

On the disjunctive Markov principle

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Abstract

In this note we show that over a strong (semi-)intuitionistic base theory, the recursive comprehension principle Δ_1^0 -CA does not imply the disjunctive Markov principle MP^\vee .

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Consider the following weak forms of the law-of-excluded-middle principle

$$\text{MP}^\vee : \forall \alpha^1, \beta^1 \left(\underbrace{\neg(\neg\exists n(\alpha(n) \neq 0) \wedge \neg\exists n(\beta(n) \neq 0)) \rightarrow \neg\neg\exists n(\alpha(n) \neq 0) \vee \neg\neg\exists n(\beta(n) \neq 0)}_{\text{MP}^\vee_{(\alpha, \beta)} \equiv} \right)$$

(introduced in [2] under the name of ‘disjunctive Markov principle’) and

$$\Delta_1^0\text{-LEM} : \forall \alpha, \beta \left(\begin{array}{l} \forall x(\exists n(\alpha(x, n) = 0) \leftrightarrow \forall n(\beta(x, n) = 0)) \\ \rightarrow \exists n(\alpha(x, n) = 0) \vee \neg\exists n(\alpha(x, n) = 0) \end{array} \right)$$

(studied e.g. in [1]) as well as the weak comprehension principle

$$\Delta_1^0\text{-CA} : \forall \alpha, \beta \left(\begin{array}{l} \forall x(\exists n(\alpha(x, n) = 0) \leftrightarrow \forall n(\beta(x, n) = 0)) \\ \rightarrow \exists f^1 \forall x(f(x) = 0 \leftrightarrow \exists n(\alpha(x, n) = 0)) \end{array} \right).$$

As shown in [2], MP^\vee follows intuitionistically from both the Markov principle MP (for numbers, also called Σ_1^0 -DNE) as well as from the Σ_1^0 -LLPO principle (called SEP in [2]). Furthermore, Δ_1^0 -CA trivially implies Δ_1^0 -LEM. For further background information see

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[2, 1], but note that in the formulations above we allow for function parameters whereas in [1] first-order versions are considered. In the following, \mathcal{S}^ω denotes the full set-theoretic model of E-HA^ω (see [3] to which we also refer for all other undefined notions in this note). We show that, over E-HA^ω , the principle $\Delta_1^0\text{-CA}$ does not imply MP^\vee . In fact this even holds for strong extensions of $\text{E-HA}^\omega + \Delta_1^0\text{-CA}$. We first observe:

Proposition 1. $\text{E-HA}^\omega + \text{AC} + \text{IP}_{ef}^\omega \vdash \Delta_1^0\text{-CA}$. In fact, instead of IP_{ef}^ω the weaker principle IP_\forall^0 suffices.

Proof: By the assumption of $\Delta_1^0\text{-CA}$ we, in particular, have

$$\forall x (\forall m (\beta(x, m) = 0) \rightarrow \exists n (\alpha(x, n) = 0))$$

and so by IP_\forall^0

$$\forall x \exists n (\forall m (\beta(x, m) = 0) \rightarrow \alpha(x, n) = 0).$$

By AC we get the existence of a function g such that

$$\forall x (\forall m (\beta(x, m) = 0) \rightarrow \alpha(x, g(x)) = 0).$$

One easily show that $f(x) := \alpha(x, g(x))$ satisfies the conclusion of $\Delta_1^0\text{-CA}$. □

Theorem 2. Let Γ be any set of \exists -free axioms which are true in \mathcal{S}^ω . Then

$$\text{E-HA}^\omega + \text{AC} + \text{IP}_{ef}^\omega + \Gamma \not\vdash \text{MP}^\vee.$$

In fact, one can extend Γ to also include \mathcal{S}^ω -valid axioms whose modified realizability (in \mathcal{S}^ω) only requires continuous functionals of type degrees ≤ 2 .

The theorem follows using the following two lemmas:

Lemma 3. There is no continuous functional $\varphi : \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$ such that

$$\mathcal{S}^\omega \models \varphi \text{ } mr \text{ } \text{MP}^\vee,$$

where mr denotes modified realizability (see e.g. [3]).

Proof:

$$\mathcal{S}^\omega \models \varphi \text{ } mr \text{ } \text{MP}^\vee$$

implies (using classical logic) that

$$(*) \mathcal{S}^\omega \models \left\{ \begin{array}{l} \forall \alpha, \beta (\exists n (\alpha(n) \neq 0) \vee \exists n (\beta(n) \neq 0)) \\ \rightarrow [\varphi(\alpha, \beta) = 0 \rightarrow \exists n (\alpha(n) \neq 0)] \wedge [\varphi(\alpha, \beta) \neq 0 \rightarrow \exists n (\beta(n) \neq 0)] \end{array} \right\}.$$

If φ is continuous then it is uniformly continuous for $\alpha, \beta \leq 1$, i.e. there exists an $N \in \mathbb{N}$ such that $\varphi(\alpha, \beta) = \varphi(\alpha', \beta')$ whenever $\bar{\alpha}(N) = \bar{\alpha}'(N)$ and $\bar{\beta}(N) = \bar{\beta}'(N)$ for all $\alpha, \alpha', \beta, \beta' \leq 1$.

Now consider $\varphi(0,0)$ (where 0 denotes the constant-0 function).

Case 1: $\varphi(0,0) = 0$. Define $\alpha := \lambda k.0$ and $\beta(n) := 0$, if $n \leq N$ and $\beta(n) := 1$, if $n > N$. Then $\varphi(\alpha, \beta) = \varphi(0,0) = 0$ and $\exists n(\alpha(n) \neq 0) \vee \exists n(\beta(n) \neq 0)$, but $\neg \exists n(\alpha(n) \neq 0)$ contradicting (*).

Case 2: $\varphi(0,0) \neq 0$. Now take $\alpha := \beta$ (with β as in case 1) and $\beta := 0$. Again we get a contradiction to (*). \square

Lemma 4. Let Γ be as in Theorem 2. $\text{E-HA}^\omega + \text{AC} + \text{IP}_{ef}^\omega + \Gamma$ has an mr -interpretation in $\text{E-HA}^\omega + \Gamma$ (and hence in \mathcal{S}^ω) by closed terms of E-HA^ω . If Γ is extended by further axioms Φ whose mr -interpretation is assumed to be satisfied by new constants \underline{c} , then the resulting system has an mr -interpretation in $\text{E-HA}^\omega + \Gamma + (\underline{c} \text{ } mr \text{ } \Phi)$ by closed terms built up by the constants of E-HA^ω and \underline{c} .

Proof: This follows from [3] (Theorem 5.8) (see also [4]). \square

Proof of Theorem 2: Suppose that MP^\vee would be derivable in the respective theory, then it would have (valid in \mathcal{S}^ω) an mr -interpretation by a functional φ^2 which is given by a term $t[\underline{c}]$ of E-HA^ω plus extra constants \underline{c} of type level 2 which are interpreted by continuous functionals. Since $t[\underline{c}]$ is primitive recursive (in the sense of Gödel's T) in \underline{c} and of type 2 it denotes (interpreted in \mathcal{S}^ω) a continuous functional (to see this one can use e.g. Scarpellini's model $\mathcal{C}^\omega \models T$ of continuous functionals and $C_2 \subset S_2$, see [3]). This, however, contradicts Lemma 3. \square

The proof of Theorem 2 above only shows that the full 2nd order version $\forall \alpha, \beta \text{MP}^\vee(\alpha, \beta)$ of MP^\vee is not provable in the respective theory since Lemma 3 crucially requires this. We now show that even for primitive recursive sequences $\alpha_x(n) := s(x, n), \beta_x(n) := t(x, n)$ of functions α, β (s, t primitive recursive terms), MP^\vee is not derivable, i.e. that the function parameter-free version $\text{MP}^{\vee-}$ is underivable, using the following

Lemma 5. There is no total recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\mathcal{S}^\omega \models f \text{ } mr \text{ } \forall x^0 \text{MP}^\vee(\alpha_x, \beta_x)$$

for α_x, β_x as defined above for suitable primitive recursive s, t .

Proof: Define (using the primitive recursive T, U from the Kleene normal form theorem)

$$s(x, n) := \begin{cases} 1, & \text{if } T(x, x, n) \wedge U(n) = 1, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \\ t(x, n) := \begin{cases} 1, & \text{if } T(x, x, n) \wedge U(n) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\exists n(\alpha_x(n) \neq 0) \leftrightarrow \{x\}(x) \simeq 1, \quad \exists n(\beta_x(n) \neq 0) \leftrightarrow \{x\}(x) \simeq 0.$$

Suppose that $f : \mathbb{N} \rightarrow \mathbb{N}$ is total recursive with $f \text{ } mr \text{ } \forall x^0 \text{MP}^\vee(\alpha_x, \beta_x)$. Then, as in the proof of Lemma 1, one gets that

$$(*) \quad \left\{ \begin{array}{l} \forall x \left(\{x\}(x) \simeq 1 \vee \{x\}(x) \simeq 0 \rightarrow \right. \\ \left. [f(x) = 0 \rightarrow \{x\}(x) \simeq 1] \wedge [f(x) \neq 0 \rightarrow \{x\}(x) \simeq 0] \right). \end{array} \right.$$

Let e be a code of the total recursive function $sg \circ f$. Then $\{e\}(e)$ is defined with values ≤ 1 , i.e. $\{e\}(e) \simeq 1 \vee \{e\}(e) \simeq 0$ and so by (*)

$$\{e\}(e) \simeq 0 \leftrightarrow \{e\}(e) \simeq 1,$$

which is a contradiction. □

Analogously to Theorem 2, Lemmas 4 and 5 imply

Theorem 6. Let Γ be any set of \exists -free axioms which are true in \mathcal{S}^ω plus any \mathcal{S}^ω -valid axioms whose modified realizability only requires total computable functions (type degree ≤ 1). Then

$$\text{E-HA}^\omega + \text{AC} + \text{IP}_{ef}^\omega + \Gamma \not\vdash \text{MP}^{\vee-}.$$

Corollary 7. $\text{E-HA}^\omega + \Delta_1^0\text{-CA} \not\vdash \text{MP}^{\vee-}$.

Proof: By Theorem 6 and Proposition 1. □

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References

- [1] Akama, Y., Berardi, S., Hayashi, S., Kohlenbach, U., An arithmetical hierarchy of the law of excluded middle and related principles. Proc. of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS'04), pp. 192-201, IEEE Press (2004).
- [2] Ishihara, H., Markov's principle, Church's thesis and Lindelöf's theorem. Indag. Mathem., N.S. **4**, pp. 321-325 (1993).
- [3] Kohlenbach, U., Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. xx+536pp., Springer Heidelberg-Berlin, 2008.
- [4] Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes in Mathematics **344** (1973).