On the disjunctive Markov principle

U. Kohlenbach*

Department of Mathematics
Technische Universität Darmstadt
Schlossgartenstraße 7, 64289 Darmstadt, Germany
kohlenbach@mathematik.tu-darmstadt.de

May 1, 2015

Abstract

In this note we show that over a strong (semi-)intuitionistic base theory, the recursive comprehension principle Δ^0_1-CA does not imply the disjunctive Markov principle $\mathrm{MP}^\vee$.

Keywords: Law-of-excluded middle, Markov principle, intuitionistic mathematics, realizability.
Mathematics Subject Classification (2010): 03F55, 03F10, 03B20

Consider the following weak forms of the law-of-excluded-middle principle

$$\mathrm{MP}^\vee: \ \forall \alpha^1, \beta^1 \Big(\underbrace{\begin{array}{l} \neg(\neg\exists n\,(\alpha(n)\neq 0) \wedge \neg\exists n(\beta(n)\neq 0)) \\ \qquad \to \neg\neg\exists n(\alpha(n)\neq 0) \vee \neg\neg\exists n(\beta(n)\neq 0) \end{array}}_{\mathrm{MP}^\vee(\alpha,\beta):\equiv} \Big)$$

(introduced in [2] under the name of 'disjunctive Markov principle') and

$$\Delta^0_1\text{-LEM}: \ \forall \alpha, \beta \Big(\begin{array}{l} \forall x(\exists n(\alpha(x,n)=0) \leftrightarrow \forall n(\beta(x,n)=0)) \\ \qquad \to \exists n(\alpha(x,n)=0) \vee \neg\exists n(\alpha(x,n)=0) \end{array} \Big)$$

(studied e.g. in [1]) as well as the weak comprehension principle

$$\Delta^0_1\text{-CA}: \ \forall \alpha, \beta \Big(\begin{array}{l} \forall x(\exists n(\alpha(x,n)=0) \leftrightarrow \forall n(\beta(x,n)=0)) \\ \qquad \to \exists f^1 \forall x(f(x)=0 \leftrightarrow \exists n(\alpha(x,n)=0)) \end{array} \Big).$$

As shown in [2], $\mathrm{MP}^\vee$ follows intuitionistically from both the Markov principle MP (for numbers, also called Σ^0_1-DNE) as well as from the Σ^0_1-LLPO principle (called SEP in [2]). Furthermore, Δ^0_1-CA trivially implies Δ^0_1-LEM. For further background information see

*The author has been supported by the German Science Foundation (DFG Project KO 1737/5-2).

[2, 1], but note that in the formulations above we allow for function parameters whereas in [1] first-order versions are considered. In the following, $\mathcal{S}^\omega$ denotes the full set-theoretic model of E-HA$^\omega$ (see [3] to which we also refer for all other undefined notions in this note). We show that, over E-HA$^\omega$, the principle Δ^0_1-CA does not imply MP$^\vee$. In fact this even holds for strong extensions of E-HA$^\omega$ + Δ^0_1-CA. We first observe:

Proposition 1. E-HA$^\omega$+AC+IP$^\omega_{ef}$ $\vdash \Delta^0_1$-CA. In fact, instead of IP$^\omega_{ef}$ the weaker principle IP$^0_\forall$ suffices.

Proof: By the assumption of Δ^0_1-CA we, in particular, have

$$\forall x\, (\forall m(\beta(x,m)=0) \to \exists n(\alpha(x,n)=0))$$

and so by IP$^0_\forall$

$$\forall x\, \exists n\, (\forall m(\beta(x,m)=0) \to \alpha(x,n)=0).$$

By AC we get the existence of a function g such that

$$\forall x\, (\forall m(\beta(x,m)=0) \to \alpha(x,g(x))=0).$$

One easily show that $f(x) := \alpha(x, g(x))$ satisfies the conclusion of Δ^0_1-CA. □

Theorem 2. Let Γ be any set of $\exists$-free axioms which are true in $\mathcal{S}^\omega$. Then

$$\text{E-HA}^\omega\text{+AC+IP}^\omega_{ef} + \Gamma \nvdash \text{MP}^\vee.$$

In fact, one can extend Γ to also include $\mathcal{S}^\omega$-valid axioms whose modified realizability (in $\mathcal{S}^\omega$) only requires continuous functionals of type degrees ≤ 2.

The theorem follows using the following two lemmas:

Lemma 3. There is no continuous functional $\varphi : \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}$ such that

$$\mathcal{S}^\omega \models \varphi\ mr\ \text{MP}^\vee,$$

where mr denotes modified realizability (see e.g. [3]).

Proof:

$$\mathcal{S}^\omega \models \varphi\ mr\ \text{MP}^\vee$$

implies (using classical logic) that

$$(*)\ \mathcal{S}^\omega \models \begin{cases} \forall \alpha, \beta \Big(\exists n(\alpha(n) \neq 0) \vee \exists n(\beta(n) \neq 0) \\ \to [\varphi(\alpha,\beta)=0 \to \exists n(\alpha(n)\neq 0)] \wedge [\varphi(\alpha,\beta)\neq 0 \to \exists n(\beta(n)\neq 0)] \Big). \end{cases}$$

If φ is continuous then it is uniformly continuous for $\alpha, \beta \leq 1$, i.e. there exists an $N \in \mathbb{N}$ such that $\varphi(\alpha,\beta) = \varphi(\alpha',\beta')$ whenever $\overline{\alpha}(N) = \overline{\alpha'}(N)$ and $\overline{\beta}(N) = \overline{\beta'}(N)$ for all $\alpha, \alpha', \beta, \beta' \leq 1$.

Now consider $\varphi(0,0)$ (where 0 denotes the constant-0 function).
Case 1: $\varphi(0,0)=0$. Define $\alpha := \lambda k.0$ and $\beta(n) := 0$, if $n \leq N$ and $\beta(n) := 1$, if $n > N$. Then $\varphi(\alpha,\beta) = \varphi(0,0) = 0$ and $\exists n(\alpha(n) \neq 0) \vee \exists n(\beta(n) \neq 0)$, but $\neg\exists n(\alpha(n) \neq 0)$ contradicting $(*)$.
Case 2: $\varphi(0,0) \neq 0$. Now take $\alpha := \beta$ (with β as in case 1) and $\beta := 0$. Again we get a contradiction to $(*)$. □

Lemma 4. Let Γ be as in Theorem 2. E-HA$^\omega$+AC+IP$^\omega_{ef}$ + Γ has an mr-interpretation in E-HA$^\omega$ + Γ (and hence in $\mathcal{S}^\omega$) by closed terms of E-HA$^\omega$. If Γ is extended by further axioms Φ whose mr-interpretation is assumed to be satisfied by new constants $\underline{c}$, then the resulting system has an mr-interpretation in E-HA$^\omega$ + Γ + ($\underline{c}$ mr Φ) by closed terms built up by the constants of E-HA$^\omega$ and $\underline{c}$.

Proof: This follows from [3] (Theorem 5.8) (see also [4]). □

Proof of Theorem 2: Suppose that MP$^\vee$ would be derivable in the respective theory, then it would have (valid in $\mathcal{S}^\omega$) an mr-interpretation by a functional φ^2 which is given by a term $t[\underline{c}]$ of E-HA$^\omega$ plus extra constants $\underline{c}$ of type level 2 which are interpreted by continuous functionals. Since $t[\underline{c}]$ is primitive recursive (in the sense of Gödel's T) in $\underline{c}$ and of type 2 it denotes (interpreted in $\mathcal{S}^\omega$) a continuous functional (to see this one can use e.g. Scarpellini's model $\mathcal{C}^\omega \models T$ of continuous functionals and $C_2 \subset S_2$, see [3]). This, however, contradicts Lemma 3.

□

The proof of Theorem 2 above only shows that the full 2nd order version $\forall \alpha, \beta\, \mathrm{MP}^\vee(\alpha,\beta)$ of MP$^\vee$ is not provable in the respective theory since Lemma 3 crucially requires this. We now show that even for primitive recursive sequences $\alpha_x(n) := s(x,n), \beta_x(n) := t(x,n)$ of functions α, β (s,t primitive recursive terms), MP$^\vee$ is not derivable, i.e. that the function parameter-free version MP$^{\vee-}$ is underivable, using the following

Lemma 5. There is no total recursive function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\mathcal{S}^\omega \models f \; mr \; \forall x^0 \; \mathrm{MP}^\vee(\alpha_x, \beta_x)$$

for α_x, β_x as defined above for suitable primitive recursive s,t.

Proof: Define (using the primitive recursive T, U from the Kleene normal form theorem)

$$s(x,n) := \begin{cases} 1, & \text{if } T(x,x,n) \wedge U(n) = 1, \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$
$$t(x,n) := \begin{cases} 1, & \text{if } T(x,x,n) \wedge U(n) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\exists n(\alpha_x(n) \neq 0) \leftrightarrow \{x\}(x) \simeq 1, \; \exists n(\beta_x(n) \neq 0) \leftrightarrow \{x\}(x) \simeq 0.$$

Suppose that $f : \mathbb{N} \to \mathbb{N}$ is total recursive with $f \; mr \; \forall x^0 \; \mathrm{MP}^\vee(\alpha_x, \beta_x)$. Then, as in the proof of Lemma 1, one gets that

$$(*) \quad \begin{cases} \forall x \Big(\{x\}(x) \simeq 1 \vee \{x\}(x) \simeq 0 \to \\ \qquad [f(x) = 0 \to \{x\}(x) \simeq 1] \wedge [f(x) \neq 0 \to \{x\}(x) \simeq 0] \Big). \end{cases}$$

Let e be a code of the total recursive function $sg \circ f$. Then $\{e\}(e)$ is defined with values ≤ 1, i.e. $\{e\}(e) \simeq 1 \vee \{e\}(e) \simeq 0$ and so by $(*)$

$$\{e\}(e) \simeq 0 \leftrightarrow \{e\}(e) \simeq 1,$$

which is a contradiction. $\square$

Analogously to Theorem 2, Lemmas 4 and 5 imply

Theorem 6. Let Γ be any set of $\exists$-free axioms which are true in $\mathcal{S}^\omega$ plus any $\mathcal{S}^\omega$-valid axioms whose modified realizability only requires total computable functions (type degree ≤ 1). Then

$$\text{E-HA}^\omega\text{+AC+IP}^\omega_{ef} + \Gamma \not\vdash \text{MP}^{\vee-}.$$

Corollary 7. $\text{E-HA}^\omega + \Delta^0_1\text{-CA} \not\vdash \text{MP}^{\vee-}$.

Proof: By Theorem 6 and Proposition 1. $\square$

Acknowledgement: The author is grateful to Makoto Fujiwara for bringing the question addressed in this note to his attention.

References

[1] Akama, Y., Berardi, S., Hayashi, S., Kohlenbach, U., An arithmetical hierarchy of the law of excluded middle and related principles. Proc. of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS'04), pp. 192-201, IEEE Press (2004).

[2] Ishihara, H., Markov's principle, Church's thesis and Lindelöf's theorem. Indag. Mathem., N.S. **4**, pp. 321-325 (1993).

[3] Kohlenbach, U., Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. xx+536pp., Springer Heidelberg-Berlin, 2008.

[4] Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes in Mathematics **344** (1973).