TERM EXTRACTION AND RAMSEY'S THEOREM FOR PAIRS

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Abstract. In this paper we study with proof-theoretic methods the function(al)s provably recursive relative to Ramsey's theorem for pairs and the cohesive principle (COH).

Our main result on COH is that the type 2 functionals provably recursive from $\mathsf{RCA}_0 + \mathsf{COH} + \mathsf{\Pi}^0_1\text{-}\mathsf{CP}$ are primitive recursive. This also provides a uniform method to extract bounds from proofs that use these principles. As a consequence we obtain a new proof of the fact that $\mathsf{WKL}_0 + \mathsf{\Pi}^0_1\text{-}\mathsf{CP} + \mathsf{COH}$ is Π^0_2-conservative over PRA.

Recent work of the first author showed that $\mathsf{\Pi}^0_1\text{-}\mathsf{CP} + \mathsf{COH}$ is equivalent to a weak variant of the Bolzano-Weierstraß principle. This makes it possible to use our results to analyze not only combinatorial but also analytical proofs.

For Ramsey's theorem for pairs and two colors (RT^2_2) we obtain the upper bounded that the type 2 functionals provable recursive relative to $\mathsf{RCA}_0 + \mathsf{\Sigma}^0_2\text{-}\mathsf{IA}+\mathsf{RT}^2_2$ are in T_1. This is the fragment of Gödel's system T containing only type 1 recursion — roughly speaking it consists of functions of Ackermann type. With this we also obtain a uniform method for the extraction of T_1-bounds from proofs that use RT^2_2. Moreover, this yields a new proof of the fact that $\mathsf{WKL}_0 + \mathsf{\Sigma}^0_2\text{-}\mathsf{IA} + \mathsf{RT}^2_2$ is Π^0_3-conservative over $\mathsf{RCA}_0 + \mathsf{\Sigma}^0_2\text{-}\mathsf{IA}$.

The results are obtained in two steps: in the first step a term including Skolem functions for the above principles is extracted from a given proof. This is done using Gödel's functional interpretation. After this the term is normalized, such that only specific instances of the Skolem functions are used. In the second step this term is interpreted using Π^0_1-comprehension. The comprehension is then eliminated in favor of induction using either elimination of monotone Skolem functions (for COH) or Howard's ordinal analysis of bar recursion (for RT^2_2).

1. Introduction

The aim of this paper is to develop a technique of program extraction for proofs that use Ramsey's theorem for pairs, the cohesive principle and other principle weaker than Ramsey's theorem for pairs. As a consequence it also gives a proof theoretic account of conservation results for those principles. This paper extends our previous treatment of Ramsey's theorem for pairs in [34], where only single instances of Ramsey's theorem are discussed, to the full second order closure of those principles.

Ramsey's theorem for pairs (RT^2_n) is the statement that every coloring of pairs of natural numbers ($[\mathbb{N}]^2$) with n colors has an infinite homogeneous set. A simple colorblindness argument shows that

$$\mathsf{RT}^2_2 \leftrightarrow \mathsf{RT}^2_n \quad \text{for every fixed } n.$$

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Ramsey's theorem for pairs and arbitrary large colorings ($\mathsf{RT}^2_{<\infty}$) is defined as $\forall n\, \mathsf{RT}^2_n$. This principle is proof-theoretically stronger than RT^2_2.

A coloring c of pairs is called *stable* if $c(\{x, \cdot\})$ eventually becomes constant for every x. The restriction of RT^2_n to stable colorings is denoted by SRT^2_n. Here a similar colorblindness argument can be applied.

A set G is called *cohesive* for a sequence $(R_i)_{i\in\mathbb{N}}$ of subsets of $\mathbb{N}$ if

$$\forall i \left(G \subseteq^* R_i \vee G \subseteq^* \overline{R_i}\right),$$

where $X \subseteq^* Y :\equiv (X \setminus Y \text{ is finite})$. The *cohesive principle* (COH) states that for every $(R_i)_{i\in\mathbb{N}}$ an infinite cohesive set exists. It is in some way the counterpart to SRT^2_n since

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_n \leftrightarrow \mathsf{SRT}^2_n \wedge \mathsf{COH}$$

for $2 \leq n \leq \infty$, see [7, 8].

The computational strength of Ramsey's theorem has been investigated since the early 70's. Specker showed 1971 that there exists a computable coloring of $[\mathbb{N}]^2$ that has no computable homogeneous set, see [47]. Jockusch improved this 1972 by showing that in general there is not even a Σ^0_2 infinite homogeneous set. He also provided an upper bound on the strength of Ramsey's theorem for pairs and showed that each computable coloring of pairs admits an infinite homogeneous set H with $H' \leq_T 0''$, see [21]. Seetapun and Slaman showed in [43] that RT^2_2 does not solve the halting problem. Cholak, Jockusch and Slaman improved both results by showing that an infinite homogeneous low_2 set exists for every computable coloring of pairs, i.e. a set H satisfying $H'' \leq_T 0''$, see [7].

From Specker's results it is clear that $\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2$. Seeptapun's and Slaman's results immediately yield an upper bound on the proof-theoretic strength, it implies that RT^2_2 does not prove Π^0_1-comprehension (or, equivalently, ACA_0). Hirst showed 1987 that RT^2_2 implies the infinite pigeonhole principle ($\mathsf{RT}^1_{<\infty}$) which is equivalent to the Π^0_1-bounded collection principle ($\mathsf{\Pi}^0_1$-CP)[1], see [16]. Cholak, Jockusch and Slaman showed along their recursion theoretic proof that RT^2_2 is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{\Sigma}^0_2$-$\mathsf{IA}$.

This leaves the question whether RT^2_2 implies $\mathsf{\Sigma}^0_2$-IA. Despite of many efforts in the last years this question has not been settled yet.

Ramsey's theorem for triples and bigger tuples is equivalent to ACA_0 and hence fully classified in the sense of reverse mathematics, see [45].

The cohesive principle has been originally considered in recursion theory, see for instance [46]. Its computational strength has been fully determined in [19]. Cholak, Jockusch and Slaman observed in [7] that Ramsey's theorem for pairs splits nicely into a stable part and the cohesive principle. They also showed that it is Π^1_1-conservative over RCA_0 and $\mathsf{RCA}_0 + \mathsf{\Sigma}^0_2$-$\mathsf{IA}$. In the course of the classification of Ramsey's theorem the cohesive principle's logical strength received attention in the last years, see for instance [10] and [9]. In [9] it was shown that the cohesive principle is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{\Pi}^0_1$-$\mathsf{CP}$. We recently showed that over $\mathsf{RCA}_0 + \mathsf{\Pi}^0_1$-$\mathsf{CP}$ the cohesive principle is equivalent to a weak form of the Bolzano-Weierstraß principle, see [37]. Thus the cohesive principle also shows up in analytic proofs.

For an extensive survey on the current status of Ramsey's theorem for pairs and weaker principles, see [15] and [44].

The purpose of this paper is to give an account to the above mentioned conservation results from the perspective of proof mining and program extraction. We provide new proofs for these conservation results which additionally yield realizing

[1]In the first order context this principle is usually denoted by $B\Pi^0_1$ which is equivalent to $B\Sigma^0_2$.

terms. Since the types of these terms rise with the complexity of the formula this is naturally bounded to Π^0_3-sentences.

Proofwise low. Define Π^0_1-comprehension as

$$(\mathsf{\Pi^0_1\text{-}CA})\colon \forall X\, \exists Y\, \forall u\ (u \in Y \leftrightarrow \forall v\ \langle u, v\rangle \in X)\,.$$

This covers the full strength of Π^0_1-comprehension since $\forall v\ \langle u,v\rangle \in X$ is a universal Π^0_1-statement (relative to the parameter u). Full arithmetical comprehension (ACA_0) follows by iteration. For a primitive recursive term t we will write $\mathsf{\Pi^0_1\text{-}CA}(t)$ if X is instantiated with the set $\{n \mid t(n) = 0\}$.[2] For a closed term t the principle $\mathsf{\Pi^0_1\text{-}CA}(t)$ is also called an *instance of* Π^0_1*-comprehension.*

The union of $\mathsf{\Pi^0_1\text{-}CA}(t)$ for all terms t containing only number variables free is the same as light-face Π^0_1-comprehension. In particular, this does not prove ACA_0.

Let $\mathcal{P}$ be a second order principle stating the existence of a set G relative to a set parameter S — that is a principle of the form

$$(\mathcal{P})\colon \forall S\, \exists G\ P(S,G).$$

Definition 1 (proofwise low)**.** Call a principle of the form $\mathcal{P}$ *proofwise low* over a system $\mathcal{T}$ if for every provably continuous[3] term φ a provably continuous term ξ exists such that

(1) $$\mathcal{T} \vdash \forall S\ \big(\mathsf{\Pi^0_1\text{-}CA}(\xi S) \to \exists G\ \big(P(S,G) \wedge \mathsf{\Pi^0_1\text{-}CA}(\varphi S G)\big)\big)\,.$$

If we additionally can prove this for a sequence of solutions, i.e.

(2) $$\mathcal{T} \vdash \forall (S_i)_{i\in\mathbb{N}}\ \big(\mathsf{\Pi^0_1\text{-}CA}(\xi (S_i)_i) \to \exists (G_i)_{i\in\mathbb{N}}\ \big(\forall i\, P(S_i,G_i) \wedge \mathsf{\Pi^0_1\text{-}CA}(\varphi (S_i)_i (G_i)_i)\big)\big)$$

then we call $\mathcal{P}$ *proofwise low in sequence* over the system $\mathcal{T}$. Here $(S_i)_i$ is (a code of) the sequence of sets S_i. It is given by the set $\{\langle i,x\rangle \mid x \in S_i\}$.

The notion of proofwise low is comparable to low_2 in the recursion theoretic setting: take for instance $\mathcal{T} = \mathsf{WKL}_0$, then a proofwise low statement in $\mathcal{T}$ satisfies

$$\mathsf{RCA}_0 \vdash \forall S\ \big(\mathsf{WKL} \wedge \mathsf{\Pi^0_1\text{-}CA}(\xi S) \to \exists G\ \big(\mathcal{P}(S,G) \wedge \mathsf{\Pi^0_1\text{-}CA}(\varphi S G)\big)\big)\,.$$

The analogous recursion theoretic statement would be that relative to an oracle of Turing degree $d \gg 0'$ — this resembles the premise — a set G satisfying the statement $P(S,G)$ and its Turing jump G' can be computed. From this follows that $G'' \equiv_T 0''$ or in other word that G is low_2.

The main results of this paper are divided into two parts:

(i) We show roughly that
  - RT^2_2 is proofwise low over WKL_0 (Corollary 46) and that
  - COH is proofwise low in sequence over WKL^*_0. The system WKL^*_0 is defined to be WKL_0 where Σ^0_1-induction is replaced by quantifier-free-induction plus the exponential function. (Corollary 33)

(ii) We show for principles $\mathcal{P}$ that
  - if $P(S,G)$ is Π^0_1 and $\mathcal{P}$ is proofwise low over WKL_0, the system $\mathsf{WKL}_0 + \Sigma^0_2\text{-}\mathsf{IA} + \mathcal{P}$ is Π^0_3-conservative over Σ^0_2-induction. (Section 10.3)
  - if $P(S,G)$ is Π^0_3 and $\mathcal{P}$ is proofwise low in sequence over WKL^*_0 the system $\mathsf{WKL}_0 + \mathsf{\Pi^0_1\text{-}CP} + \mathcal{P}$ is Π^0_3-conservative over RCA_0 and Π^0_2-conservative over PRA. (This covers COH. See Theorem 36.)

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[2] Strictly speaking RCA_0 does not contain terms. Here and in the following we silently assume that we work in the conservative extension of RCA_0 by all primitive recursive functions.

[3] Continuous means here continuous in the sense of Baire space, i.e. φ is continuous if

$$\forall f\, \exists n\, \forall g\ (\forall x < n\, f(x) = g(x) \to \varphi(f) = \varphi(g))\,.$$

Such functionals can be coded into primitive recursive functions. For details see definitions 6 and 7 below.

This simplifies the results slightly. The actual results require a suitable finite type extension of WKL_0 and WKL_0^*, they also allow a function parameter to the Π^0_3-formula and provide extraction of type 2 functionals, see below.

The first part of the results is based on the proofs by "first jump control" for SRT^2_2 and COH of Cholak, Jockusch and Slaman, see [7], showing that these principles have low_2 solutions. (See proposition 31 with corollary 33 and proposition 44 with corollary 46.) To our knowledge these proofs have not been used before to obtain conservativity results for RT^2_2. Cholak, Jockusch and Slaman developed in this paper a different, more complicated proof needing Π^0_2-comprehension that can be used in a forcing construction to show conservativity of RT^2_2 over Σ^0_2-induction.

For the second part we use Gödel's functional interpretation (always combined with a negative translation) to extract a term t from a proof of an arbitrary statement of the following form

$$\mathcal{P} \to \forall x\, \exists y\, A(x, y),$$

where A is quantifier-free and $\mathcal{P}$ is a proofwise low principle. (See the proof of proposition 35 and proposition 50.) For an oracle solution $\mathcal{P}$ of the functional interpretation of $\mathcal{P}$ this term will then satisfy

$$\forall x\, A(x, t(\mathcal{P}, x)).$$

We normalize t so that every application of $\mathcal{P}$ in the proof is of a specific form and one can read off from the term and the proof how much of $\mathcal{P}$ is used (section 8). The functional $\mathcal{P}$ is then eliminated from t by interpreting every specific application of $\mathcal{P}$. This is done either by (2) or the functional interpretation of (1) in a way that retains the instance of comprehension. If this retained instance of comprehension is used for the next interpretation of $\mathcal{P}$ then an inductive treatment of every application of $\mathcal{P}$ yields that

(i) in the first case one instance of the functional interpretation of $\mathsf{\Pi}^0_1\text{-}\mathsf{CA}$ suffices to prove to totality of t and hence $\forall x\, \exists y\, A(x, y)$, see proposition 52,
(ii) in the second case one instance of $\mathsf{\Pi}^0_1\text{-}\mathsf{CA}$ proves the totality of t and hence $\forall x\, \exists y\, A(x, y)$, see proposition 35.

The instance of comprehension is then eliminated in favor of induction:
In the case (i) the solution to this functional interpreted instance of comprehension is provided by an instance of Spector's bar recursion (in fact by an application of the rule of bar recursion). This usage of bar recursion is then eliminated using Howard's ordinal analysis of bar recursion in favor of Σ^0_2-induction, see section 7.

In the case (ii) the instance of comprehension is eliminated through elimination of Skolem functions for monotone formulas (section 5) yielding that $\forall x\, \exists y\, A(x, y)$ is provable in primitive recursive arithmetic. For this it is crucial that $\mathcal{P}$ is proofwise low over a system that does *not* contain Σ^0_1-induction, for instance WKL_0^*.

These techniques of elimination of instances of comprehension can be viewed as a proof-theoretic refinement of the arithmetical conservativity of ACA_0 over PA, see [4], [11], [50] and [45, IX.1.6].

Comparison to conservation results by syntactic forcing. Syntactic forcing is a method to prove conservativity result. It is commonly used in reverse mathematics.

To show that a second order principle $\mathcal{P}$ is conservative over $\mathcal{T}$ it proceeds by first taking an arbitrary countable model of $\mathcal{T}$. This model is then extended through a forcing argument to include sets solving all instances of $\mathcal{P}$ without altering the first order part. The conservativity then follows by Gödel's completeness theorem. For details and further information see [2].

The elimination of monotone Skolem functions and Howard's elimination of bar recursion are constructive: a careful analysis of the proofs would yield a uniform method to obtain a term of $\mathcal{T}$ for each function provable total using $\mathcal{P}$. Whereas the forcing argument essentially uses a reductio ad absurdum argument (if $\mathcal{P}$ would not be conservative then by the completeness theorem there would be a model that could not be extended).

Forcing yields in many cases full Π^1_1-conservativity whereas the functional interpretation usually stops at Π^0_3-conservativity. This is a consequence of the way the functional interpretation works: it transforms every statement in a functional, where for every additional quantifier alternation the type-level rises, making it more complex to analyze. For instance, Π^0_3-statements correspond to type 2 functionals (i.e. functionals essentially of the form $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$).

This makes it easier to handle principles implying the Π^0_1-bounded collection principle ($\mathsf{\Pi}^0_1$-CP). Due to the well-known fact that $\mathsf{\Pi}^0_1$-CP is Π^0_3-conservative over $\mathsf{\Sigma}^0_1$-IA the base theory for the functional interpretation does not change. This circumvents the problems forcing experiences when proving conservativity over $\mathsf{\Pi}^0_1$-CP, see [15, §6].

The original proof that RT^2_2 or COH is Π^1_1-conservative over Σ^0_2-induction uses syntactic forcing, also the proof that COH is Π^1_1-conservative over Σ^0_1-induction uses it, see [7]. The original proof of the fact that COH is Π^1_1-conservative over $\mathsf{\Pi}^0_1$-CP is done using a complicated double forcing, see [9]. Our proof of the fact that $\mathsf{COH} + \mathsf{\Pi}^0_1$-$\mathsf{CP}$ is Π^0_3-conservative over $\mathsf{\Sigma}^0_1$-IA is similar to the proof of [7] since we show conservativity over RCA_0 (without $\mathsf{\Pi}^0_1$-CP) and therefore do not face the problems forcing experiences with $\mathsf{\Pi}^0_1$-CP and that Chong, Slaman and Yang in [9] deal with. Additionally, our proof is open for proof mining, which means it provides a method for program extraction.

2. Logical systems

We will work in a setting based on fragments of Heyting and Peano arithmetic in all finite types introduced in [51], for details see also [32].

In general, theories will be written with a superscript ω which indicates that this is a finite type theory. Axioms and rules will not have an ω. The only exceptions to this are the theories of reverse mathematics (RCA_0, WKL_0, ACA_0, RCA^*_0, WKL^*_0) and PRA.

2.1. **Finite types.** The set of all finite types $\mathbf{T}$ is inductively defined as

$$0 \in \mathbf{T}, \qquad \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T},$$

where 0 denotes the type of natural numbers and $\tau(\rho)$ the type of functions from ρ to τ. The set of pure types $\mathbf{P} \subset \mathbf{T}$ is defined as

$$0 \in \mathbf{P}, \qquad \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}.$$

They will often be denoted by natural numbers:

$$0(n) := n + 1,$$

e.g. $0(0) = 1$. The degree $deg(\rho)$ of a type ρ is inductively defined as

$$deg(0) := 0, \qquad deg(\tau(\rho)) := \max(deg(\tau), deg(\rho) + 1).$$

We will often denote the type of a term or variable by a superscribed index. For two types ρ, τ we will write $\rho \leq \tau$ if $deg(\rho) \leq deg(\tau)$.

Equality $=_0$ for type 0 objects will be added as primitive notion to the systems. Higher type equality $=_{\tau\rho}$ will be treated as abbreviation:

$$x^{\tau\rho} =_{\tau\rho} y^{\tau\rho} :\equiv \forall z^\rho\, xz =_\tau yz.$$

2.2. **Gödel's system T.** Define the λ-combinators $\Pi_{\rho,\sigma}, \Sigma_{\rho,\sigma,\tau}$ to be the functionals satisfying

$$\Pi_{\rho,\sigma} x^\rho y^\sigma =_\rho x, \qquad \Sigma_{\rho,\sigma,\tau} x^{\tau\sigma\rho} y^{\sigma\rho} z^\rho =_\tau xz(yz).$$

Similar define the recursor R_ρ of type ρ to be the functional satisfying

$$R_\rho 0yz =_\rho y, \qquad R_\rho(Sx^0)yz =_\rho z(R_\rho xyz)x.$$

Let *Gödel's system T* be the $\mathbf{T}$-sorted set of closed terms that can be build up from 0^0, the successor function S^1, the λ-combinators and, the recursors R_ρ for all finite types ρ. Using the λ-combinators one easily sees that T is closed under λ-abstraction, see [51].

T_n denotes the subsystem of Gödel's system T, where primitive recursion is restricted to recursors R_ρ with $deg(\rho) \leq n$. The system T_0 corresponds to the extension of Kleene's primitive recursive functionals to mixed types, see [24], whereas full system T corresponds to Gödel's primitive recursive functionals, see [13].

2.3. **Heyting and Peano arithmetic.** Define the *neutral Heyting/Peano arithmetic* ($\mathsf{N\text{-}HA}^\omega$, $\mathsf{N\text{-}PA}^\omega$) to be the extension of the term system T to a $\mathbf{T}$-sorted intuitionistic resp. classical logical system plus the schema of full induction and the equality axioms for type 0, i.e.

- $x =_0 x,\ \ x =_0 y \to y =_0 x,\ \ x =_0 y \wedge y =_0 z \to x =_0 z,$
- $x_1 =_0 y_1 \wedge \dots \wedge x_n =_0 y_n \to t(x_1, \dots, x_n) =_0 t(y_1, \dots, y_n)$ for any n-ary term t of suitable type,

and substitution schemata for λ-combinators and the recursors, i.e.

$$(\mathsf{SUB})\colon \begin{cases} t[\Pi xy] =_0 t[x] \\ t[\Sigma xyz] =_0 t[xz(yz)] \\ t[R0yz] =_0 t[y] \\ t[R(Sx)yz] =_0 t[z(Rxyz)x] \end{cases} \quad \text{for all } t \text{ of type } 0.$$

For a formal definition see [52, I.1.6.15] (there $\mathsf{N\text{-}HA}^\omega$ is called HA^ω).

These theories are neutral with respect to an intensional or an extensional interpretation of higher type objects. However, for type 0 objects the usual equality axioms hold. Higher type equality is of no effect except for the SUB -rule. Later we will add functionals yielding cohesive and homogeneous set which are not extensional (in the presence of extensionality they would prove full arithmetical comprehension, see [31]) and therefore can only be analyzed in a neutral context.

Let *weakly extensional Heyting/Peano arithmetic* ($\mathsf{WE\text{-}HA}^\omega$, $\mathsf{WE\text{-}PA}^\omega$) be $\mathsf{N\text{-}HA}^\omega$ resp. $\mathsf{N\text{-}PA}^\omega$ plus the quantifier-free rule of extensionality, i.e.

$$(\mathsf{QF\text{-}ER})\colon \frac{A_{qf} \to s =_\rho t}{A_{qf} \to r[s/x^\rho] =_\tau r[t/y^\rho]},$$

where A_{qf} is quantifier-free and s^ρ, t^ρ, r^τ are terms of $\mathsf{WE\text{-}HA}^\omega$. Note that the addition of SUB here is redundant, since $\mathsf{QF\text{-}ER}$ together with the axioms for Π, Σ, R proves it. The systems with *full extensionality*, i.e. $\mathsf{N\text{-}HA}^\omega$, $\mathsf{N\text{-}PA}^\omega$ plus the extensionality axioms

$$(\mathsf{E}_{\rho,\tau})\colon \forall z^{\tau\rho}, x^\rho, y^\rho\, (x =_\rho y \to zx =_\tau zy)$$

for all $\tau, \rho \in \mathbf{T}$, will be denoted by $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{E\text{-}PA}^\omega$. For a detailed definition of these systems, see [32, section 3].

The weakly extensional and neutral theories allow functional interpretation in themselves, which is not possible in the presence of full extensionality. Later we will eliminate the usage of extensionality (see proposition 3 below), hence neither the interpretation of constants yielding cohesive/homogeneous sets nor the functional interpretation will lead to problems. For a discussion of these systems and the connection to functional interpretation we refer to [51].

It is also important to note that in presence of only $\mathsf{QF\text{-}ER}$ the deduction theorem in general fails, see [32, theorem 9.11]. To overcome this we will restrict the use of principles in premises of $\mathsf{QF\text{-}ER}$. This will be denote by the $\oplus$-sign, e.g. $\mathsf{WE\text{-}PA}^\omega \oplus \mathsf{WKL}$ denote the system $\mathsf{WE\text{-}PA}^\omega + \mathsf{WKL}$, where WKL may not be used in the premise of $\mathsf{QF\text{-}ER}$. The weak extensional systems satisfy the deduction theorem with respect to $\oplus$.

We now introduce fragments of neutral and (weakly) extensional Heyting/Peano arithmetic corresponding to T_n:
Define $\mathsf{N\text{-}HA}^\omega_n\!\restriction$ to be the logical system extending T_n plus Σ^0_{n+1}-IA and plus the case-distinction functionals $(\mathrm{Cond}_\rho)_{\rho\in\mathbf{T}}$ and its substitution axioms

$$(\mathsf{SUB}_{\mathrm{Cond}})\colon \begin{cases} t[\mathrm{Cond}_\rho(0^0, x^\rho, y^\rho)] =_0 t[x] \\ t[\mathrm{Cond}_\rho(Su, x^\rho, y^\rho)] =_0 t[y] \end{cases} \qquad \text{for all } t \text{ of type } 0.$$

These case distinction functionals are needed for the functional interpretation and cannot be defined in these fragments of $\mathsf{N\text{-}HA}^\omega$, see [41, 3]. In the full system T they can be simulated by the recursors. Instead of $\mathsf{N\text{-}HA}^\omega_0\!\restriction$ we also write $\widehat{\mathsf{N\text{-}HA}}^\omega\!\restriction$. The classical systems $\mathsf{N\text{-}PA}^\omega_n\!\restriction$, $\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction$ are defined similarly. In the same way also the (weakly) extensional systems $\mathsf{(W)E\text{-}HA}^\omega_n\!\restriction$, $\widehat{\mathsf{(W)E\text{-}HA}}^\omega\!\restriction$, $\mathsf{(W)E\text{-}PA}^\omega_n\!\restriction$, $\widehat{\mathsf{(W)E\text{-}PA}}^\omega\!\restriction$ are defined.[4] However for the classical systems defined here one does not need to add Cond to the system since it is provably definable with the λ-combinators and R_0, see [41]. Note that Σ^0_{n+1}-induction is provable with the recursor R_n and quantifier-free induction and full $\mathsf{QF\text{-}AC}$ in all types (definition below) over the classical systems defined here. Hence the addition of it to the classical systems is actually superfluous. This follows from [41] and Kreisel's characterization theorem, see [32, proposition 10.13].

2.4. **Grzegorczyk arithmetic.** We moreover need weaker fragments of Heyting and Peano arithmetic containing only quantifier-free induction.

Let *weakly extensional Grzegorczyk arithmetic of level n in all finite types* $\mathsf{G}_n\mathsf{A}^\omega_{(\mathsf{i})}$ be the (intuitionistic) system containing $=_0$-axioms, $\mathsf{QF\text{-}ER}$, λ-abstraction, the n-th branch of the Ackermann-function, bounded search and bounded primitive recursion. For a detailed definition see [26].[5] The neutral variant will be denoted by $\mathsf{N\text{-}G}_n\mathsf{A}^\omega$, the extensional one by $\mathsf{E\text{-}G}_n\mathsf{A}^\omega$.

Let $\mathsf{G}_\infty\mathsf{A}^\omega$ be the union of all these systems. This system contains all primitive recursive functions but not all primitive recursive functionals (in the sense of Kleene). For instance R_0 is not contained in $\mathsf{G}_\infty\mathsf{A}^\omega$. Thus it also contains no Σ^0_1-induction. The set of all closed terms of $\mathsf{G}_n\mathsf{A}^\omega$ is called $\mathsf{G}_n\mathsf{R}^\omega$. See [26] and [32, Chapter 3] for all of this.

2.5. **Quantifier-free axiom of choice.** Let $\mathsf{QF\text{-}AC}$ be the schema

$$\forall x\, \exists y\, A_{qf}(x,y) \to \exists f\, \forall x\, A_{qf}(x, f(x)),$$

where A_{qf} is a quantifier-free formula. If the types of x, y are restricted to α, β we write $\mathsf{QF\text{-}AC}^{\alpha,\beta}$.

[4]For a formal definition let $\widehat{\mathsf{(W)E\text{-}HA}}^\omega\!\restriction$ be defined as in [32, section 3.4] and define $\mathsf{(W)E\text{-}HA}^\omega_n\!\restriction$ to be $\widehat{\mathsf{(W)E\text{-}HA}}^\omega\!\restriction$ plus Σ^0_{n+1}-IA and the defining axioms and constants for the recursors R_ρ with $deg(\rho) \le n$. The neutral variants are defined in the same way but without the rule of extensionality.

[5]In [32] the system $\mathsf{G}_n\mathsf{A}^\omega$ is defined to include all $\mathbb{N}, \mathbb{N}^\mathbb{N}, \mathbb{N}^{\mathbb{N}^\mathbb{N}}$-true $\forall$-sentences. In a pure proof-mining context these sentences do not matter because they have no impact on the provable recursive functions in the system. We only add quantifier-free induction ($\mathsf{QF\text{-}IA}$), to be able to later establish conservativity over PRA.

The scheme $\mathsf{QF\text{-}AC}^{0,0}$ corresponds to recursive comprehension (Δ^0_1-CA) in a second order context. Thus $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ and RCA_0 share the same proof theoretic strength. RCA_0 can easily be embedded into $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ and $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ is conservative over RCA_0 modulo this embedding, see [31]. For this reason $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ is called RCA^ω_0.

The system RCA^*_0 is RCA_0, where Σ^0_1-induction is replaced by quantifier-free-induction and the exponential function, see [45, X.4.1]. This system can be embedded into $\mathsf{G}_3\mathsf{A}^\omega + \mathsf{QF\text{-}AC}^{1,0}$ and both systems are Π^0_2-conservative over Kalmar elementary arithmetic.

In ordinary mathematics higher types usually do not occur and second order arithmetic is sufficient to formalize most of it. We require here a system containing all finite types to be able to carry out a functional interpretation and thus cannot use a second order system.

2.6. **The quantifier-free subsystems.** In order to exploit the full subtlety of the functional interpretation we will also need the *quantifier-free subsystems of* $\mathsf{N\text{-}G}_n\mathsf{A}^\omega_\mathsf{i}$ *and* $\mathsf{N\text{-}HA}^\omega_n\!\restriction$. The quantifier-free subsystems are denote by qf-$\mathsf{N\text{-}G}_n\mathsf{A}^\omega$ resp. qf-$\mathsf{N\text{-}PA}^\omega_n\!\restriction$. (The quantifier free subsystems satisfy the law of excluded middle and are therefore classical.)

They are obtained from the full systems as follows:

- The quantifier-rules and -axioms are dropped from logic.
- For all axioms of the form $A(x_1^{\rho_1},\dots,x_n^{\rho_n})$, where A is quantifier-free, the following axioms are added to the system:
$$A(t_1^{\rho_1},\dots,t_n^{\rho_n}),$$
where t_i are arbitrary terms.
- The induction schema is replaced by the (quantifier-free) induction rule:
$$\frac{A(0^0),\quad A(x^0)\to A(Sx^0)}{A(t^0)},$$
where A is quantifier-free, x does not occur free in the assumption and t is an arbitrary term.

These quantifier-free systems contain only prime formulas of the form
$$t_0 =_0 t_1,$$
where t_0, t_1 are terms in $\mathsf{N\text{-}G}_n\mathsf{A}^\omega_\mathsf{i}$ resp. $\mathsf{N\text{-}HA}^\omega_n\!\restriction$. Formulas are logical combinations of these predicates. Obviously, qf-$\mathsf{N\text{-}G}_n\mathsf{A}^\omega$ and qf-$\mathsf{N\text{-}PA}^\omega_n\!\restriction$ are subsystems of $\mathsf{N\text{-}G}_n\mathsf{A}^\omega_\mathsf{i}$ resp. $\mathsf{N\text{-}HA}^\omega_n\!\restriction$. (For a detailed discussion of these systems we also refer the reader to [51, 1.6.5]. For technical reason we use here the variant of the systems described remark 1.5.8.)

Observe, that in these system we can only instantiate type 0 variables (via the induction rule) and not higher type variables, hence we immediately obtain the following lemma:

Lemma 2. *Let A be a sentence and*
$$\mathcal{T} \vdash A,$$
where $\mathcal{T} =$ qf-$\mathsf{N\text{-}G}_n\mathsf{A}^\omega$, qf-$\mathsf{N\text{-}PA}^\omega_n\!\restriction$.

Then there exists a derivation of A in $\mathcal{T}$ that contains only the variables of A plus some fresh variables of type 0.

Proof. In a derivation of A in $\mathcal{T}$ replace every variable not of type 0 and not occurring in A by constant 0^ρ of suitable type. Since higher type variables cannot be instantiated the derivation remains valid. $\square$

2.7. **Functional interpretation.** *Functional interpretation* will denote in this paper a negative translation followed by Gödel's Dialectica translation.

Gödel's Dialectica translation is a proof interpretation that translates proofs from (a fragment of) $\mathsf{WE\text{-}HA}^\omega$ or $\mathsf{N\text{-}HA}^\omega$ into its quantifier-free subsystem, see [13].

Let $\mathcal{T}$ be such a system. The Dialectica translation associates to each formula A of $\mathcal{T}$ a $\exists\forall$-formula

$$A^D :\equiv \exists x\, \forall y\, A_D(x,y),$$

where A_D is quantifier-free. In particular, for a Σ^0_2 sentence A the formula A_D is the quantifier-free matrix of A.

From a proof of A one then can extract a term t, such that

$$\text{qf-}\mathcal{T} \vdash A_D(t,x).$$

A *negative translation* is a proof translation that translates classical proofs into intuitionistic proofs. It also proceeds by associating each formula A a formula A^N such that

$$\mathcal{S} \vdash A \leftrightarrow A^N \qquad \text{and} \qquad \mathcal{S} \vdash A \quad \Longrightarrow \quad \mathcal{S}_{\mathrm{i}} \vdash A^N.$$

Here $\mathcal{S}$ is any of $(\mathsf{W})\mathsf{E\text{-}PA}^\omega$, $\widehat{(\mathsf{W})\mathsf{E\text{-}PA}}^\omega\!\restriction$, $\mathsf{G}_n\mathsf{A}^\omega$ or its neutral variants and $\mathcal{S}_{\mathrm{i}}$ is its intuitionistic counterpart. (To be specific, Kuroda's negative translation A^N is obtained from A by inserting $\neg\neg$ after each $\forall$ and in front of the whole formula.)

Thus we denote by functional interpretation a proof translation from (a fragment of) $\mathsf{WE\text{-}PA}^\omega$ or $\mathsf{N\text{-}PA}^\omega$ into its quantifier-free part. We abbreviate the functional interpretation by ND. The ND-translation of a formula A will be denoted by A^{ND} and the quantifier-free matrix of it by A_{ND}.

The functional interpretation in particular has the property to extract a term for each provable recursive function, i.e. from a proof of a $\forall\exists$-statement (in $\mathsf{WE\text{-}PA}^\omega$ or any other fragment for which the functional interpretation holds)

$$\mathsf{WE\text{-}PA}^\omega \vdash \forall u\, \exists v\, A_{qf}(u,v)$$

it extracts a term t such that

$$\text{qf-}\mathsf{WE\text{-}HA}^\omega \vdash \underbrace{A_{qf}(u,tu)}_{\equiv A_{ND}(t,u)}.$$

For an introduction to the functional interpretation see [32, 3, 51].

Since the functional interpretation does not interpret full extensionality it is often combined with the elimination of extensionality.

Proposition 3 (Elimination of extensionality, [38])**.** *Let A be a formula containing only free variables and quantification of type ≤ 1. If*

$$\mathsf{E\text{-}PA}^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} \vdash A$$

then

$$\mathsf{N\text{-}PA}^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} \vdash A.$$

The same holds also for the fragments $\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction$ and $\mathsf{N\text{-}G}_n\mathsf{A}^\omega$.

Proof. Proposition 10.45 and lemma 10.41 of [32]. These lemma and proposition actually do not make use of weak extensionality and therefore show conservativity even over a neutral theory. $\square$

2.8. **Additional notation and definitions.** We denote sets by capital letters. Unless otherwise noted they are represented by characteristic functions. Sometimes capital letters also denote higher type functionals. It will be clear from the context what is meant.

It is important to note that in systems not containing Σ^0_1-induction it is in general not provable that every infinite set — that is a set X satisfying $\forall k\, \exists n > k \;\, n \in X$ — can be strictly increasingly enumerated, i.e. there exists a strictly monotone function f such that $rng(f) = X$. The system $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\upharpoonright + \mathsf{QF\text{-}AC}^{0,0}$ proves that the first statement implies the second. The converse — every strictly increasingly enumerable set is infinite — is already provable without Σ^0_1-induction, for instance $\mathsf{G_3A}^\omega$ suffices.

Sequence codes are denoted by $\langle x_0, \dots, x_n\rangle$. The corresponding projection functions and length function are denoted by $(\cdot)_i$ and $\mathrm{lth}(\cdot)$. We encode sequences using a bijective and monotone (in each component) sequence-coding based on the Cantor pairing, see [32, definition 3.30]. This coding is definable in every system containing qf-$\mathsf{N\text{-}G_3A}^\omega$.

We model in our systems n-colorings of $[\mathbb{N}]^2$ as functions $c\colon \mathbb{N}\times\mathbb{N} \to n$ with $c(x,y) = c(y,x)$.

Further we define the following notions:

- $\bar f$ denotes the course-of-value function of f^1, i.e. $\bar f(n) = \langle f(0), \dots, f(n-1)\rangle$.
- $x \sqsubset X$ iff x is an initial segment of a strictly monotone enumeration of X.
- $x \subseteq^{fin} X$ iff x is an code for a finite subset of X.

Definition 4 (Bounded type 1 recursor, $\tilde R_1$)**.** The bounded type 1 recursor $\tilde R_1$ is defined as

$$\tilde R_1 0yzhu =_0 \min(y(u), h(0,u))$$
$$\tilde R_1 (x+1)yzhu =_0 \min(z(\tilde R_1 xyzh)xu, h(x,u)).$$

We will denote by $(\tilde R_1)$ the defining axioms. Note that they are purely universal and that $\tilde R_1$ can be trivially majorized.

Definition 5 (Uniform weak König's lemma, UWKL, [30])**.** Uniform weak König's lemma is the statement

$$\exists \Phi \;\leq_{1(1)} 1 \forall f\, (T^\infty(f) \to \forall x^0\, f(\overline{\Phi f} x) = 0),$$

where T^∞ expresses that f describes an infinite $0/1$-tree.

We can modify (in $\mathsf{G_\infty A}^\omega$) every function f such that it describes an infinite $0/1$-tree and is not altered if it already described such a tree. We will write $\check f$ for this modification, see [25, 32].

With this we can restate UWKL equivalently as

$$\exists \Phi \;\leq_{1(1)} 1 \forall f^1\, \forall x^0\, \check f(\overline{\Phi \check f} x) = 0.$$

Note that the condition $\leq_{1(1)}$ is superfluous because the modified tree contains only $0/1$-sequences.

By Skolemization we add a weak König's Lemma functional constant $\mathcal{B}$ described by the (purely universal) axiom

$$\forall f\, \forall x^0\, \check f(\overline{\mathcal{B}\check f} x) = 0. \tag{3}$$

This axiom will be denoted by $(\mathcal{B})$. Note that $\mathcal{B}$ can be trivially majorized.

In a system containing full extensionality UWKL implies Π^0_1-CA, see [30], hence it is too strong for our purpose. But in a weakly extensional system it often can be handled like WKL, for instance it vanishes under a monotone functional interpretation like WKL and can be added to the elimination of monotone Skolem functions, see [30]. Note that proposition 3 does not cover UWKL.

3. Continuous functionals

Recall that a type 2 functional φ is continuous if

$$\forall g^1\, \exists n^0\, \forall h^1\ \left(\bar{g}n = \bar{h}n \rightarrow \varphi(g) = \varphi(h)\right). \tag{4}$$

Definition 6 (Associate, [23, 33])**.** For every continuous type 2 functional φ we will denote by α_φ an associate of φ, i.e. a type 1 function with the properties

$$\begin{gathered}\forall f\, \exists n\ \alpha_\varphi(\bar{f}n) \neq 0,\\ \forall f, n\ \left(\alpha_\varphi(\bar{f}n) \neq 0 \rightarrow \varphi(f) = \alpha_\varphi(\bar{f}n) \dot{-} 1\right).\end{gathered} \tag{5}$$

The value of φ is uniquely determined through α_φ. For every continuous functional there exists an associate, though it is not uniquely determined. For details see also [39].

Definition 7. A functional given by a closed term φ^ρ of $\mathcal{T}$ is called *provably continuous* if for some term α_φ (containing at most the free variables of φ) of type 1 (if $\rho > 0$) resp. 0 (if $\rho = 0$), the following holds:

$$\mathcal{T} \vdash \varphi \approx_\rho \alpha_\varphi.$$

Here, for general x^ρ and $\alpha^{0/1}$, the relation $x \approx_\rho \alpha$ is defined by induction on ρ:

$$\begin{aligned} &x \approx_0 \alpha :\equiv x =_0 \alpha,\\ &x \approx_{\tau\rho} \alpha :\equiv \alpha \in \mathsf{ECF}_{\tau\rho} \wedge \forall y^\rho\, \forall \beta \in \mathsf{ECF}_\rho\ (y \approx_\rho \beta \rightarrow xy \approx_\tau \alpha \upharpoonright \beta)\,,\end{aligned}$$

where ECF is the model of extensional hereditarily continuous functionals formalized in $\mathcal{T}$ and $\upharpoonright$ denotes the application in ECF. (See [24, 33, 51], for a definition see also [32, definitions 3.58, 3.59].)

Especially, in the case of $\rho = 2$ a functional φ is provably continuous in $\mathcal{T}$ if it has an associate α_φ in $\mathcal{T}$ and (5) is provable.

Proposition 8. *For every term $t^2 \in \mathsf{G}_n\mathsf{R}^\omega, T_0, T_1$ there exists provably in $\mathsf{G}_n\mathsf{A}^\omega$ resp. $\widehat{\mathsf{WE}\text{-}\mathsf{PA}}^\omega\!\upharpoonright$, $\mathsf{WE}\text{-}\mathsf{PA}^\omega_1\!\upharpoonright$ a (primitive recursive) associate α_t. In other words t is provably continuous.*

Proof. We first consider the case of $\widehat{\mathsf{WE}\text{-}\mathsf{PA}}^\omega\!\upharpoonright = \mathsf{WE}\text{-}\mathsf{PA}^\omega_0\!\upharpoonright$ and $\mathsf{G}_n\mathsf{A}^\omega$. Here the only functional constants having no trivial associate are the λ-combinators and R_0 (in the case of $\widehat{\mathsf{WE}\text{-}\mathsf{PA}}^\omega\!\upharpoonright$) and the course-of-value functional (in the case of $\mathsf{G}_n\mathsf{A}^\omega$). The associates of R_0 and the course-of-value functional can easily be computed and (5) be proven in the respective systems. By normalization one can find a term $\tilde{t} =_2 t$ that does not include λ-abstraction of type ≥ 1. The proposition for $\widehat{\mathsf{WE}\text{-}\mathsf{PA}}^\omega\!\upharpoonright$ and $\mathsf{G}_n\mathsf{A}^\omega$ follows from this.

In the case of $\mathsf{WE}\text{-}\mathsf{PA}^\omega_1\!\upharpoonright$ we prove by induction over the structure of t that t is provably continuous. For this it is sufficient to prove that every functional constant is provably continuous and to observe that this property is retained under composition. The associates for the λ-combinators are easily definable and provable in these systems, see [51].

Here we only show that the existence of an associate for R_1 is provable in $\mathsf{WE}\text{-}\mathsf{PA}^\omega_1\!\upharpoonright$, since we are only interested in this case. For the other recursors the proof is similar. Let

$$\alpha_{R_1}(0, y', z', u) := \begin{cases} (y')_u + 1 & \text{if } u < \mathrm{lth}\, y', \\ 0 & \text{otherwise,} \end{cases}$$

$$\alpha_{R_1}(x+1, y', z', u) := \begin{cases} (z')_{\langle x, \overline{(\lambda k.\alpha_{R_1}(x,y',z',k) \dot{-} 1)k} \rangle} & \begin{array}{l}\text{if } \exists k < \mathrm{lth}\, y', \text{ such that} \\ \alpha_{R_1}(x, y', z', k) > 0 \\ \text{and this is } > 0,\end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Using Π^0_2-induction one shows that

$$\forall x\ (\forall u\, \exists n\, \alpha_{R_1}(x, \bar{y}n, \overline{\alpha_{\lambda r.\bar{z}\bar{r}n}}n, u) = R_1(x, y, z, u) + 1)$$

and hence that α_{R_1} is an associate of R_1. □

4. Properties of instances of comprehension

Remark 9. A sequence of Π^0_1-comprehension instances $\left(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(f_i)\right)_i$ may be reduced to the single instance of $\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(f')$ with $f'xy := f_{(x)_1}(x)_2 y$, see [27, remark 3.8].

Lemma 10 ([27, 28]). *For suitable terms ξ_i of $\mathsf{G_3A^\omega}$ we have*
(i) $\mathsf{G_3A^\omega} + \mathsf{QF\text{-}AC^{0,0}} \vdash \forall f\ \left(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\xi_1 f) \rightarrow \mathsf{\Pi}^0_1\text{-}\mathsf{AC}(f)\right)$,
(ii) $\mathsf{G_3A^\omega} + \mathsf{QF\text{-}AC^{0,0}} \vdash \forall f\ \left(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\xi_2 f) \rightarrow \mathsf{\Delta}^0_2\text{-}\mathsf{CA}(f)\right)$,
(iii) $\mathsf{G_3A^\omega} + \mathsf{QF\text{-}AC^{0,0}} \vdash \forall f\ \left(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\xi_3 f) \rightarrow \mathsf{\Delta}^0_2\text{-}\mathsf{IA}(f)\right)$,
(iv) $\mathsf{G_3A^\omega} + \mathsf{QF\text{-}AC^{0,0}} \vdash \forall f\ \left(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\xi_4 f) \rightarrow \mathsf{\Pi}^0_1\text{-}\mathsf{CP}(f)\right)$,
(v) $\mathsf{G_3A^\omega} + \mathsf{QF\text{-}AC^{0,0}} + \mathsf{WKL} \vdash \forall f\ \left(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\xi_5 f) \rightarrow \mathsf{\Pi}^0_2\text{-}\mathsf{WKL}(f)\right)$.

Here the principle $\mathcal{K}$-AC denotes the scheme of axiom of choice, where the base formula is of type $\mathcal{K}$. Similarly $\mathcal{K}$-WKL denotes weak König's lemma where the tree is given by a predicate of type $\mathcal{K}$. The principles $\mathcal{K}$-IA and $\mathcal{K}$-CA are defined likewise.

If $\mathcal{K} = \Pi^0_n, \Sigma^0_n$ then an instance of those principles is given by a function f coding the quantifier-free part of the Π^0_n resp. Σ^0_n formula. For instance

$$\mathsf{\Pi}^0_1\text{-}\mathsf{AC}(f) \equiv \forall x\, \exists y\, \forall z\, f(x, y, z) = 0 \rightarrow \exists Y\, \forall x\, \forall z\, f(x, Y(x), z) = 0.$$

Similar a Δ^0_2-formula is given by an f coding a function for a Π^0_n and a function for a Σ^0_n formula.

Proof of lemma 10. For (i), (ii) see [28, lemma 4.2]. The statements (iii), (iv) are immediate consequences of these. Note that we require here $\mathsf{G_3A^\omega}$ and not only $\mathsf{G_2A^\omega}$ as in the reference, since we do not add the true universal sentences to the system, see footnote 5.

For (v) let ξ_5 be such that the instance of $\mathsf{\Pi}^0_1\text{-}\mathsf{CA}$ yields the comprehension function for the innermost quantifier of the tree predicate reducing $\mathsf{\Pi}^0_2\text{-}\mathsf{WKL}$ to $\mathsf{\Pi}^0_1\text{-}\mathsf{WKL}$. This is equivalent to WKL and thus included in the system, see for instance [45]. □

For the ordinal analysis of terms we will need the following abbreviation:

$$\omega^\mu_0 = \mu \quad \text{and} \quad \omega^\mu_{k+1} = \omega^{\omega^\mu_k},$$

where $k \in \mathbb{N}$ and μ is an ordinal.

Lemma 11. *Let $n \in \mathbb{N}$ and let $t[g]$ be a type 1 term with the only free variable g such that $\lambda g.t[g] \in T_n$. Then for every term φ in T_{n-1} or in $\mathsf{G_\infty R^\omega}$ if $n = 0$ there exists a term ξ in the same system such that $\mathsf{WE\text{-}PA}^\omega_{n-1}\!\restriction + \mathsf{QF\text{-}AC}$ or $\mathsf{G_\infty A^\omega} + \mathsf{QF\text{-}AC}$ in the case of $n = 0$ proves*

$$\forall g\ \left(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\xi g) \rightarrow \exists f^1\ \left(f \text{ satisfies the defining axioms of } t[g] \wedge \mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\varphi f g)\right)\right).$$

Defining axioms of $t[g]$ *are a formula* A*, such that* $\forall g, x, y\ (A(g, x, y) \leftrightarrow t[g]x = y)$*. (Since* t^1 *can be defined by (unnested) ordinal recursion of order* $< \omega_{n+1}^{\omega}$*, one can take for* A *the formula describing this recursion.)*

Proof. First fix a suitable encoding for ordinals smaller than ε_0 in this system, see for instance [14].

Every term $t^1 \in T_n$ can be defined through (unnested) ordinal recursion of order $< \omega_{n+1}^{\omega}$; the totality of such a recursion can be proven using a suitable instance of Σ_{n+1}^0-IA, see [40] and theorem 17 below. Such an instance is included in the system because a suitable instance of Π_1^0-CA reduces it to Σ_n^0-IA. This proves the claim that there is a total function f satisfying the definition of $t[g]$.

For the second part note that the defining axioms of unnested ordinal primitive recursion of order type α are given by

$$f(0) := f_0, \qquad f(n) := h(n, f(l(n))), \tag{6}$$

where l satisfies

$$l(n) \prec n \qquad \text{for } n > 0 \tag{7}$$

and $\prec$ defines a well-ordering on $\mathbb{N}$ of order type α.

We say a finite sequence s *satisfies the defining axioms* (6) *up to* n if

$$(s)_0 = f_0, \qquad (s)_i = h(i, (s)_{l(i)}) \quad \text{for all } i \in \bigcup_{n' \le n} \bigcup_k \{l^k(n')\} \setminus \{0\}$$

For notational ease we assume here that $l(0) = 0$. Note that because of (7) the set $\bigcup_k \left\{l^k(n')\right\}$ defines an $\prec$-descending chain and is therefore provably finite.

For the second part we have to prove a comprehension of the form

$$\exists H\, \forall k\ (k \in H \leftrightarrow \forall x\, \varphi(f, g, k, x) = 0)\,. \tag{8}$$

We use the imposed instance of comprehension to prove the following comprehension

$$\exists H\, \forall k\ \big(k \in H \leftrightarrow \forall x\, \forall s, n\, \big(s \text{ satisfies the defining axioms of } t[g] \text{ up to } n \\ \to \alpha_{\lambda f.\varphi(f,g,k,x)}(s) \le 1\big)\big).$$

Note that this comprehension is equivalent to (8) if f is total. □

The proof of the comprehension above is similar to the construction of a 1-generic set: If the statement

$$\forall x\, \varphi(f, g, k, x) = 0$$

for a fixed k fails, then there is an x such that $\varphi(f, g, k, x) \neq 0$. Since φ is continuous this depends only on an initial segment of f. We express this by using associates, i.e. this statement is equivalent to

$$\exists n\, \alpha_{\lambda f.\varphi(f,g,k,x)}(\bar{f}n) > 1.$$

Hence it suffices to consider only finite initial segments.

We will use this technique in most proofs of instances of comprehension in this paper. This is the reason why we require φ to be provably continuous in the definition of proofwise low.

5. Elimination of monotone Skolem functions

Let Δ be a set of sentences of the form $\forall a\, \exists b < ra\, \forall c^0\, B_{qf}(a, b, c)$, where r is a closed term and B_{qf} is quantifier-free and contains any further free variables than those shown. Let $\tilde{\Delta}$ be the corresponding set of Skolem normal form of the sentence of Δ, i.e. the corresponding formulas of the form $\exists B < r\, \forall a, c^0\, B_{qf}(a, Ba, c)$.

Theorem 12 ([27, 3.8])**.** *Let γ be an arbitrary type and let A_{qf} be a quantifier-free statement where only the shown variables are free and let s be a term in $\mathsf{G}_\infty\mathsf{R}^\omega$. If*

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \Delta \vdash \forall u^1\, \forall v \leq_\gamma su\, \left(\Pi^0_1\text{-}\mathsf{CA}(\xi uv) \to \exists w^0\, A_{qf}(u,v,w)\right)$$

then one can extract from a proof a term $t \in T_0$ such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus\, \tilde{\Delta} \vdash \forall u^1\, \forall v \leq_\gamma su\, \exists w \leq_0 tu\ A_{qf}(u,v,w).$$

Especially, in case that $A_{qf} \in \mathcal{L}(\mathsf{PRA})$, u of type 0, v absent and $\Delta = \emptyset$ we have

$$\mathsf{PRA} \vdash \forall u^0\, A_{qf}(u,tu).$$

Corollary 13. *Let γ, ξ, s, A_{qf} be as in theorem 12. However ξ may contain $\mathcal{B}$ but s and A_{qf} must not. Then the following holds: If*

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \vdash \forall u\, \forall v \leq_\gamma su\, \left(\Pi^0_1\text{-}\mathsf{CA}(\xi uv) \to \exists w^0\, A_{qf}(u,v,w)\right)$$

then one can extract from a proof a term $t \in T_0$ such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \vdash \forall u^1\, \forall v \leq_\gamma su\, \exists w \leq_0 tu\ A_{qf}(u,v,w).$$

Proof. First note that due to [27, remark 2.10] we may add the (majorizable) constant $\mathcal{B}$ to $\mathsf{G}_\infty\mathsf{A}^\omega$ in theorem 12.

Apply this theorem to $\Delta := \left\{\forall f\, \forall x\, \check{f}(\mathcal{B}\check{f}x) = 0\right\}$, cf. definition 5 and (3) on p. 10. The premise of the corollary implies that

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \Delta \vdash \forall u\, \forall v \leq_\gamma su\, \left(\Pi^0_1\text{-}\mathsf{CA}(\xi uv) \to \exists w^0\, A_{qf}(u,v,w)\right).$$

Theorem 12 and noticing that $\Delta \equiv \tilde{\Delta}$ yields

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus\, \Delta \vdash \forall u^1\, \forall v \leq_\gamma su\, \exists w \leq_0 tu\ A_{qf}(u,v,w)$$

and so

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \vdash \Delta \to \forall u^1\, \forall v \leq_\gamma su\, \exists w \leq_0 tu\ A_{qf}(u,v,w).$$

Since the constant $\mathcal{B}$ only occurs in Δ, we may replace it with a new variable and so replace Δ with UWKL. The corollary now follows from [32, corollary 10.34]. □

6. Bar recursor

With bar recursion (even of the lowest type) one can interpret the functional interpretation of (instances of) Π^0_1-CA. This will be discussed in detail in proposition 48 below. In this section we will show that the bar recursor $B_{0,1}$ can be majorized provably in $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$.

Definition 14 (bar induction of type 0)**.** Let bar induction of type 0 be

$$(\mathsf{BI}_0)\colon \begin{cases} \forall x^1\, \exists n^0_0\, \forall n \geq n_0\, Q(\overline{x,n};n)\ \wedge \\ \forall x^1, n^0\ (\forall d\, Q(\overline{x,n} * d; n+1) \to Q(\overline{x,n};n)) \\ \to \forall x^1, n^0\, Q(\overline{x,n};n), \end{cases}$$

where

$$(\overline{x,n})k := \begin{cases} x(k), & \text{if } k < n, \\ 0, & \text{otherwise,} \end{cases} \qquad (\overline{x,n} * d)k := \begin{cases} x(k), & \text{if } k < n, \\ d, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

If Q is restricted to formulas in $\mathcal{K}$, we write $\mathcal{K}$-BI_0.

Lemma 15.

$$\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{0,0} \vdash \Pi^0_1\text{-}\mathsf{BI}_0$$

Proof. Let $Q(\overline{x,n};n) \equiv \forall k\, Q_{qf}(\overline{x,n};n;k)$. Suppose that Π^0_1-BI_0 does not hold, i.e. the premises of Π^0_1-BI_0 are true and

$$\exists x^1_0, n^0_0\, \neg\forall k^0_0\, Q_{qf}(\overline{x_0,n_0};n_0;k_0),$$

which is equivalent to

$$\exists x^1_0, n^0_0, k^0_0\, \neg Q_{qf}(\overline{x_0,n_0};n_0;k_0). \tag{9}$$

The second premise yields

$$\forall x^1, n^0, k^0\, \exists d, k' \,\left(\neg Q_{qf}(\overline{x,n};n;k) \to \neg Q_{qf}(\overline{x,n} * d; n+1; k')\right).$$

Since the whole statement only depends on an initial segment of x^1, it can be coded in a type 0 object x'^0. For instance let $x' := \bar{x}n$ then $\overline{\lambda i.(x')_i, n} = \overline{x,n}$.

Using $\mathsf{QF\text{-}AC}^{0,0}$ we then obtain functions $D(x,n,k)$, $K(x,n,k)$ with

(10)

$$\forall x^0, n, k\, \left(\neg Q_{qf}(\overline{\lambda i.(x)_i, n};n;k) \to \neg Q_{qf}(\overline{\lambda i.(x)_i, n} * D(x,n,k); n+1; K(x,n,k)\right).$$

Then define using simultaneous course-of-value recursion (n_0, x_0, k_0 are from (9)) the functions D_0, K_0:

$$\left.\begin{aligned} D_0(n) &:= x_0(n) \\ K_0(n) &:= k_0 \end{aligned}\right\} \text{ for } n \le n_0, \qquad \left.\begin{aligned} D_0(n) &:= D(\overline{D_0,n}, n, K_0(n-1)) \\ K_0(n) &:= K(\overline{D_0,n}, n, K_0(n-1)) \end{aligned}\right\} \text{ for } n > n_0.$$

The definition of D_0 and (9),(10) yield

$$\forall n \ge n_0\, \neg Q(\overline{D_0,n};n)$$

and hence a contradiction to the first premise of Π^0_1-BI_0. □

Proposition 16. $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{0,0}$ *proves that there exists a majorant* $B^*_{0,1}$ *of* $B_{0,1}$.

Proof. Define $B^*_{0,1}$ like in [32, proof of theorem 11.17]. By the cited proof it suffices to show Π^0_1-BI_0. (Note that in that proof Q is a Π^0_1 formula in the case where $\rho = 0$.) Hence the proposition is an immediate consequence of lemma 15. See also [5]. □

7. Ordinal analysis of terms

7.1. **Ordinal Peano/Heyting arithmetic.** In this section we will investigate the strength of induction along ordinals the systems $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$, $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction$.

We will code ordinals using the ordinal coding of [14, II.3.a]. (This coding uses the Cantor normal form for ordinals to define primitive recursive codes for ordinals.) For convenience we repeat the definition of ω^μ_k:

$$\omega^\mu_0 = \mu \quad \text{and} \quad \omega^\mu_{k+1} = \omega^{\omega^\mu_k}$$

Here $k \in \mathbb{N}$ and μ is an arbitrary ordinal number.

Theorem 17 ([40], [53])**.** *The functions and functionals of level* 2 *that are ordinal recursive (unnested) in an ordering* $< \omega^\omega_{k+1}$ *are exactly the functions and functionals in* T_k.

Theorem 18 ([14, II.3.18])**.**

$$\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \Sigma^0_{m+k-1}\text{-}\mathsf{IA} \vdash \Sigma^0_m\text{-}\mathsf{LNP}(< \omega^\omega_k)$$

for every $m, k \in \mathbb{N}$, *where* LNP *denotes the least number principle.*
In particular, $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \Sigma^0_{k+1}\text{-}\mathsf{IA}$ *proves the totality of* $< \omega^\omega_{k+1}$*-recursive functionals of type* ≤ 2.

Proof. See [14, II.3.18] and [40]. □

7.2. **Application to bar recursion.** Our goal is now to use the equivalences between ordinal induction and Σ^0_k-induction and an ordinal analysis of bar recursion to establish conservation results of bar recursion over induction along ω.

Definition 19 (Howard's bar recursor)**.** Define the bar recursor $B_{\rho,\tau}$ as

$$B_{\rho,\tau}AFGt :=_\tau \begin{cases} Gt, & \text{if } A[t] < \operatorname{lth} t, \\ Ft(\lambda u^\rho . B_{\rho,\tau}AFG(t * u)), & \text{otherwise,} \end{cases}$$

where $[t] := \lambda x.(t)_x$.

Definition 20 (restricted bar recursor)**.**

$$\Phi'_\tau AFGt :=_\tau \begin{cases} Gt, & \text{if } A[t] < \operatorname{lth} t, \\ Ft(\Phi'_\tau AFG(t * 0))(\Phi'_\tau AFG(t * 1)), & \text{otherwise.} \end{cases}$$

The bar recursor Φ'_0 can be used to solve the functional interpretation of WKL, see [18]. (Φ'_τ is the restricted bar recursor schema 1 from there.)

We call a term *semi-closed* if it contains only variables of degree ≤ 1 free. Howard introduced the notion of computational size for semi-closed terms, see [17, 18]. Roughly speaking the computation size of a semi-closed term of type 0 is an upper bound on the number of term reductions on has to apply to obtain a numeral. The computational size of a degree 1 term is the computational size of $t(H_0, \dots, H_n)$, where H_i are fresh variables such that the terms is of type 0.

Theorem 21 ([18, 2.2, 2.3])**.** *Let $\Phi'_0 AFGc$ resp. $B_{0,1}AFGc$ be a semi-closed term and let A, F, G have the computational sizes a, f, g then*

(i) *$\Phi'_0 AFGc$ has computational size $\sigma := (f + g + h)\omega + \omega(h + 1)$, where $h := \omega a + \omega$ and,*
(ii) *$B_{0,1}AFGc$ has computational size $\sigma := \omega^{g+f2h}$, where $h := \omega a + \omega$.*

This equivalence can be proven in Σ^0_1-$\mathsf{LNP}(\sigma)$.

Proof. See the proofs in [18, 2.2, 2.3]. Note that these proofs actually define a counting function for the computation-tree through transfinite recursion. This recursion is essentially a transfinite primitive recursion over σ. Hence this proof may be carried out in Σ^0_1-$\mathsf{LNP}(\sigma)$. □

Remark 22. If we apply the rule of bar recursion to semi-closed, primitive recursive terms (in the sense of Kleene, i.e. terms of computation size ω^n for $n \in \omega$) we obtain a term with computation size $< \omega^{m\omega}$ for an $m \in \omega$ and therefore a term that is provably definable already in $\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction_{\omega^l_2}$ for an $l \in \omega$ or in $\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction + \Sigma^0_2$-$\mathsf{IA}$. We can carried out the proof of the equivalence, theorem 21, in the same system, see theorem 17. Hence in each of these systems we can also proof the equivalence of both terms.

If we apply the rule of restricted bar recursion to primitive recursive terms, which contain only free variable of type 0, we even end up with a primitive recursive term.

8. Term-normalization

Denote by $T_{(k)}[F_0, \dots, F_{n-1}]$ the extension of the system T_k resp. T with the constants $F_0, \dots, F_{n-1}$. Further we treat here R_ρ as an unspecified constant (without R_ρ axioms) in the case of qf-$\mathsf{G}_n\mathsf{A}^\omega$.

In the following we will call the reduction of

$$\mathrm{Cond}_{\rho(\tau)}(x, y, z)u^\tau \quad \text{to} \quad \mathrm{Cond}_\rho(x, yu, zu)$$

a Cond-*reduction*. These Cond-reductions are provably valid in qf-N-$\mathsf{G}_n\mathsf{A}^\omega$.

Theorem 23 (term-normalization for type 2)**.** *Let F_i be constants of type ≤ 2.*

For every term $t^1 \in T_0[(\mathrm{Cond}_\rho)_{\rho\in\mathbf{T}}, F_0, \ldots, F_{n-1}]$ there is provably in $\mathsf{qf\text{-}N\text{-}G_3A^\omega}$ *a term $\tilde{t} \in T_0[\mathrm{Cond}_0, F_0, \ldots, F_{n-1}]$ for which*

$$\forall x\ tx =_0 \tilde{t}x$$

and where every occurrence of an F_i is of the form

$$F_i(\tilde{t}_0[y^0], \ldots, \tilde{t}_{k-1}[y^0]).$$

Here k is the arity of F_i, and $\tilde{t}_j[y^0]$ are fixed terms whose only free variable is y^0.

Proof. Without loss of generality we take the system $T_0[F]$ where F is of type 2. For notational simplification we assume that the recursor R_0 can be obtained from F. This can always be achieved with coding.

Let t^1 be a term in $T_0[F]$. The term tx, where x is a fresh variable, is $=_0$-equal to a term $t'[x]$ where t' results from tx by carrying out all possible Π-, Σ-, and Cond-reductions. The outermost symbol of t' cannot be Π, Σ, or Cond_ρ with $\rho \neq 0$, since otherwise in t' either not all Π-, Σ-reductions had been carried out or t' would not be of type 0.

Hence one of the following holds:
1) $t'[x] = 0^0$
2) $t'[x] = S(t_a^0[x])$
3) $t'[x] = F(t_b^1[x])$
4) $t'[x] = \mathrm{Cond}_0(t_c^0[x], t_d^1[x], t_e^1[x])$

In the first case we are done, $\lambda x.t'[x]$ satisfies the theorem. In the second case we proceed the same way with the term t_a. In the third case we proceed with the term $t_b y^0$ where y^0 is a new variable making t_b to type 0 and in the fourth case we proceed with the terms $t_c, t_d y^0, t_e y^0$. Note that we can code the variables x and y in one type 0 variable. Also note that since we applied all Cond-reductions only Cond_0 occurs.

By the strong normalization theorem this process stops, yielding the desired term, see e.g. [12]. □

Theorem 24 (term-normalization for type 3)**.** *Now let G_i be constants of type ≤ 3. For every term $t^1 \in T_0[(\mathrm{Cond}_\rho)_{\rho\in\mathbf{T}}, G_0, \ldots, G_{n-1}]$ there is provably in* $\mathsf{qf\text{-}N\text{-}G_3A^\omega}$ *a term $\tilde{t} \in T_0[\mathrm{Cond}_0, G_0, \ldots, G_{n-1}]$ for which*

$$\forall x\ tx =_0 \tilde{t}x$$

and where every occurrence of an G_i is of the form

$$G_i(\tilde{t}_0[f^1], \ldots, \tilde{t}_{k-1}[f^1]).$$

Here k is the arity of G_i, and $\tilde{t}_j[f^1]$ are fixed terms whose only free variable is f^1.

Proof. Analogous to proof of theorem 23. See also [29, proof of proposition 4.2]. □

Note that the equality between $t, \tilde{t}$ is only pointwise. Therefore one needs (weak) extensionality to conclude that $s[t] =_0 s[\tilde{t}]$ for an arbitrary term s.

Application to proofs in quantifier-free systems. For a term t call the term where every maximal type 0 subterm (i.e. every subterm of type 0 which is not included in a different subterm of type 0) is replaced by a fresh type 0 variable *skeleton.* Obviously, t can be regained from its skeleton by substitution of type 0 terms.

Lemma 25. *Let $\mathcal{T}$ be* $\mathsf{qf\text{-}N\text{-}G}_n\mathsf{A}^\omega$ *with $n \geq 3$ or* $\mathsf{qf\text{-}}\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction$ *augmented with arbitrary constants $H_0, H_1, \ldots$, let $t_0, t_1 \in T_0[\mathrm{Cond}_0, H_0, H_1, \ldots]$ and in t_0, t_1 all possible Π-, Σ-reductions have been carried out.*

Then the following are equivalent:

(i) *The terms* t_0, t_1 *are provable equal in every term context* ($\mathcal{T} \vdash s[t_0] =_0 s[t_1]$ *for every term* s).
(ii) $\mathcal{T} \vdash P(t_0) =_0 P(t_1)$, *where* P *is a variable of suitable type.*
(iii) *The terms* t_0, t_1 *have the same skeleton (modulo renaming of type 0 variables) and* t_0, t_1 *are obtained from the skeleton by substitution of* $=_0$*-equal terms.*

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (i) follows from the fact that one can replace P by any term in the derivation and so in particular by $\lambda x.s[x]$. By definition of the axioms of a quantifier-free system the axioms of this new derivation are also in $\mathcal{T}$. (iii) $\Rightarrow$ (i) follows from the $=_0$-axioms.

For (ii) $\Rightarrow$ (iii) observe that the only way to prove the equality in (ii) are the SUB rule, the $\mathsf{SUB}_{\mathrm{Cond}}$ rule for Cond_0, or the $=_0$-axioms. The Π-, and Σ-reductions commute with applications of $=_0$-axioms and in t_0, t_1 all possible Π- and Σ-reductions have been carried out we may assume that only the $=_0$-axioms, $\mathsf{SUB}_{\mathrm{Cond}}$-axioms for Cond_0, and the SUB-axioms for R_0 are used. These axioms only change type 0 values and, therefore, the skeletons have to be the same. The lemma follows. $\square$

Proposition 26. *Let* $\mathcal{T}$ *be* qf-N-$\mathsf{G}_n\mathsf{A}^\omega$ *where* $n \geq 3$ *or* qf-$\widehat{\mathsf{N}\text{-}\mathsf{PA}}^\omega\!\upharpoonright$ *augmented by a type* 2 *constant* F. *Further let* A *be a formula containing only type* 0 *variables free and satisfying* $\mathcal{T} \vdash A$.

Then there exists a formula $\tilde{A}$ *such that the weakly extensional intuitionistic system* $\mathcal{T}_{\mathrm{WE}}$ *corresponding to* $\mathcal{T}$ *(i.e.* $\mathsf{G}_n\mathsf{A}^\omega_{\mathsf{i}}$ *or* $\widehat{\mathsf{WE}\text{-}\mathsf{HA}}^\omega\!\upharpoonright$*) proves* $A \leftrightarrow \tilde{A}$ *and that there is a derivation* $\tilde{\mathcal{D}}$ *of* $\mathcal{T} \vdash \tilde{A}$ *where every occurring term is normalized according to theorem 23, i.e. each occurrence of* F *is of the form* $F(t_i[x])$.

Moreover, these applications of F *can be chosen independently from each other in the sense that*

$$\mathcal{T} \nvdash P[F(t')] =_0 P[F(t'')] \qquad \textit{for a fresh variable } P$$

for all type 0 substitution instances t', t'' *of* t_i *resp.* t_j *with* $i \neq j$. *(In other words, the theory* $\mathcal{T}$ *does not see that the* $F(t'), F(t'')$ *are applications of* F *and not just an arbitrary term of suitable type and with the same free variables. Hence they may be replaced independently.)*

Using coding we may also allow finitely many constants F_i *of type* ≤ 2.

Proof. Let $\mathcal{D}$ be a derivation of $\mathcal{T} \vdash A$. By lemma 2 we may assume that only the variables of A and some free type 0 variables occur in $\mathcal{D}$. Hence every term showing up in $\mathcal{D}$ satisfies the premise of theorem 23.

We obtain a new derivation $\tilde{\mathcal{D}}$ by replacing every term in $\mathcal{D}$ with its normal form as defined in the proof of theorem 23 (in particular all possible Π-, and Σ-reductions have been carried out and only Cond_0 occurs in $\tilde{t}$). The derivation $\tilde{\mathcal{D}}$ is still valid because the used logical axioms and rules, SUB-axioms for the recursor and Cond, $=_0$-axioms, and quantifier-free induction rule are translated into other instances of themselves. The used SUB-axioms for Π and Σ become trivial since in all terms all possible Π- and Σ-reductions have been carried out.

Let $\tilde{A}$ be the result of $\tilde{\mathcal{D}}$. Each term occurring in $\tilde{A}$ is just the normal form of the term at the same position in A and therefore weakly extensional equal to it. Hence

$$\mathcal{T}_{\mathrm{WE}} \vdash A \leftrightarrow \tilde{A}.$$

Obviously, the derivation $\tilde{\mathcal{D}}$ contains only finitely many applications t_i of F. Each of the t_i contains only type 0 variables free. However, these applications of F are not independent from each other because there might be equalities between them provable in $\mathcal{T}$.

Passing to the skeletons of t_i we obtain applications of F which are by lemma 25 pairwise independent or literally equal and which still contain only type 0 parameters. $\square$

Remark 27. If in the above theorem one adds a type 3 constant G instead of F to the system and uses theorem 24 instead of 23 one obtains a similar result with the exception that the applications t_i now also depend on function variables f_i. (These variables result from the normalization defined in theorem 24. They can be coded together into one variable f such that the derivation $\tilde{\mathcal{D}}$ may be contains only the variables occurring in $\tilde{A}$ plus some fresh type 0 variables.)

9. Cohesive principle (COH)

Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$. A set G is *cohesive* for $(R_n)_{n\in\mathbb{N}}$ if $\forall n \left(G \subseteq^* R_n \vee G \subseteq^* \overline{R_n}\right)$, i.e.

$$\forall n\, \exists s\, (\forall j \geq s\ (j \in G \to j \in R_n) \vee \forall j \geq s\ (j \in G \to j \notin R_n))\,.$$

A set G is *strongly cohesive* for $(R_n)_{n\in\mathbb{N}}$ if

$$\forall n\, \exists s\, \forall i < n\ (\forall j \geq s\ (j \in G \to j \in R_i) \vee \forall j \geq s\ (j \in G \to j \notin R_i))\,.$$

The *cohesive principle* (COH) is the statement that for every sequence of sets an infinite cohesive set exists. Similarly the *strong cohesive principle* (StCOH) is the statement that for every sequence of sets an infinite strongly cohesive set exists. We denote by $(\mathsf{St})\mathsf{COH}(r, G)$ the statement that G is a set that satisfies the (strong) cohesive principle for the sets given by the characteristic functions $(\lambda x.r(i, x))_i$ where $r\colon \mathbb{N} \times \mathbb{N} \to 2$.

Proposition 28 ([15, 4.4])**.**

(i) $\mathsf{G_3A}^\omega \vdash \mathsf{StCOH} \to \mathsf{COH}$
(ii) $\mathsf{G_3A}^\omega \vdash \mathsf{StCOH} \to \Pi^0_1\text{-}\mathsf{CP}$
(iii) $\mathsf{G_3A}^\omega \vdash \mathsf{StCOH} \leftrightarrow \mathsf{COH} \wedge \Pi^0_1\text{-}\mathsf{CP}$

Proof. The first statement is clear and the third statement is an immediate consequence of the first and second.

For the second we prove the infinite pigeonhole principle $\mathsf{RT}^1_{<\infty}$ from StCOH. The infinite pigeonhole principle is equivalent to Π^0_1-CP, over Σ^0_1-induction. This was shown in [16]. The proof can even be carried out in $\mathsf{G_3A}^\omega$, see [36]:
Let $f\colon \mathbb{N} \to n$ be a coloring. Define $R_i := \{x \mid f(x) = i\}$. Let G be an infinite, strongly cohesive set for R_i. By definition there is an s with

$$\forall i < n\ (\forall j \geq s\ (j \in G \to j \in R_i) \vee \forall j \geq s\ (j \in G \to j \notin R_i))\,.$$

By the totality of f there is exactly one i such that the first disjunction holds, i.e. the color i occurs infinitely often on G and thus on $\mathbb{N}$. $\square$

Lemma 29. $\mathsf{G_3A}^\omega$ *proves that a countable number of instances of* $(\mathsf{St})\mathsf{COH}$ *is uniformly equivalent to a single instance of* $(\mathsf{St})\mathsf{COH}$.

Proof. Let $(R_{j,i})_{j,i\in\mathbb{N}}$ be a sequence of sequences of sets. A set which is (strongly) cohesive for all of these sets is obviously also (strongly) cohesive for the sets $(R_{j,i})_{i\in\mathbb{N}}$ for each j. Hence a single application of $(\mathsf{St})\mathsf{COH}$ is sufficient to solve the sequence of instance of $(\mathsf{St})\mathsf{COH}$ given by $(R_{j,i})_{i\in\mathbb{N}}$ for each j. $\square$

Proposition 30.

$$\mathsf{G_\infty A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL} \vdash \forall r\colon \mathbb{N} \times \mathbb{N} \to 2\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi r) \to \exists G\, \mathsf{StCOH}(r, G)\right),$$

where ξ is a suitable term.

Proof. Define

$$R_n := \lambda x.r(n,x), \qquad\qquad R^x := \bigcap_{i<\mathrm{lth}(x)} \begin{cases} R_i & \text{if } x_i = 0, \\ \overline{R_i} & \text{otherwise.} \end{cases}$$

Here and in the following let x be the code of the sequence $\langle x_0, \dots, x_{\mathrm{lth}(x)-1}\rangle$.

For every n the set (of sets) $\{R^x \mid x \in 2^n\}$ is a partition of $\mathbb{N}$, i.e.

$$\forall n\, \forall z\; \exists! x \in 2^n \;\; z \in R^x. \tag{11}$$

This statement can be proved with an instance of quantifier-free induction (the tuple $\langle x_0, \dots, x_{n-1}\rangle$ is bounded by $\overline{1}n$ and z is a parameter).

We construct an infinite Π^0_2-0/1-tree T deciding at level n whether for the solution set G either $G \subseteq^* \overline{R_n}$ or $G \subseteq^* R_n$ holds: Let

$$T(\langle x_0, \dots, x_n\rangle) \quad \text{iff} \quad R^{\langle x_0,\dots,x_n\rangle} \text{ is infinite.}$$

The statement "R^x is infinite" is Π^0_2. The predicate T clearly defines a tree. The tree is infinite because otherwise

$$\exists n\, \forall x \in 2^n\, \exists y\, \forall z > y \;\; z \notin R^x$$

and this together with an instance of $\mathsf{\Pi}^0_1$-CP yields a contradiction to (11). (x can be bounded by $\overline{1}n$.)

With an application of an instance of Σ^0_1-induction we prove

$$\forall x \left(R^x \text{ infinite} \to \forall n\, \exists \langle l_0, \dots, l_{n-1}\rangle \; (\forall i < n-1 \;\; l_i < l_{i+1} \wedge \forall i < n \;\; l_i \in R^x)\right)$$

and then conclude

$$\begin{aligned} \forall n\, \forall x \big(&\mathrm{lth}(x) = n \\ &\wedge R^x \text{ infinite} \to \exists \langle l_0, \dots, l_{n-1}\rangle\, \forall i < n-1 \;\; l_i < l_{i+1} \wedge \forall i < n \;\; l_i \in R^x\big). \end{aligned} \tag{12}$$

An instance of $\mathsf{\Pi}^0_2$-WKL yields an infinite branch b of T, i.e. $\forall n \left(R^{\bar{b}(n)} \text{ infinite}\right)$. Using (12) we obtain

$$\forall n\, \exists \langle l_0, \dots, l_{n-1}\rangle \left(\forall i < n-1 \;\; l_i < l_{i+1} \wedge \forall i < n \;\; l_i \in R^{\bar{b}n} \subseteq R^{\bar{b}i}\right). \tag{13}$$

An application of QF-AC yields an enumeration $n \mapsto \langle l_0, \dots, l_{n-1}\rangle$ of finite tuples. Searching for the least code of a tuple and the properties of (13) assure that every tuple is extended by the following. Hence we may diagonalize to obtain an the set $G := \{l_0, l_1, \dots\}$. This set is strongly cohesive and solves the proposition.

Note that the instances of $\mathsf{\Sigma}^0_1$-IA, $\mathsf{\Pi}^0_1$-CP and $\mathsf{\Pi}^0_2$-WKL can be reduced to an instance of $\mathsf{\Pi}^0_1$-CA using lemma 10 and remark 9 yielding a suitable term ξ. $\square$

We now strengthen this proposition to

Proposition 31. *For every closed term φ one can construct a closed term ξ such that*

$$\begin{aligned} &\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF}\text{-}\mathsf{AC} \oplus \mathsf{WKL} \vdash \\ &\qquad \forall r\colon \mathbb{N} \times \mathbb{N} \to 2 \left(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\xi r) \to \exists G \left(\mathsf{StCOH}(r, G) \wedge \mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\varphi r G)\right)\right). \end{aligned}$$

Proof. We construct an infinite Π^0_2-0/1-tree, in which we decide at level

- $2n$ — whether $G \subseteq^* \overline{R_n}$ or $G \subseteq^* R_n$ and at level,
- $2n+1$ — the n-th value of the instance of Π^0_1-comprehension, i.e. whether $\forall k\, (\varphi r G)nk = 0$ is true.

We assign to every element of the tree a finite (potential) initial segment L^x of G. At level $2n$ we add — as in the previous proposition — the next element of R^x; at level $2n+1$ we only add the smallest counterexample (extending our old initial segment of G with elements from R^x) to the statement $\forall k\,(\varphi rG)nk = 0$ if it is false and nothing otherwise. Define:

$$
\begin{aligned}
T(\langle x_0,\dots,x_{2n}\rangle) &\quad \text{iff} \quad R^{\langle x_0,x_2,\dots,x_{2n}\rangle} \text{ is infinite},\\
T(\langle x_0,\dots,x_{2n},0\rangle) &\quad \text{iff} \quad \forall l \subseteq^{fin} R^{\langle x_0,x_2,\dots,x_{2n}\rangle}\, \forall k\ \alpha_\varphi(L^{\langle x_0,\dots,x_{2n}\rangle} * l, n, k) \le 1,\\
T(\langle x_0,\dots,x_{2n},1\rangle) &\quad \text{iff} \quad \exists l \subseteq^{fin} R^{\langle x_0,x_2,\dots,x_{2n}\rangle}\, \exists k\ \alpha_\varphi(L^{\langle x_0,\dots,x_{2n}\rangle} * l, n, k) > 1,\\
L^{\langle\rangle} &:= \langle\rangle,\\
L^{\langle x_0,\dots,x_{2n}\rangle} &:= L^{\langle x_0,\dots,x_{2n-1}\rangle} * \left\langle \min\left\{ x \in R^{\langle x_0,x_2,\dots,x_{2n}\rangle} \,\middle|\, x > \max L^{\langle x_0,\dots,x_{2n-1}\rangle} \right\}\right\rangle,\\
L^{\langle x_0,\dots,x_{2n},0\rangle} &:= L^{\langle x_0,\dots,x_{2n}\rangle},\\
L^{\langle x_0,\dots,x_{2n},1\rangle} &:= L^{\langle x_0,\dots,x_{2n}\rangle} * l,\\
k^{\langle x_0,\dots,x_{2n},1\rangle} &:= k,\\
k^x &:= 0 \quad \text{for all } x \text{ not of this form},
\end{aligned}
$$

where $\langle l,k\rangle$ minimal with

$$l \sqsubset R^{\langle x_0,x_2,\dots,x_{2n}\rangle} \wedge \alpha_\varphi(L^{\langle x_0,\dots,x_{2n}\rangle} * l, n, k) > 1.$$

For notational simplification we omitted the requirements to make T closed under prefix, but we can simply add the conditions of the previous levels to the definition of T making it a tree.

L^x and k^x is clearly defined if $T(x)$ is true (use an instance of Σ^0_1-induction to show this — weaken the Π^0_2-statement "R^x is infinite" in the definition of T to $\exists z \in R^x$).

Using the same argument as in the previous proposition we see that the tree is infinite. But we cannot apply Σ^0_1-$\mathsf{WKL}(\xi r)$, because this instance contains L, which is in general not computable in r (in the sense of $\mathsf{G}_\infty\mathsf{A}^\omega$).

The graph of $x \mapsto (L^x, k^x)$ is definable and Δ^0_1. For notational ease we define the graph of its course-of-value function:

$$(\langle x_0,\dots,x_n\rangle, \langle L_0,\dots,L_n\rangle, \langle k_0,\dots,k_n\rangle) \in \mathcal{G}_{\bar{L},\bar{k}} \qquad \text{iff}$$

$n = 0$**:** $L_n = \langle\rangle$, $k = 0$,
n **even:** $L_n = L_{n-1} * \langle y\rangle$, $k_n = 0$
where y minimal with $y \in R^{\langle x_0,\dots,x_{2m-1}\rangle} \wedge y > \max(L_{n-1})$,
n **odd and** $x_n = 0$**:** $L_n = L_{n-1}$, $k_n = 0$,
n **odd and** $x_n = 1$**:** $L_n = L_{n-1} * l$ and $\langle l, k_n\rangle$ minimal with $l \sqsubset R^{\langle x_0,x_2,\dots,x_{2n}\rangle} \wedge \alpha_\varphi(L_n * l, (n-1)/2, k_n) > 2$.

(Note that equations like $L_n = L_{n-1} * l$ we omitted for notational ease the bounded quantifier $\exists l < L_n$ for l.) So we can replace every reference to L^x in the definition of T by

$$\exists k, y\,(x,(y,k)) \in \mathcal{G}_{L,k} \qquad \text{or} \qquad \forall k, y\,(x,(y,k)) \in \mathcal{G}_{L,k}.$$

The resulting tree is still Π^0_2 so we may apply an instance of Π^0_2-WKL and obtain an infinite branch b.

Setting $G := \bigcup_n L^{\bar{b}(n)}$ now enumerates an infinite strongly cohesive set and from b we can decide $\forall k\,(\varphi rG)nk = 0$ for every n. $\square$

Corollary 32 (to the proof)**.** *For every system $\mathcal{T}$ containing $\mathsf{G}_\infty\mathsf{A}^\omega$ and every provably continuous term φ there exists a term ξ, such that*

$$\mathcal{T} + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL} \vdash$$
$$\forall r \colon \mathbb{N}\times\mathbb{N}\to 2 \left(\Pi^0_1\text{-}\mathsf{CA}(\xi r) \to \exists G \left(\mathsf{StCOH}(r,G) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi r G)\right)\right).$$

Corollary 33. (St)COH *is proofwise low in sequence over* $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}\oplus\mathsf{WKL}$.

Proof. Lemma 29 and proposition 31 (with corollary 32). □

Our goal is now to interpret consequences (of the form $\forall x^1\, \exists y^0\, A_{qf}(x,y)$) of a principle $\mathcal{P}$ that is proofwise low in sequence. For this we will strengthen $\mathcal{P}$ to the statement that there exists a uniform solution functional $\mathcal{P}$ for $\mathcal{P}$. The functional $\mathcal{P}$ must be of type ≤ 2, such that after extracting terms using the functional interpretation one can normalizing them with the tools of Section 8. With this we will see that $\mathcal{P}$ is only used finitely many times and can be replaced using the lowness property in favor of an instance of Π^0_1-CA.

The properties of the solution functional $\mathcal{P}$ must be axiomatizable universally, since they will become an implicative assumption. After prenexation they will become purely existential and the functional interpretation will extract terms witnessing them. Existential quantifier in the axiomitation of $\mathcal{P}$ would become universal after prenexation and therefore would need to be presented afterward.

If $\mathcal{P}$ is of the form

$$\forall S\, \exists G\, \underbrace{\forall x\, P_{qf}(S,G,x)}_{\equiv: P(S,G)}, \tag{14}$$

where P_{qf} is quantifier-free. Then one can take for $\mathcal{P}$ the Skolem functional for G, i.e. a functional $\mathcal{P}$ satisfying

$$\forall S\, \forall x\, P_{qf}(S, \mathcal{P}(S), x).$$

With the help of the following lemma we can obtain a functional for $\mathcal{P}$ where P is a Π^0_3 formula. This is sufficient for StCOH.

Lemma 34. *Let $\mathcal{P}$ be a principle proofwise low in sequence over* $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}\oplus\mathsf{WKL}$, *that has the form*

$$(\mathcal{P})\colon \forall S\, \exists G\, \underbrace{\forall x\, \exists y\, \forall z\, P_{qf}(S,G,x,y,z)}_{\equiv: P(S,G)}, \tag{15}$$

where P_{qf} is quantifier-free.

Then the principle

$$(\mathcal{P}')\colon \forall S\, \exists G, Y\, \forall Z^1\, \forall x\;\; P_{qf}(S,G,x,Y(x,Z),Z(Y(x,Z))) \tag{16}$$

is proofwise low in sequence, in the sense that for every closed term φ a closed term ξ exists, such that $\Pi^0_1\text{-}\mathsf{CA}(\xi(S_i)_i(Z_i)_i)$ proves

$$\exists (G_i)_i, (Y_i)_i \left(\forall i, Z', x\;\; P_{qf}(S_i, G_i, x, (Y)_i(x,Z'), Z'(Y_i(x,Z')))\; \wedge \right.$$
$$\left. \Pi^0_1\text{-}\mathsf{CA}(\varphi(S_i)(Z_i)(G_i)(\lambda x.Y_i(x,Z_i)_i))\right).$$

Proof. The lowness of $\mathcal{P}$ provides that for every term φ' an instance of Π^0_1-comprehension $\Pi^0_1\text{-}\mathsf{CA}(\xi S Z)$ proves

$$\exists G\, \left(\forall x^0\, \exists y^0\, \forall z^0\, P_{qf}(S,G,x,y,z) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi' S Z G)\right).$$

Hence it also proves

$$\exists G\, \left(\forall x, Z\, \exists y\, P_{qf}(S,G,x,y,Z(y)) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi' S Z G)\right).$$

By searching for the least y we may assume that there exists a unique y for each x, Z. Let $Y(x,Z)$ be the choice function for y obtained using $\mathsf{QF\text{-}AC}$. To show that $\mathcal{P}'$ is proofwise low it suffices to show for every closed φ that there is a closed φ' (and thus a closed ξ) such that $\Pi^0_1\text{-}\mathsf{CA}(\varphi SZG(\lambda x.Y(x,Z)))$ is provable from $\Pi^0_1\text{-}\mathsf{CA}(\varphi SZ)$.

Since Y is computable in S, G a suitable φ can easily be constructed with the same generic construction used in the proof of lemma 11.

One also easily verifies that the whole argumentation is stable under sequences and hence that $\mathcal{P}'$ is proofwise low in sequence. $\square$

It is easy to see that $\mathcal{P}'$ is equivalent to $\mathcal{P}$ over $\mathsf{QF\text{-}AC}^{0,0}$. For such principle we could then use a solution functional $\mathcal{P} = (\mathcal{P}_G, \mathcal{P}_Y)$ that codes together the Skolem functions for G, Y in (16), i.e.

$$\forall S\, \forall Z\, \forall x\ \underbrace{P_{qf}(S, \mathcal{P}_G(S), x, \mathcal{P}_Y(G,x,Z), Z(\mathcal{P}_G(G,x,Z))}_{\equiv: P_S(\mathcal{P},(Z,x))}. \tag{17}$$

Proposition 35. *Let $A_{qf} \in \mathcal{L}(\mathsf{G}_\infty\mathsf{A}^\omega)$ be a quantifier-free formula that contains only the shown variables free and let $\mathcal{P}$ be a principle proofwise low in sequence over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$ of the form (15). If*

$$\widehat{\mathsf{E\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \Pi^0_1\text{-}\mathsf{CP} + \mathcal{P} + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y),$$

then one can find a term ξ such that

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \vdash \forall x^1\, \left(\Pi^0_1\text{-}\mathsf{CA}(\xi x) \to \exists y^0\, A_{qf}(x,y)\right).$$

Proof. We first prove the proposition without $\Pi^0_1\text{-}\mathsf{CP}$.

Note that due to

- the deduction theorem (which holds for $\widehat{\mathsf{E\text{-}PA}}^\omega{\restriction}$),
- the elimination of extensionality (proposition 3),
- the strengthening of WKL to UWKL and
- the strengthening of $\mathcal{P}$ to the Skolem normal-form of $\mathcal{P}'$, i.e. the statement there exists an $\mathcal{P}$ satisfying (17)

we obtain

$$\mathsf{N\text{-}G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \vdash (\exists \mathcal{P}\, \forall u^1\, P_S(\mathcal{P}, u)) \wedge (R_0) \wedge (\mathcal{B}) \to \forall x^1\, \exists y^0\, A_{qf}(x,y),$$

where u codes the pair (Z,x) from (17) and (R_0) are the defining axioms for the recursor R_0. Note that also the formulas (R_0), $(\mathcal{B})$ can be written in the form $\exists R_0\, \forall u^1\, (\mathsf{R}_0)_{qf}(R_0, u)$ resp. $\exists \mathcal{B}\, \forall u^1\, (\mathcal{B})_{qf}(\mathcal{B}, u)$ for quantifier free $(\mathsf{R}_0)_{qf}$, $(\mathcal{B})_{qf}$.

Applying the functional interpretation to this yields terms $t_y, t_P, t_{R_0}, t_{\mathcal{B}} \in \mathsf{G}_\infty\mathsf{R}^\omega$ such that

$$\text{qf-}\mathsf{N\text{-}G}_\infty\mathsf{A}^\omega \vdash \big(P_S(\mathcal{P}, t_P(x,\mathcal{P},R_0,\mathcal{B})) \wedge \tag{18}$$
$$(R_0)_{qf}(R_0, t_{R_0}(x,\mathcal{P},R_0,\mathcal{B})) \wedge (\mathcal{B})_{qf}(\mathcal{B}, t_{\mathcal{B}}(x,\mathcal{P},R_0,\mathcal{B})) \to A_{qf}(x, t_y(x,\mathcal{P},R_0,\mathcal{B}))\big),$$

see [51, 26].

The terms $t_P(x,\mathcal{P},R_0,\mathcal{B})$, $t_{R_0}(x,\mathcal{P},R_0,\mathcal{B})$, $t_{\mathcal{B}}(x,\mathcal{P},R_0,\mathcal{B})$, $t_y(x,\mathcal{P},R_0,\mathcal{B})$ have type ≤ 1. By proposition 26 we obtain a new derivation in qf-$\mathsf{N\text{-}G}_\infty\mathsf{A}^\omega$ of a sentence which is equivalent to (18) over qf-$\mathsf{G}_\infty\mathsf{A}^\omega$ and where each application of $\mathcal{P}$ is of the form $\mathcal{P}(t_i[z^0])$ or a substitution instance of $\mathcal{P}(t_i[z^0])$ and $\mathcal{P}(t_i[z^0])$ and $\mathcal{P}(t_j[z^0])$ are independent (in the sense of proposition 26). Same for $R_0, \mathcal{B}$.

Our goal is now to replace these occurrences of $\mathcal{P}$, R_0, and $\mathcal{B}$ in the normalized derivation of (18) by a low solution to those principles, such that the premise of (18) becomes provable.

We proceed by inductively over the nesting-depth of $\mathcal{P}$, R_0, and $\mathcal{B}$ replacing the applications (and their substitution instances) with low solutions retaining the instance of comprehension. This operation leaves the derivation valid since the different applications are independent. Concretely we replace $\mathcal{P}, R_0, \mathcal{B}$ by the following:

- $R_0(t_i[z^0])$ simply defines a primitive recursive function, which is provably total using an instance of Σ^0_1-induction. This instance can be obtained from $\mathsf{QF\text{-}IA}$ and an instance of Π^0_1-comprehension. Then lemma 11 yields a new instance of comprehension (which allows $R_0(t_i[z^0])$ as parameter).
- $\mathcal{P}(t_i[z^0])$ can be handled using the assumption that $\mathcal{P}$ is proofwise low in sequence (lemma 34)
- $\mathcal{B}(t_i[z^0])$ can trivially be handled because it is present in the verifying system.

For the construction of these replacements we work in the system $\mathsf{G}_\infty\mathsf{A}^\omega$, i.e. with weak extensionality and quantifiers. After this the premise of (18) becomes provable. Quantifying over all x and coding x, z together into a new variable x, yields the proposition without Π^0_1-CP.

To prove the full proposition note that we can add StCOH to the system since it is proofwise low in sequence, see corollary 33, and that StCOH implies Π^0_1-CP, see proposition 28. This completes the proof. $\square$

Theorem 36 (Conservation for proofwise low in sequence)**.** *Let $\mathcal{P}$ be a principle of the form* (15) *that is proofwise low in sequence over* $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$. *In particular, this includes all principles of this form proofwise low in sequence over* WKL_0^*. *If*

$$\widehat{\mathsf{E\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \Pi^0_1\text{-}\mathsf{CP} + \mathcal{P} + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y)$$

then one can extract a primitive recursive term t such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega{\restriction} \vdash \forall x^1\, A_{qf}(x, tx).$$

In particular, if $A_{qf} \in \mathcal{L}(\mathsf{PRA})$ and x is of type 0 we have $\mathsf{PRA} \vdash \forall x\, A_{qf}(x, tx)$.

Proof. We may assume that $A_{qf} \in \mathcal{L}(\mathsf{G}_\infty\mathsf{A}^\omega)$. Otherwise it would contain R_0. If this is the case we normalize every term occurring in A_{qf} and replace every occurrence of R_0uvw by a fresh variable that will be $\exists$-quantified. There are no other occurrence of R_0 in A_{qf} since it contains (beside Π, Σ) no constant of type > 2. These fresh variables will hold the value of R_0uvw. This values exists provably with Σ^0_1-IA and can be expressed in a quantifier-free way.

Apply now elimination of Skolem function for monotone formulas (corollary 13) to the result of proposition 35. $\square$

Corollary 37. *Especially from a proof of*

$$\widehat{\mathsf{E\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH} + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y)$$

one can extract a primitive recursive term t such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega{\restriction} \vdash \forall x^1\, A_{qf}(x, tx).$$

Proof. Theorem 36 and corollary 33. $\square$

Corollary 38. *The system* $\mathsf{WKL}_0^\omega + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH}$ *is Π^0_2-conservative over* PRA. *Additionally, for every Π^0_2-sentence one can extract uniformly a primitive recursive (provably) realizing term.*

Further $\mathsf{WKL}_0^\omega + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH}$ *is conservative over* RCA_0^ω *for sentences of the form $\forall x^1\, \exists y^0\, \forall z^0\, A_{qf}(x,y,z)$.*

As consequence we also obtain that $\mathsf{WKL}_0 + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH}$ *is conservative over* RCA_0 *for sentences of the form $\forall X\, \exists y\, \forall z\, A(X,y,z)$, where A is Δ^0_0, and thus in particular is Π^0_3-conservative.*

Proof. The first statement is clear from the preceding corollary and the definition of WKL_0. The second statement follows also from this corollary by noting that over $\mathsf{QF\text{-}AC}^{0,0}$ every formula of the given form $\forall x^1\, \exists y^0\, \forall z^0\, A_{qf}(x,y,z)$ is equivalent to $\forall x^1, Z^1\, \exists y^0\, A_{qf}(x,y,Zy)$.

The last claim follows from the former since RCA_0^ω is conservative over the second-order fragment, which can be simulated in RCA_0, see [31]. □

This in some sense is the best possible result since $\mathsf{RCA}_0 + \mathsf{\Pi}^0_1\text{-}\mathsf{CP}$ is not Σ^0_3-conservative over a theory containing only Σ^0_1-induction, see [1].

Remark 39. Recall that BW is the statement that every bounded sequence $(y_i)_{i\in\mathbb{N}}$ of real numbers contains a subsequence $(y_{f(i)})_{i\in\mathbb{N}}$ converging with the rate 2^{-n}, i.e. $\forall n\, \forall i,j \geq n\ |y_{f(i)} - y_{f(j)}| < 2^{-n}$, see [45]. It turns out that StCOH is equivalent to a natural variant of this principle, namely the statement that each bounded sequence $(y_i)_{i\in\mathbb{N}}$ of reals contains a Cauchy subsequence $(y_{f(i)})_{i\in\mathbb{N}}$. This means a sequence which convergences but possibly without a computable rate of convergence, i.e. $\forall n\, \exists k\, \forall i,j \geq k\ |y_{f(i)} - y_{f(j)}| < 2^{-n}$, see [37]. Hence the term extraction results we obtain below for StCOH also apply to this variant of the Bolzano-Weierstraß principle.

10. Ramsey's theorem for pairs

10.1. **Stable Ramsey's theorem for pairs (SRT^2_2).** An n-coloring $c\colon [\mathbb{N}]^2 \to n$ is called *stable* if

$$\forall x\, \exists k\, \forall y > k\, c(x,k) = c(x,y).$$

The point k is called a *stability point for* x.

We call an n-coloring *strongly stable* if

$$\forall x\, \exists k\, \forall y > k\, \forall x' \leq x\, c(x',k) = c(x',y).$$

Over $\mathsf{\Pi}^0_1\text{-}\mathsf{CP}$ strongly stable and stable coincide. Even an instance of the collection principle of the form $\mathsf{\Pi}^0_1\text{-}\mathsf{CP}(\xi c)$ where ξ is a suitable term and c the coloring suffices to prove this equivalence.

Let SRT^2_n be the statement expressing that every *stable* n-coloring of pairs has an infinite homogeneous set and let $\mathsf{SRT}^2_{<\infty} :\equiv \forall n\, \mathsf{SRT}^2_n$. For a stable n-coloring c the statement $\mathsf{SRT}^2_n(c,H)$ denotes that H is a homogeneous set for c.

The principle SRT^2_2 is over Σ^0_1-induction equivalent to the statement that for every Δ^0_2-set X there exists an infinite set Y with $Y \subseteq X$ or $Y \subseteq \overline{X}$, see [7, 8].

Before we go on with the main result we need some auxiliary lemmata:

Lemma 40 ([7, lemma 4.2]). *For every fixed n, let $(\xi_{k,i})_{k<n,i\in\mathbb{N}}$ be a sequence of Π^0_1-sentences of the form $\xi_{k,i} \equiv \forall x\, A(k,i,x)$ for a quantifier-free A such that $\forall i\, \exists k < n\ \xi_{k,i}$. Then* WKL *proves that there exists a choice function $g\colon \mathbb{N} \to n$ satisfying $\forall i\, \xi_{g(i),i}$.*

If WKL *is replaced by* $\mathsf{\Sigma}^0_1\text{-}\mathsf{WKL}$ *the same holds for Π^0_2-sentences.*

Proof. Define

$$f(\langle x_0,\ldots,x_n\rangle) = 0 \quad \text{iff} \quad \bigwedge_{i\leq n} \xi_{x_i,i}.$$

The function f clearly defines a Π^0_1-0–n-tree and is by assumption infinite.

Via the equivalence of 0–n-trees and 0/1-trees and of $\mathsf{\Pi}^0_1\text{-}\mathsf{WKL}$ and WKL (see [45]), weak König's lemma yields a infinite branch g solving the lemma. □

Lemma 41 (and definition, Π^0_1-class, [22]). *A Π^0_1-class $\mathcal{A}$ of 2^ω is a set of functions of the form*

$$\mathcal{A} = \{f \in 2^\omega \mid \forall n\, A(\bar{f}n)\},$$

where A is a quantifier-free formula.

WKL *proves that a Π^0_1-class $\mathcal{A}$ is not empty if*

$$\forall n^0\, \exists s \in 2^n\, \forall s' \sqsubseteq s\ A(s'). \tag{19}$$

(The definition of Π^0_1-class induces an infinite tree in which every $f \in \mathcal{A}$ codes an infinite path through it.) The statement (19) *is equivalent to a Π^0_1-statement.*

Note that one may also allow A to be a Π^0_1-formula as the $\forall$-quantifier can be coded into the quantification over n (see for instance [45]*).*

Remark 42 (Treatment of Π^0_1-0/1-trees). Let $T(w) :\equiv (\forall k\, T_{qf}(w,k) = 0)$ be a Π^0_1-predicate. Using the UWKL functional $\mathcal{B}$ one can define the functional

$$\mathcal{B}_{\Pi^0_1}(T_{qf}) := \mathcal{B}\left(\min_{w' \sqsubseteq w, k \leq \mathrm{lth}\, w} T_{qf}(w',k)\right)$$

that yields an infinite branch of T, if T defines an infinite 0/1-tree.

Furthermore, an instance of Π^0_1-CA decides whether the tree T is infinite, since

$$\forall n\, \exists w \in 2^n\, \forall k\, T_{qf}(w,k)$$

is equivalent a Π^0_1-statement (over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}$).

Hence one can treat Π^0_1-0/1-trees mostly like quantifier free trees.

Proposition 43.

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \vdash \forall c \colon \mathbb{N}\times\mathbb{N} \to 2\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi c) \to \exists H\, \mathsf{SRT}^2_2(c,H)\right),$$

where ξ is a suitable term.

Proof. Assume that the coloring c is stable. Define for $i < 2$

$$A_i := \{\, x \mid \forall k\, \exists y \geq k\ \ c(x,y) = i \,\}.$$

By stability $A_i = \{\, x \mid \exists k\, \forall y \geq k\ \ c(x,y) = i \,\}$. Hence each A_i is a Δ^0_2-set.

At least for one i the set A_i is infinite (by RT^1_2). Fix such an i. With an instance of Π^0_1-CP we obtain strong stability, i.e.

$$\forall x\, \exists k\, \forall y > k\, \forall x' \leq x\ \ c(x',k) = c(x',y).$$

This instance of Π^0_1-CP follows from a suitable instance of Π^0_1-CA, see lemma 10.(iv).

Together with the infinity of A_i we get

$$\forall x\, \exists k \in A_i\, \forall x' \leq x\ (x' \in A_i \to c(x',k) = i)\,.$$

Define the set H inductively by

$$x \in H \quad \text{iff} \quad x \in A_i \text{ and } c(x',x) = i \text{ for all } x' < x \text{ with } x' \in H.$$

This definition only uses bounded course-of-value recursion in the characteristic function of A_i which can be obtained from a suitable instance of Π^0_1-CA, see lemma 10.(ii). (The characteristic function χ_H of H is clearly bounded and hence also its course-of-value function $\overline{\chi_H}$, which is actually defined in the recursion.)

The set H is clearly infinite and homogeneous. (The two instances of Π^0_1-CA can be coded into one term ξ, see remark 9.) □

Proposition 44. *Let φcH be a term that is provably continuous in H, where $\alpha_{\varphi c}(\cdot, n, k)$ is an associate for $\lambda H.\varphi(c,H,n,k)$. Then there exists a term ξ, such that*

$$\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash$$
$$\forall c \colon \mathbb{N}\times\mathbb{N}\to 2\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi c) \to \exists H\, \mathsf{SRT}^2_2(c,H) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi c H)\right).$$

If φcH is moreover provably continuous in c the term ξ can be chosen such that it is provably continuous.

Sketch of proof. We assume that each A_i is unbounded, otherwise we are done.

We will build a set G such that $G \cap A_0$ and $G \cap A_1$ are infinite, homogeneous and at least for one $i < 2$ the comprehension $\Pi^0_1\text{-CA}(\varphi c(G \cap A_i))$ is decided. The set $H := G \cap A_i$ then solves this proposition.

We will construct the set G in steps such that at each step n we will assure that

$$|G \cap A_i| \geq n \quad \text{for every } i < 2$$

and for some $i < 2$ the comprehension for $G \cap A_i$ at the position $(n)_i$ will be decided, i.e. whether the statement

$$\forall k\, (\varphi c(G \cap A_i)(n)_i)k = 0 \tag{20}$$

holds. More precisely, we will construct functions $I, J \colon \mathbb{N} \to 2$, such that

$$\exists I, J\, \forall n\, \big(\forall k\, (\varphi c(G \cap A_{I(n)})(n)_{I(n)})k = 0 \leftrightarrow J(n) = 0\big)\,.$$

With these functions we can then obtain a comprehension function for one of the sets $G \cap A_i$, because either

$$\forall m\, \exists n\, \big(m = (n)_{I(n)} \wedge I(n) = 0\big) \tag{21}$$

and then $J(N(m))$, where $N(m)$ is some choice function for n obtained by QF-AC, decides the comprehension for $G \cap A_0$ or

$$\exists m\, \forall n\, \big(m \neq (n)_{I(n)} \vee I(n) = 1\big)\,. \tag{22}$$

By choosing $n = \langle m, m' \rangle$ we obtain $\forall m'\, I(\langle m, m' \rangle) = 1$ and therefore the function $\lambda m'.J(\langle m, m' \rangle)$ decides the comprehension for $G \cap A_1$.

The set G and the functions I, J will be constructed by recursion. We will first give a sketch of the argument and later show that R_0 and the imposed comprehension suffice for the construction.

By induction we construct (d_n, L_n), such that the sequence (d_n) is an ascending sequence of finite sets and (L_n) is a descending sequence of infinite sets of possible candidates to extend d_n (i.e. $d_{n+1} \setminus d_n \subset L_n$ and $\min(L_n)$ is greater than the stability point of d_n). Each set L_n is *low*, in the sense that it can be described by a term containing $\mathcal{B}$ and $\tilde{R}_1$. The set G will be given by $\bigcup_n d_n$.

We start with $(\emptyset, \mathbb{N})$. Assume (d_n, L_n) is already defined. We distinguish two cases:

Case i) A partition Z_0 and Z_1 of L_n exists such that

$$\forall z \subseteq^{fin} Z_i\, \big(z \text{ is } i\text{-homogeneous} \to \forall k\ \alpha_{\varphi c}(d^i_n \cup z)(n)_i k \leq 1\big)\,, \tag{23}$$

where $d^i_n = d_n \cap A_i$, holds for all $i < 2$. (If we extend the initial segment d_n with elements from Z_i the comprehension remains true.)

At least one of Z_0 and Z_1 is infinite because L_n is infinite. We take this set as L_{n+1}, forcing (20) to be true for this i on all further extensions and let $d_{n+1} := d_n$.

Case ii) No partition satisfying (23) exists.

We know then that especially $L_n \cap A_0$ and $L_n \cap A_1$ is no such partition. So we can find for one i a finite i-homogeneous set $d' \subseteq^{fin} A_i$ such that

$$\exists k\ \alpha_{\varphi c}(d^i_n \cup d')(n)_i k > 1.$$

Setting $d_{n+1} := d_n \cup d'$ and $L_{n+1} := \{x \in L_n \mid x > \max d'\}$ forces the comprehension function to be $\neq 0$ at $(n)_i$.

Note that (23) defines a Π^0_1-class of 2^ω. (We view here a partition of $\mathbb{N}$ into two sets Z_0, Z_1 as a function $f \in 2^\omega$ with $f(n) = i$ iff $n \in Z_i$.) Thus we may assume that the Z_i are low and we can decide which case holds by asking if a certain $0/1$-tree is infinite (this is a Π^0_1-statement).

The size requirements are met by extending d_{n+1} with suitable elements of L_n.

The set $G := \bigcup_n d_n$ then satisfies the proposition. □

Proof. Define

$$L^{\langle\rangle}(w) := 0, \tag{24}$$

$$L^{\langle x_0,\dots,x_{n-1},(d,k,y)\rangle}(w) := \begin{cases} 1 & \text{if } w \le y, \\ sg\left|\mathcal{B}_{\Pi^0_1}(\theta(L^{\langle x_0,\dots,x_{n-1}\rangle}, d)) - (k-1)\right| & \text{if } k \ge 1 \wedge w > y, \\ L^{\langle x_0,\dots,x_{n-1}\rangle} w & \text{if } k = 0 \wedge w > y. \end{cases}$$

(d is just an auxiliary parameter used to build the tree, it will be set to d_{n-1} defined below; k denotes the case, $k = 0$ for case ii), $k \ge 1$ for case i) and Z_{k-1} infinite in the sketch; y is a lower bound for L.)

Here $\theta(B, d^0, d^1)wk$ will be the characteristic function of the predicate

$$\forall i < 2 \ \forall y \subseteq^{fin} B \cap \{x < \mathrm{lth}(w) \mid (w)_x = i\} \qquad \big(y \text{ is } i\text{-homogeneous} \rightarrow \alpha_{\varphi c}(d^i \cup y)(n)_i k \le 1\big), \tag{25}$$

where the variables w, y are numerals coding finite sets. The statement

$$T_{B,d^0,d^1}(w) :\equiv \forall k\, \theta(B, d^0, d^1)wk = 0$$

defines the Π^0_1-0/1-tree build in (23) in the sketch.

We will write $T_{B,d}$ and $\theta(B,d)wk$ for T_{B,d^0,d^1} resp. $\theta(B, d\cap A_0, d\cap A_1)wk$. This will not lead to problems because $d\cap A_i$ is just a number computable from d relative to the imposed instance of comprehension. Note that L^x can be defined in $\mathcal{B}$ and θ using the bounded iterator $\tilde{R}_1$. Thus the function L^x can be described by a term in this system.

We assume that for all x and i the set $L^x \cap A_i$ is infinite if L^x is infinite. Otherwise the set $L^x \cap [k, \infty]$ for a suitable k would be an infinite subset of A_{1-i} and therefore solve the proposition.

Using this and an instance of Δ^0_2-comprehension (over L) we generate functions g_i such that

$$g_i(x) := \min(L^x \cap A_i). \tag{26}$$

With an application of Π^0_1-AC and taking a maximum we obtain a function $h(\langle x_1, \dots, x_n\rangle)$ giving a common stability point of $x_1, \dots, x_n$.

We now define (d_n, l_n) by recursion. (L^{l_n} should match L_n from sketch above.) We use primitive recursion in the sense of Kleene, i.e. the recursion can be defined with the recursor R_0.

Let $d_0 := \langle\rangle$ and $l_0 := \langle\rangle$. For the recursion step we distinguish the cases:

Case i) The tree $T_{L^{l_n},d_n}(w)$ is infinite, i.e.

$$\forall m\, \exists w \in 2^m\, \forall k\ \theta(L^{l_n}, d_n)wk = 0.$$

By RT^1_2 there is at least one $j < 2$ such that $\{x \in \mathbb{N} \mid \mathcal{B}_{\Pi^0_1}(\theta(L^{l_n}, d_n))x = j\}$ is infinite. An index j can be chosen constructively relative to Σ^0_1-WKL, see lemma 40. Set

$$d'_{n+1} := d_n \quad \text{and} \quad k'_{n+1} := j + 1.$$

Case ii) The tree $T_{L^{l_n},d_n}(w)$ is finite, i.e.

$$\exists m\, \forall w \in 2^m\, \exists k\ \theta(L^{l_n}, d_n)wk \ne 0.$$

Then especially the set A_0 does not code a path through the tree, i.e. for this m

$$\exists k\, \theta(L^{l_n}, d_n)(\overline{\chi_{A_0}} m)k \ne 0,$$

where χ_{A_0} is the characteristic function of A_0. So there is an i and a finite i-homogeneous set $y \subseteq^{fin} A_i \cap \{0, \dots, m-1\} \cap L^{l_n}$ such that

$$\exists k\, \alpha_{\varphi c}(d^i \cup y)(n)_i k > 1.$$

Set

$$d'_{n+1} := d \cup y \quad \text{and} \quad k'_{n+1} := 0.$$

Note that this case distinction is constructive relative to the given instance of comprehension (the second quantifier of the formula is bounded).

Now we extend d'_{n+1} with suitable elements, such that the size requirements are met:

$$\begin{aligned} d_{n+1} &:= d_n \cup \bigcup_{i<2} \{ g_i(l_n * \langle d_n, l', h(d'_{n+1}) + 1 \rangle) \} \\ l_{n+1} &:= l_n * \langle d_n, k'_{n+1}, h(d_{n+1}) + 1 \rangle \end{aligned}$$

Applying RT^1_2 yields an i such that all comprehension instances are decided. From the d_n and the given comprehension one can easily obtain a enumeration of the set $G \cap A_i =: H$.

This solves the proposition. The term ξc is continuous in c because the only discontinuous functional in this system is $\mathcal{B}$ but it is only used to define L^x and to prove WKL. Hence ξ can be chosen such that c does not occur as a parameter to $\mathcal{B}$. More precisely ξc is of the form $\xi'[t_1 c, \lambda x. L^x]$ with $\xi', t \in T_0$ and therefore continuous. $\square$

Proposition 45. *Let $\varphi c H$ be a term that is provably continuous in H and let α_{φ_c} be as in proposition 44. Then there exists a term ξ such that*

$$\begin{aligned} &\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \Sigma^0_2\text{-}\mathsf{IA} + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \\ &\qquad \forall c \colon \mathbb{N} \times \mathbb{N} \to n \left(\Pi^0_1\text{-}\mathsf{CA}(\xi c) \to \exists H\, \mathsf{SRT}^2_{<\infty}(c, H) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi c H) \right). \end{aligned}$$

If φ is moreover provably continuous in c the term ξ can be chosen such that it is provably continuous in c.

Proof. Analogous to Proposition 44.

The applications of RT^1_2 become applications of $\mathsf{RT}^1_{<\infty}$, which is equivalent to Π^0_1-CP and thus provable using Σ^0_2-IA. The 0/1-trees will become 0–n-trees; but these trees can be constructively transformed into 0/1-trees, see [45].

The only difficult part is adopt the assumption that

$$\forall x\, \forall i < n\ (L^x \text{ infinite} \to L^x \cap A_i \text{ infinite})\,, \tag{27}$$

which leads to the definition of g_i in (26) because we cannot simply deduce the existence of a solution from the failure of (27).

First note that (27) due to the minimal element parameter (y in (24)) is equivalent to

$$\forall x\, \forall i < n\ (L^x \text{ infinite} \to L^x \cap A_i \text{ not empty})\,. \tag{28}$$

If (27) resp. (28) does not hold, our goal is to find a set L^x on which — provided we neglect colors that do not occur — the assumption holds. This can be done by finding a maximal set $K \subseteq n$, such that there is an x with $L^x \cap \bigcup_{k \in K} A_k$ is empty. Then for all $x' \sqsupseteq x$ and $i \notin K$ the sets $A_i \cup L^x$ are not empty. Thus if we relativize our argumentation to L^x and the colors $n \setminus K$ the condition (27) holds.

To find such a K and x define

$$\eta(\langle s_0, \dots, s_{n-1} \rangle) :\equiv \exists x \left(L^x \text{ infinite} \wedge \bigwedge_i (s_i = 0 \to L^x \cap A_i = \emptyset) \right).$$

η is clearly Σ^0_3. Finding a minimal tuple $\langle s_0, \dots, s_{n-1}\rangle$ satisfying η yields a suitable solution. A minimal tuple can be obtained using an instance of Σ^0_3-induction, which is provable from Σ^0_2-IA and an instance of Π^0_1-comprehension. □

Corollary 46. *Let $\varphi c H$ be a term that is provably continuous in H. Then there exists a term ξ such that*

$$(29)\quad \widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c\colon \mathbb{N}\times\mathbb{N}\to n\ \big(\mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\xi c) \to \exists H\, \mathsf{RT}^2_2(c,H) \wedge \mathsf{\Pi}^0_1\text{-}\mathsf{CA}(\varphi c H)\big).$$

The term ξ can be chosen such that c does not occur as a subterm of a parameter of $\mathcal{B}$.

If Σ^0_2-IA is added to the system, RT^2_2 may be replaced by $\mathsf{RT}^2_{<\infty}$.

Hence RT^2_2 is proofwise low over $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1)$ and $\mathsf{RT}^2_{<\infty}$ is proofwise low over $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) + \Sigma^0_2$-IA.

Proof. Let $R_i = \{x \in \mathbb{N} \mid c(i,x) = 0\}$ and let g be a strictly increasing enumeration of a cohesive set for R_i. The coloring $c'(x,y) := c(gx, gy)$ is stable and for each homogeneous set H' of c' the set gH' is homogeneous for c. See [7].

Hence the corollary follows from corollary 32 and proposition 44 resp. proposition 45. □

By the proposition below RT^2_2 implies $\mathsf{\Pi}^0_2$-LEM. Therefore, sequences of instances of RT^2_2 imply $\mathsf{\Pi}^0_2$-CA. Hence it is not possible to show that RT^2_2 is proofwise low in sequence.

Proposition 47. $\mathsf{iRCA}^*_0 \vdash \mathsf{RT}^2_2 \to \mathsf{\Pi}^0_2\text{-}\mathsf{LEM}$, *where* iRCA^*_0 *is the intuitionistic system corresponding to* RCA^*_0 *and* $\mathsf{\Pi}^0_2$-LEM *is the Π^0_2-law of excluded middle.*

More precisely, for every Π^0_2-statement $\forall x\, \exists y\, A_{qf}(x,y)$ there is a coloring such that one can decide constructively from a homogeneous set whether the Π^0_2-statement is true or not.

Proof. We show for an arbitrary quantifier-free formula A_{qf} that $\forall x\, \exists y\, A_{qf}(x,y) \vee \exists x\, \forall y\, \neg A_{qf}(x,y)$. First note that iRCA^*_0 proves that $\forall x\, \exists y\, A_{qf}(x,y) \leftrightarrow \forall x\, \exists y\, \forall x' \le x\, \exists y' \le y\, A_{qf}(x',y')$. Hence we may assume that A_{qf} is monotone in the sense that $A_{qf}(x,y) \to \forall u \le x\, \forall v \ge y\, A_{qf}(u,v)$. Now color each pair $\{x,y\}$ with $x < y$ red if $A_{qf}(x,y)$ holds and blue otherwise. It is easy to see that there exists an infinite red homogeneous set iff $\forall x\, \exists y\, A_{qf}(x,y)$ is true. □

To overcome this problem we switch to the functional interpretation (i.e. ND-interpretation) where the need for $\mathsf{\Pi}^0_2$-LEM vanishes.

10.2. **ND-Interpretation of RT^2_2.** We now formulate the ND-interpretation of RT^2_2 and of corollary 46. For notational simplification we sometimes will not apply the last application of QF-AC to the ND-interpretation. This corresponds to the so-called Shoenfield translation, see [49]. For RT^2_2 we use the formalization

$$\mathsf{RT}^2_2 :\equiv \forall c\colon [\mathbb{N}]^2 \to 2\, \exists H\, \forall u < v\ c(Hu, Hv) = c(H0, H1).$$

The ND-interpretation then yields

$$(30)\qquad \mathsf{RT}^{2\,ND}_2 :\equiv \forall c\colon [\mathbb{N}]^2 \to 2\, \forall U < V\, \exists H\ \underbrace{c(H(UH), H(VH)) = c(H0, H1)}_{\equiv:\mathsf{RT}^2_{2ND}(H;c,U,V)}.$$

Here the set H is given as an enumeration, i.e. H is strictly monotone and Hn is the n-th element of H, and $U < V$ is defined pointwise.[6] Sometimes the parameters c, U, V in $\mathsf{RT}^2_{2ND}(H; c, U, V)$ will be coded into a single parameter.

For the ND-interpretation of Π^0_1-comprehension we use an ε-calculus like formulation:

$$\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi) :\equiv \exists f\, \forall x, y\, \underbrace{(\varphi(x, f(x)) = 0 \vee \varphi(x, y) \neq 0)}_{\equiv: \left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi)\right)_{QF}(f,x,y)}. \tag{31}$$

This leads to following ND-interpretation (modulo a last application of QF-AC)

$$\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi)\right)^{ND} \equiv \forall X, Y\, \exists f\, \left(\varphi(Xf, f(Xf) = 0) \vee \varphi(Xf, Yf) \neq 0\right).$$

Because RT^2_2 and Π^0_1-$\widehat{\mathsf{CA}}(\varphi)$ are only $\forall\exists\forall$-statements, the ND-interpretation coincides with the no-counterexample interpretation. So one might view a solution to $\mathsf{RT}^{2\,ND}_2$, i.e. a term $t(c, U, V)$ that yields for every c, U, V a set H that may not be homogeneous in total but for which $c(H0, H1) = c(H(UH), H(VH))$ holds, as a procedure that disproves every possible counterexample to RT^2_2. Same for Π^0_1-$\widehat{\mathsf{CA}}(\varphi)$.

Proposition 48 ([48], [32, 42]). *The solution to $\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi)\right)^{ND}$ can be defined with a single use of Φ_0, this is Spector's bar recursor for type 0:*

$$t_f := \Phi_0 X u 0 (\lambda k^0.0), \qquad unv := \begin{cases} 1 & \text{if } \varphi(n, Y(v1)), \\ Y(v1) & \text{otherwise.} \end{cases}$$

The bar recursor Φ_0 is defined as in [32]. *It is primitive recursively and instance-wise definable in the bar recursor $B_{0,1}$, see definition 19 below.*

The statement from corollary 46 spelled out is

$$\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c \left(\exists f_\xi\, \forall x_\xi, y_\xi \left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi c)\right)_{QF}(f_\xi, x_\xi, y_\xi) \rightarrow \right.$$
$$\left.\exists H \left(\forall u < v\, c(Hu, Hv) = c(H0, H1) \wedge \exists f_\varphi\, \forall x_\varphi, y_\varphi \left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi c H)\right)_{QF}(f_\varphi, x_\varphi, y_\varphi)\right)\right).$$

An ND-interpretation leads then to

Theorem 49 (ND-interpretation of corollary 46). *For every provably continuous (in c, H) term $\varphi \in T_0[\mathcal{B}, \tilde{R}_1]$ a term $\xi \in T_0[\mathcal{B}, \tilde{R}_1]$ (that is continuous in c) exists such that*

$$\begin{aligned} &\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c \forall f_\xi\, \forall U < V\, \forall X_\varphi, Y_\varphi\, \exists x_\xi, y_\xi\, \exists H\, \exists f_\varphi \\ &\quad \Big(\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi c)\right)_{QF}(f_\xi, x_\xi, y_\xi) \rightarrow \big(c(H(UHf_\varphi), H(VHf_\varphi)) = c(H0, H1) \\ &\qquad\qquad \wedge\, \Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi c H)\big)_{QF}(f_\varphi, X_\varphi H f_\varphi, Y_\varphi H f_\varphi)\Big). \end{aligned} \tag{32}$$

Moreover, there exist terms $t_{x_\xi}, t_{y_\xi}, t_H, t_{f_\varphi} \in T_0[\mathcal{B}, \tilde{R}_1]$ (with the given parameters) satisfying this formula.

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[6] Officially, quantification over functions like $c\colon [\mathbb{N}]^2 \to 2$ or strictly monotone increasing functions like H are not included in our system as primitive notions, but we can enforce the same behavior by quantifying over $c\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $H\colon \mathbb{N} \to \mathbb{N}$ and replacing every occurrence of c, H with

$$\tilde{c}(x, y) := \min\left(1, \begin{cases} c(x, y) & \text{if } x < y \\ c(y, x) & \text{otherwise} \end{cases}\right), \qquad \tilde{H}(x) := x + \Sigma_{k \leq x} H(k).$$

Proof. The system $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}$ has an ND-interpretation into $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$. This also extends to additions of new constants and universal axioms. See e.g. [3, 32]. □

The term t_H and t_{f_φ} can be seen as procedures transforming the no-counterexample interpretation of the premise to the no-counterexample interpretation of the conclusion; the terms t_{x_ξ} and t_{y_ξ} yield which instance of the premise is needed to prove the conclusion.

Note that the counter-functions of RT^2_2 and $\Pi^0_1\text{-}\widehat{\mathsf{CA}}$ have access to both t_H and t_{f_φ}. The proof of proposition 50 bellow will use this.

To show that the no-counterexample interpretation of the conclusion (and hence the conclusion) holds we have to provide an f_ξ that satisfies $\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi c)\right)_{QF}(f_\xi, t_{x_\xi}, t_{y_\xi})$. This can be done using $B_{0,1}$, see proposition 48.

Note that here the application of $\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi)\right)^{ND}$ in the premise is not fully interpreted. We obtain this form by applying logical simplifications after the negative translation. This leads to fixed terms in the second and third parameter of the premise and will reduce the need for the bar recursor $B_{0,1}$ to the rule of $B_{0,1}$.

10.3. **Application to Ramsey's theorem.**

Proposition 50. *Let $t^1[g]$ be a term such that $\lambda g.t^1[g] \in T_0[\mathcal{R}]$, where $\mathcal{R}$ is a functional solving $\mathsf{RT}^{2\,ND}_2$, and every occurrence of $\mathcal{R}$ is of the form*

$$\mathcal{R}(t_c[g], t_u[g], t_v[g]).$$

Then there exist terms $t_x, t_y, \xi \in T_0[\tilde{R}_1, \mathcal{B}]$, such that one can inductively replace every occurrence of $\mathcal{R}$ in t with a new term

$$r(f, g; \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g])$$

(here r is a term and $\tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g]$ are the results of replacing $\mathcal{R}$ in $t_c[g], t_u[g]$, $t_v[g]$), such that

$$\begin{aligned}\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall g^1, f \left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi g)\right)_{QF}(f, t_x g, t_y g)&\\ \to \mathsf{RT}^2_{2ND}(r(f, g; \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g]);\ \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g])&.\end{aligned}$$

The formula RT^2_{2ND} denotes the quantifier-free part of $\mathsf{RT}^{2\,ND}_2$, see (30) *on p. 30.*

Proof. We use theorem 49 to inductively interpret the term t. For convenience we repeat (32), the existential quantified variables are replaced by their realizing terms constructed in that theorem:

$$\begin{aligned}(33)\quad &\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c \forall f_\xi \forall U < V\, \forall X_\varphi, Y_\varphi\\ &\Big(\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi c)\right)_{QF}(f_\xi, t_{x_\xi}, t_{y_\xi}) \to c(t_H(U t_H t_{f_\varphi}), t_H(V t_H t_{f_\varphi})) = c(t_H 0, t_H 1)\\ &\qquad \wedge \left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi c t_H)\right)_{QF}(t_{f_\varphi}, X_\varphi(t_H t_{f_\varphi}), Y_\varphi(t_H t_{f_\varphi}))\Big)\end{aligned}$$

It is clear that in case of $t_c, t_u, t_v \in T_0$, i.e. there are no nested applications of $\mathcal{R}$, every application of $\mathcal{R}$ in the term t can be interpreted using (33). (Just set $c = t_c$, $U = \lambda f_\varphi.t_u$, $V = \lambda f_\varphi.t_v$ and the others variable to 0.) Using contraction of Π^0_1-comprehension, see remark 9, a term containing multiple such occurrence of $\mathcal{R}$ can be interpreted.

If the term t_c contains a single occurrence of $\mathcal{R}$ then we first interpret this inner $\mathcal{R}$ but now we will take advantage of φ and set $\varphi, X_\varphi, Y_\varphi$ so that the resulting instance of ND-comprehension suffices to interpret the outer occurrence of $\mathcal{R}$ in t.

Iterating this process allows us to interpret all terms $t \in T_0[\mathcal{R}]$ where every occurrence of $\mathcal{R}$ is of the form $\mathcal{R}(t_c[g], t_u[g], t_v[g])$ with $t_u, t_v \in T_0$.

Now inductively assume that t_u, t_v are terms for which this proposition holds, i.e. there exists terms $\tilde{t}_u, \tilde{t}_v$ equal to t_u, t_v modulo a given instance of ND-comprehension with the parameter H. The problem is now that the instances of comprehension cannot be generated parallel to t_c because they include the parameter H. But we take advantage of the argument t_{f_φ} of U and V. Coding the instances of ND-comprehension together (ND-interpretation of remark 9) we can find $\varphi', X'_\varphi, Y'_\varphi$ such that

$$\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi' cH)\right)_{QF}(f_\varphi, X'_\varphi(Hf_\varphi), Y'_\varphi(Hf_\varphi))$$

proves the original ND-instance of Π^0_1-$\widehat{\mathsf{CA}}$ for φ and those needed for t_u, t_v.

This proves the proposition. □

Corollary 51 (Extension to R_1, Φ'_0)**.** *The statement of proposition 50 also holds for terms $t^1[g]$ with $\lambda g.t[g] \in T_0[\mathcal{R}, R_1, \Phi'_0] = T_1[\mathcal{R}, \Phi'_0]$, where every occurrence of $\mathcal{R}$ is of the form required in proposition 50 and every occurrence of R_1 or Φ'_0 is of the form*

$$R_1(t_1[g], t_2[g], t_3[g]) \quad \textit{resp.} \quad \Phi'_0(t_1[g], t_2[g], t_3[g]).$$

Proof. The proof proceeds like in proposition 50:

To interpret R_1 while retaining the instance of ND-comprehension, we will essentially use a functional interpretation of the proof of lemma 11 (for $n = 1$). First note that $s := R_1(t_1[g], t_2[g], t_3[g])$ defines a type 1 function in $T_1[g]$. Arguing as in lemma 11, it is clear that over $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction$ a suitable instance of Π^0_1-CA with the parameter g proves that s is total ($\forall x\, \exists y\, \langle x, y\rangle \in \mathcal{G}_s[g]$, where $\mathcal{G}_s$ is the graph of s). An ND-interpretation of this statement yields that even an instance of the ND-interpretation of Π^0_1-CA is sufficient to prove that s is total. Another instance of ND-comprehension proves the ND-interpretation of the Π^0_1-CA-instance in (8) on p. 13. This instance is modulo the totality of s equivalent to an instance of ND-comprehension with the parameter s. The two instances of ND-comprehension used can be coded together, see remark 9.

The functional Φ'_0 can be replaced by a function in $T_1[g]$, see theorem 21 and remark 22, and hence can also be interpreted. □

Proposition 52. *Let A_{qf} be a quantifier-free formula that contains only the shown variables free. If*

$$\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} + \Sigma^0_2\text{-}\mathsf{IA} + \mathsf{RT}^2_2 + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x, y) \tag{34}$$

then one can find a terms $t_y, t_u, t_v, \xi \in T_0[\mathcal{B}, \tilde{R}_1]$ such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus (\mathcal{B}) \oplus (\tilde{R}_1)$$
$$\vdash \forall x^1\, \forall f\, \left(\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi x)\right)_{QF}(f, t_u f x, t_v f x) \to A_{qf}(x, t_y f x)\right).$$

Proof. A functional interpretation of the statement (34) yields closed terms resp. term tuples $t_y, t_{R_1}, t_{\mathcal{R}}, t_{\Phi'_0} \in T_0$, such that

$$\text{qf-}\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction \vdash \big((R_1)_{ND}(R_1, t_{R_1} R_1 \mathcal{R} \Phi'_0 x) \wedge \mathsf{RT}^2_{2ND}(\mathcal{R}, t_{\mathcal{R}} R_1 \mathcal{R} \Phi'_0 x)$$
$$\wedge\, \mathsf{WKL}_{ND}(\Phi'_0, t_{\Phi'} R_1 \mathcal{R} \Phi'_0 x)\big) \to A_{qf}(x, t_y R_1 \mathcal{R} \Phi'_0 x).$$

Here we use that $(\Sigma^0_2\text{-}\mathsf{IA})^{ND}$ can be solved by R_1, see [41].

Apply now proposition 26 and remark 27 to this derivation to normalize it such that only finitely many independent applications of $\mathcal{R}, R_1, \Phi'_0$ occur, where each of them is of the form

$$\mathcal{R}^*(t_1[g], t_2[g], t_3[g]) \quad \text{resp.} \quad R_1(t_1[g], t_2[g], t_3[g]), \quad \Phi'_0(t_1[g], t_2[g], t_3[g])$$

and t_1, t_2, t_3 are semi-closed.

The terms occurring in this normalized derivation can be interpreted using corollary 51. (Applications to literally equal terms are replaced by the same interpretation.)

The instances of ND-comprehension needed for corollary 51 can be coded together in one instance using remark 9. □

The application of $\Pi^0_1\text{-CA}$ can be interpreted by a non-iterated use R-$(B_{0,1})$ of the rule of bar-recursion — this means we substitute f with a solution t_f to $\left(\Pi^0_1\text{-}\widehat{\text{CA}}\right)^{ND}$:

$$\begin{aligned}&\widehat{\text{WE-HA}}^\omega{\restriction} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \oplus \text{R-}(B_{0,1})\\ &\qquad\vdash \forall x^1 \, \Big(\left(\Pi^0_1\text{-}\widehat{\text{CA}}(\xi x)\right)_{QF}(t_f[x], t_u t_f[x]x, t_v t_f[x]x) \to A_{qf}(x, t_y t_f[x]x)\Big)\end{aligned}$$

The term $t_f \in T_0[\mathcal{B}, \tilde{R}_1, B_{0,1}]$ is defined as in proposition 48. Note that t_f depends on ξ, t_u, t_v and that it is of type 2 containing only *one application* of $B_{0,1}$ to semi-closed terms defining a type 2 object.

Since t_f solves the instance of comprehension we obtain:

$$\widehat{\text{WE-HA}}^\omega{\restriction} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \oplus \text{R-}(B_{0,1}) \vdash \forall x^1 \, A_{qf}(x, t_y t_f[x]x).$$

The term $t := \lambda x.t_y t_f[x]x \in T_0[\mathcal{B}, \tilde{R}_1, B_{0,1}]$, contains only majorizable constants; the majorants to $\mathcal{B}$, $\tilde{R}_1$ are trivial and $B_{0,1}$ is essentially majorized by itself, see proposition 16, hence we can find a majorant $t^* \in T_0[B_{0,1}]$ to t containing also only one application of $B_{0,1}$ to semi-closed terms. Now we can apply bounded search to obtain a new realizer t' for y not containing $\mathcal{B}$ or $\tilde{R}_1$:

$$t'x := \begin{cases}\text{minimal } y \le t^*x \text{ with } A_{qf}(x,y), & \text{if such a } y \text{ exists,}\\ 0, & \text{otherwise.}\end{cases}$$

Since t' now does not contain $\mathcal{B}$ anymore we may weaken $(\mathcal{B})$ to UWKL and then eliminate it from the system using a monotone functional interpretation, see [25, 32]. Hence we obtain a term $t' \in T_0[B_{0,1}]$ containing after normalization only one occurrence of $B_{0,1}$ defining a type 2 object, such that with the rule R-$(B_{0,1})$ of $B_{0,1}$

$$\widehat{\text{WE-HA}}^\omega{\restriction} \oplus (\tilde{R}_1) \oplus \text{R-}(B_{0,1}) \vdash \forall x^1 \, A_{qf}(x, t'x).$$

Using ordinal analysis of the $B_{0,1}$-rule (cf. theorem 21 and remark 22) yields a new term t'' definable with ordinal primitive recursion up to ω_2^ω such that

$$\widehat{\text{WE-HA}}^\omega{\restriction}_{\omega_2^\omega} \oplus (\tilde{R}_1) \vdash \forall x^1 \, A_{qf}(x, t''x).$$

Combining this with theorem 17 and noting that $\tilde{R}_1$ is included in $\text{WE-HA}^\omega_1{\restriction}$ and that $\widehat{\text{WE-PA}}^\omega{\restriction} + \Sigma^0_2\text{-IA}$ has an ND-interpretation in $\text{WE-HA}^\omega_1{\restriction}$ we obtain the following theorem:

Theorem 53 (Conservation for RT^2_2)**.** *If*

$$\widehat{\text{N-PA}}^\omega{\restriction} + \text{QF-AC} + \Sigma^0_2\text{-IA} + \text{RT}^2_2 + \text{WKL} \vdash \forall x^1 \, \exists y^0 \, A_{qf}(x,y)$$

then one can extract a term $t \in T_1$ such that

$$\text{WE-HA}^\omega_1{\restriction} \vdash \forall x^1 \, A_{qf}(x, tx).$$

10.3.1. *Extension to* $\mathsf{RT}^2_{<\infty}$. Proposition 50 holds analogously for $\mathsf{RT}^2_{<\infty}$ if one adds R_1 and Σ^0_2-IA to the verifying system; corollary 51 holds if one replaces R_1 by R_2.

But in contrast to the previous the technique used in remark 27 to extract terms that meet the requirements of these propositions can only be applied to terms in $T_1[\mathcal{R}_\infty]$ and not to terms $T_2[\mathcal{R}_\infty]$, because $deg(R_2) = 4$ and therefore we could not apply the term normalization. The mathematical reason is that R_2 is strong enough to iterate $B_{0,1}$ and $\mathcal{R}_\infty$.

This will hinder us to achieve full conservativity for full Σ^0_3-IA over a system in all finite types but a restricted variant of Σ^0_3-induction can be handled. Define the rule of Σ^0_3-induction Σ^0_3-IR as

$$(\Sigma^0_3\text{-}\mathsf{IR})\colon\ \frac{\exists x\,\forall y\,\exists z\,A_{qf}(0,x,y,z,\underline{a}) \qquad \begin{array}{l}\forall n\,\big(\exists x\,\forall y\,\exists y\,A_{qf}(n,x,y,z,\underline{a})\\ \qquad\to\exists u\,\forall v\,\exists w\,A_{qf}(n+1,u,v,w,\underline{a})\big)\end{array}}{\forall n\,\exists x\,\forall y\,\exists z\,A_{qf}(n,x,y,z,\underline{a})},$$

where A_{qf} is quantifier-free and contains only the variables shown, u, v, w, x, y, z, n are type 0 variables and $\underline{a}$ denotes an arbitrary tuple of parameters. Let Σ^0_3-IR_2 be the restriction of Σ^0_3-IR to parameters $\underline{a}$ of type ≤ 2 then

Theorem 54 (Conservation for $\mathsf{RT}^2_{<\infty}$)**.** *If*

$$\mathsf{N\text{-}PA}^\omega_1{\restriction} + \mathsf{QF\text{-}AC} + \Sigma^0_3\text{-}\mathsf{IR}_2 + \mathsf{RT}^2_{<\infty} + \mathsf{WKL} \vdash \forall x^1\,\exists y^0\,A_{qf}(x,y) \tag{35}$$

then one can extract a term $t \in T_2$ such that

$$\mathsf{WE\text{-}HA}^\omega_2{\restriction} \vdash \forall x^1\,A_{qf}(x,tx).$$

Proof. The ND-interpretation of the conclusion of Σ^0_3-IR_2 is given by

$$\forall n^0\,\forall Y^2\,\exists x^0, Z^1\,A_{qf}(n,x,YxZ,Z(YxZ),\underline{a}^2).$$

One immediately see that Σ^0_3-IR_2 introduces only type 3 terms (t_Z, t_x ranging over $n^0, Y^2, \underline{a}^2$). Hence we can ND-interpret (35) in

$$\text{qf-}\mathsf{N\text{-}PA}^\omega_1{\restriction} + (G_1) + \cdots + (G_n)$$

where (G_i) are defining axioms and constants of type ≤ 3 introduced by the rule Σ^0_3-IR_2. The terms occurring in the derivation can be viewed as terms in $T_1[\mathcal{R}_\infty, \Phi'_0, G_1, \ldots, G_n]$. The requirements of theorem 24 in remark 27 are met and we obtain a normalized derivation.

By [41], $(\Sigma^0_3\text{-}\mathsf{IA})^{ND}$ can be solved by R_2. Since Σ^0_3-IA implies Σ^0_3-IR_2 the constants G_i may be chosen to be in $T_2[\mathcal{R}_\infty, \Phi'_0]$. These terms can be handled like in proposition 52.

This completes the proof. □

Corollary 55. *If*

$$\widehat{\mathsf{E\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \Sigma^0_2\text{-}\mathsf{IA} + \mathsf{RT}^2_2 + \mathsf{WKL} \vdash \forall x^1\,\exists y^0\,A_{qf}(x,y)$$

one can extract a term $t \in T_1$ such that

$$\mathsf{WE\text{-}HA}^\omega_1{\restriction} \vdash \forall x^1\,A_{qf}(x,tx).$$

If $\mathsf{RT}^2_{<\infty} + \Sigma^0_3$-$\mathsf{IR}_2$ *is added to the above system then one can extract a term $t \in T_2$ realizing y provably in* $\mathsf{WE\text{-}HA}^\omega_2{\restriction}$ *instead of* $\mathsf{WE\text{-}HA}^\omega_1{\restriction}$.

Proof. Apply elimination of extensionality (proposition 3) and use theorem 53.

For the second statement use theorem 54. To be able to use the elimination of extensionality the induction rule Σ^0_3-IR_2 has to be altered to include the premise that the parameters are extensional. Since this is a formula of the form $\forall u^1\,\exists v^0\,B_{qf}(u,v)$, the functional interpretation does not introduce terms of type > 3 and the rule which still follows from Σ^0_3-IA can be interpreted like in the proof of theorem 54. □

Corollary 56.

- $\mathsf{WKL}_0^\omega + \Sigma^0_2\text{-}\mathsf{IA} + \mathsf{RT}^2_2$ *is conservative over* $\mathsf{RCA}_0^\omega + \Sigma^0_2\text{-}\mathsf{IA}$ *for sentences of the form* $\forall x^1\, \exists y^0\, \forall z^0\, A_{qf}(x, y)$.

  As consequence, $\mathsf{WKL}_0 + \Sigma^0_2\text{-}\mathsf{IA} + \mathsf{RT}^2_2$ *is conservative over* $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA}$ *for sentences of the form* $\forall X\, \forall x\, \exists y\, \forall z\, A(X, x, y, z)$, *where* A *is* Δ^0_0, *and thus, in particular,* Π^0_3*-conservative.*
- $\mathsf{WKL}_0^\omega + \Sigma^0_2\text{-}\mathsf{IA} + \Sigma^0_3\text{-}\mathsf{IR}_2 + \mathsf{RT}^2_{<\infty}$ *is conservative over* $\mathsf{RCA}_0^\omega + \Sigma^0_3\text{-}\mathsf{IA}$ *for sentences of the form* $\forall x^1\, \exists y^0\, \forall z^0\, A_{qf}(x, y)$.

  Hence, $\mathsf{WKL}_0 + \Sigma^0_2\text{-}\mathsf{IA} + \Sigma^0_3\text{-}\mathsf{IR} + \mathsf{RT}^2_{<\infty}$ *is conservative over* $\mathsf{RCA}_0 + \Sigma^0_3\text{-}\mathsf{IA}$ *for sentences of the form* $\forall X\, \forall x\, \exists y\, \forall z\, A(X, x, y, z)$, *where* A *is* Δ^0_0, *and thus, in particular,* Π^0_3*-conservative.*

Moreover from of $\forall x^1\, \exists y^0\, A_{qf}(x, y)$ *in the above theories one can extract terms in* T_1 *resp.* T_2 *realizing* y.

Proof. The former statements follow from the previous theorem with the fact every sentence of the form $\forall x^1\, \exists y^0\, \forall z^0\, A_{qf}(x, y, z)$ is over $\mathsf{QF}\text{-}\mathsf{AC}^{0,0}$ equivalent to a sentence of the form $\forall x^1\, \exists y^0\, B_{qf}(x, y)$.

The conservativity over RCA_0 follows from the fact that RCA_0^ω is conservative over its second order fragment, which can be simulated in RCA_0, see [31]. The quantification over X and x can be coded into a quantification over a function. The restrict on the rule of Σ^0_3-induction is automatically met in a second order system. $\square$

11. Possible extensions

The question arises whether RT^2_2 also is proofwise low in sequence over $\mathsf{WKL}_0^{\omega *}$ (or $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF}\text{-}\mathsf{AC} + (\mathcal{B})$) and hence does not imply Σ^0_2-induction.

The first obstacle to show this is that the proof of the lowness-property crucially depends on full Σ^0_1-induction which renders $\mathsf{G}_\infty\mathsf{A}^\omega$ or equivalently $\mathsf{RCA}_0^{\omega *}$ insufficient. The other obstacle is that RT^2_2 implies Π^0_2-LEM so that sequences of solutions would imply Π^0_2-CA. Thus RT^2_2 cannot be proofwise low in sequence over a theory which does not include Π^0_2-CA, see proposition 47. Actually even the so called stable chain-antichain principle (SCAC) implies Π^0_2-LEM (for a definition see [15]).

In [35] we refined the method based on the bar recursion (section 10.3) and could show that the type 2 functionals that are provable from principles which are proofwise low over $\mathsf{WKL}_0^{\omega *}$ are primitive recursive. We also show that CAC is proofwise low in sequence and thus that this theorem applies to it, see also [9]. However, we were not able to show that RT^2_2 is proofwise low in sequence over $\mathsf{WKL}_0^{\omega *}$. (In other words we could overcome the second obstacle but not the first one.) Still the question remains whether one could do the same with RT^2_2 or any other principle which is stronger than CAC. The principle RT^2_2 splits into the so called Erdős-Moser principle (EM) and CAC (actually even ADS), see [6]. Therefore EM seems to be a good candidate for further investigations.

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