y) := F(f_y)$. Then $g(0) = F(f_0) \in [0, \frac{1}{2}]$ and

$$\forall y \in (0,1]\big(g(y) = F(f_y) = 1\big).$$

Hence, $\widehat{g} : \mathbb{R} \to \mathbb{R}, \widehat{g}(y) := g(\min_{\mathbb{R}}(1, \max_{\mathbb{R}}(0,y)))$ is not sequentially continuous at $y := 0$.

Case 2: $F(f_0) \in [\frac{1}{2}, 1]$. Analogously to case 1 but with $f_y(x) := yx$.

[5] For notational simplicity we write here $\frac{i}{2^n}$ and 2^{-l} instead of their codes.

In both case we have constructed (Kleene-)primitive recursively in F a function $g : \mathbb{R} \to \mathbb{R}$ which is not everywhere sequential continuous. Hence, proposition 3.12 yields the existence of $(\exists^2)$. $\dashv$

Above we saw that certain principles which in their non-uniform version are different with respect to the set existence axioms needed to prove them are equivalent in their uniform formulation. We now indicate that also the opposite phenomenon can occur: consider again the attainment of the maximum principle (for simplicity only for dimension 1)

$$(a)\ \forall f \in C[0,1] \exists x \in [0,1] \forall y \in [0,1] \big(f(x) \geq_{\mathbb{R}} f(y)\big)$$

and also the existence of the supremum

$$(b)\ \forall f \in C[0,1] \exists y \in \mathbb{R} \big(y =_{\mathbb{R}} \sup_{x \in [0,1]} f(x)\big).$$

From ordinary reverse mathematics it is well-known that both principles are equivalent to WKL (relative to $\mathbf{RCA}_0$ and using the encoding of such functions as pointwise continuous functions as in [17], i.e. without a modulus of uniform continuity). We saw above that the uniform version of (a) is equivalent to $(\exists^2)$ (independently of whether f is assumed to be uniformly continuous or even given with a modulus of uniform continuity or not). Let's consider the uniform version of (b). The status now depends on the representation: it is easy to define a functional in $\mathbf{RCA}_0^\omega$ which computes the supremum of f uniformly in f and a modulus of uniform continuity of f. If, however, f is just given as a pointwise continuous function one has to compute a modulus of uniform continuity first. This can be achieved uniformly in f (given as a functional $\varphi^{1\to 1}$ which is extensional w.r.t. $=_{\mathbb{R}}$) by the following so-called fan functional

$$(\mathrm{MUC}):\ \exists \Omega^3 \forall \varphi^2 \forall f_1, f_2 \leq_1 1 \big(\overline{f_1}(\Omega(\varphi)) =_0 \overline{f_2}(\Omega(\varphi)) \to \varphi(f_1) =_0 \varphi(f_2)\big).$$

(MUC) is inconsistent with $(\exists^2)$ but consistent relative to $\mathbf{RCA}_0^\omega$. Moreover, adapting the proof of theorem 2.6.6 in [18] one can show

Proposition 3.15. $\mathbf{RCA}_0^\omega$+MUC *is conservative over* $\mathbf{RCA}_0^2$+WKL *(and hence* Π^0_2*-conservative over* $\mathbf{PRA}$*).*

Since the uniform version of (b) (for pointwise continuous functions)[6] can be proved in $\mathbf{RCA}_0^\omega$+MUC it is proof-theoretically weaker than the uniform version of (a).
This difference in strength for the uniform versions is due to the fact that

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[6] Actually, MUC even suffices to prove this for arbitrary functions $f : \mathbb{R} \to \mathbb{R}$. We don't know whether the (classically valid) restriction of MUC to pointwise continuous functionals φ^2 suffices to prove the uniform version of (b). The problem is, that the pointwise continuity of $f : \mathbb{R} \to \mathbb{R}$ does not imply that a functional $\varphi^{1\to 1}$ representing f is pointwise continuous in the sense of the Baire space.

(b) can be proved intuitionistically from the contraposition of WKL (i.e. 'every finite binary tree is bounded', which is a form of the so-called fan principle), whereas (a) uses essentially classical logic. This use of classical logic results in a discontinuity on the uniform level and hence the derivability of $\exists^2$. Thus we have another reason for investigating higher order reverse mathematics: differences between principle which are only visible in an intuitionistic setting become visible classically on the higher order uniform versions of these principles.

Final Comments:

1. The equivalence results established in this paper also hold for the subsystem $\mathbf{RCA}_0^{\omega^*} :=$ $\mathbf{E}$-$\mathbf{G}_3\mathbf{A}^\omega$+QF-AC^{1,0} of $\mathbf{RCA}_0^\omega$ with elementary recursive functionals only (i.e. $\mathbf{E}$-$\mathbf{G}_3\mathbf{A}^\omega$ only contains $0, S, +, \cdot, \exp$ and bounded predicative recursion). The absence of Φ_{it} blocks the derivability of Σ_1^0-IA and, in fact, $\mathbf{RCA}_0^{\omega^*}$ (which is a higher order extension of the system $\mathbf{RCA}_0^*$ from [17]) is Π_2^0-conservative over Kalmar-elementary arithmetic $\mathbf{EA}$.
2. The results in this paper depend crucially on the fact that our system $\mathbf{RCA}_0^\omega$ contains full extensionality (for type-2-objects). In [14] we have shown that in a setting where (E) is replaced by Spector's weak quantifier-free rule of extensionality e.g. UWKL is as weak as WKL.
3. One could argue to use instead of systems based on a fixed system of finite types more flexible systems like Feferman's systems of explicit mathematics are appropriate subsystems of (classical versions of) Martin-Löf type theories. However, in neither of these settings has been formulated a natural equivalent to the system $\mathbf{WKL}_0$, i.e. a system with the same mathematical strength then $\mathbf{WKL}_0$ but which at the same time allows a finitistic reduction to primitive recursive arithmetic $\mathbf{PRA}$. The problem here seems to be that these frameworks treat a principle like WKL automatically in its uniform version UWKL which, however, is (in an extensional setting) proof-theoretically as strong as $(\exists^2)$ as we saw above. In our view it is one of the most interesting outcomes of reverse mathematics that large parts of mathematics can be carried out in a $\mathbf{PRA}$-reducible system like $\mathbf{WKL}_0$.

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