A logical uniform boundedness principle for abstract metric and hyperbolic spaces

Ulrich Kohlenbach

Department of Mathematics,
Darmstadt University of Technology,
Schlossgartenstraße 7,
D-64289 Darmstadt, Germany

Abstract

We extend the principle Σ^0_1-UB of uniform Σ^0_1-boundedness introduced earlier by the author to a uniform boundedness principle $\exists$-UB^X for abstract bounded metric and hyperbolic spaces which are not assumed to be compact. Despite the fact that this principle implies numerous results which in general are true only for compact spaces (and continuous functions) we can prove that for a large class $\mathcal{K}$ of such consequences A the conclusion A is true in arbitrary bounded spaces even when $\exists$-UB^X is used to facilitate the proof of A. For a somewhat more restricted class of sentences A even effective uniform bounds can be extracted from such proofs.

1 introduction

In [15] and [8] general metatheorems are proved which have the form of rules of the following type: If certain $\forall\exists$-sentences are proved in classical analysis $\mathcal{A}^\omega$ augmented by abstract structures X ($\mathcal{A}^\omega[X,\ldots]$) as 'Urelements' such as metric, hyperbolic (in the sense of Kirk and Reich see [15]) or CAT(0) spaces (in the sense of Gromov),[1] then from a given proof one can extract an effective uniform bound which holds in arbitrary such structures and only depends on parameters from X via bounds on the metric ([15]) or even just the distances of some relevant elements ([8]). So whereas for the general class of Polish spaces as well as for individual effectively represented Polish spaces such a uniformity is guaranteed only under a compactness assumption (essentially due to the separability of the space involved, see below and [16]), in the case of proofs from general axioms for abstract classes of spaces as the ones mentioned above, metric boundedness is sufficient.

[1] These papers also treat normed, uniformly convex and inner product spaces (and in [17] this has been adapted also to hyperbolic spaces in the sense of Gromov and $\mathbb{R}$-trees) but in the present article we restrict ourselves due to the limited space to the ones mentioned.

In this paper we extend $\mathcal{A}^\omega[X,\ldots]$ by a strong uniform boundedness principle $\exists$-UB^X which states the above uniformity as an implication (rather than a rule). Despite the fact that this principle allows one to derive many consequences which are only true for compact spaces (and continuous functions), for a large class $\mathcal{K}$ of consequences the correctness in arbitrary bounded metric resp. hyperbolic spaces can be proved. $\exists$-UB^X extends the principle Σ^0_1-UB (introduced in [12] and further studied in [14,5]) for the Cantor space $\mathcal{C}$ by including abstract bounded metric spaces X in addition to $\mathcal{C}$. Σ^0_1-UB has proved to be useful in the cause of proof mining in the context of compact Polish spaces (see [13] and [1]) as it allows one to give very short and coding free proofs of many of the usual applications of weak König's lemma WKL. In addition Σ^0_1-UB proves various classically false theorems such as the uniform continuity (with modulus of continuity) of all extensional functionals $\Phi : 2^{\mathbb{N}} \to \mathbb{N}$ which makes it possible to treat continuous functions without explicitly having to refer to moduli of continuity. In the case of $\exists$-UB^X, which applies even in the absence of compactness conditions so that WKL is not applicable at all, the benefits are even bigger. As one of the applications we will show that it proves (relative to the extension of $\mathcal{A}^\omega$ by the axioms for an abstract bounded hyperbolic space (X,d,W)) that every nonexpansive function $f : X \to X$ has fixed points, where 'nonexpansive' means that

$$\forall x, y \in X (d(f(x), f(y)) \leq d(x,y)).$$

Although in general it is only true that such functions have approximate fixed points (but not necessarily fixed points) this allows one to make free use of fixed points to facilitate proofs of sentences in $\mathcal{K}$ and nevertheless obtain correct results (see [15] for a discussion of the relevance of this point). To achieve similar benefits, often ultrapowers of spaces X are used in functional analysis which, however, in contrast to our method usually prevent one from getting effective bounds on the conclusion.

2 Basic notions

Definition 2.1 1) The set $\mathbf{T}$ of all finite types over 0 is defined inductively by the clauses

$$(i)\ 0 \in \mathbf{T},\ (ii)\ \rho, \tau \in \mathbf{T} \Rightarrow (\rho \to \tau) \in \mathbf{T}.$$

2) The set $\mathbf{T}^X$ of all finite types over the two ground types 0 and X is defined by

$$(i)\ 0, X \in \mathbf{T}^X,\ (ii)\ \rho, \tau \in \mathbf{T}^X \Rightarrow (\rho \to \tau) \in \mathbf{T}^X.$$

3) A type $\rho \in \mathbf{T}$ has degree $(\leq)1$ if $\rho = 0 \to \ldots \to 0$ (including $\rho = 0$). A type $\rho \in \mathbf{T}^X$ has degree $(0, X)$ if $\rho = 0 \to \ldots \to 0 \to X$ (including $\rho = X$). A type $\rho \in \mathbf{T}^X$ has degree $(1, X)$ if it has the form $\tau_1 \to \ldots \to \tau_k \to X$ (including $\rho = X$), where τ_i has degree 1 or $(0, X)$.

A type $\rho \in \mathbf{T}^X$ has degree $(\cdot, X)$ if it has the form $\tau_1 \to \ldots \to \tau_k \to X$ (including $\rho = X$), where $\tau_1, \ldots, \tau_k \in \mathbf{T}^X$ are arbitrary.
A type $\rho \in \mathbf{T}^X$ has degree $(\cdot, 0)$ if it has the form $\tau_1 \to \ldots \to \tau_k \to 0$ (including $\rho = 0$), where $\tau_1, \ldots, \tau_k \in \mathbf{T}^X$ are arbitrary.

In the following we often denote tuples $x_1^{\rho_1}, \ldots, x_n^{\rho_n}$ by $\underline{x}^{\underline{\rho}}$.

Definition 2.2 For $\rho \in \mathbf{T}^X$ with $\rho = \rho_1 \to \ldots \to \rho_k \to 0$ (i.e. for ρ of degree $(\cdot, 0)$) we define a functional $\min_\rho$ of type $\rho \to \rho \to \rho$ by

$$\min_\rho(x^\rho, y^\rho) := \lambda v_1^{\rho_1}, \ldots, v_k^{\rho_k}. \min_0(x\underline{v}, y\underline{v})$$

and a relation $\leq_\rho$ between objects of type ρ by

$$x \leq_\rho y :\equiv \forall v_1^{\rho_1}, \ldots, v_k^{\rho_k}(x\underline{v} \leq_0 y\underline{v})$$

with the usual primitive recursively defined $\min_0$ and $\leq_0$.

The theory $\mathcal{A}^\omega$ for classical analysis is the extension of the weakly extensional Peano arithmetic in all types WE-PA$^\omega$ by the schemata of quantifier-free choice QF-AC and dependent choice DC for all types in $\mathbf{T}$ (formulated for tuples of variables). [2] The theories $\mathcal{A}^\omega[X, d]$ and $\mathcal{A}^\omega[X, d, W]$ result by extending $\mathcal{A}^\omega$ to all types in $\mathbf{T}^X$ and adding axioms for an abstract bounded metric (in the case of $\mathcal{A}^\omega[X, d]$) resp. bounded hyperbolic (in the case of $\mathcal{A}^\omega[X, d, W]$) space. $\mathcal{A}^\omega[X, d, W, \mathrm{CAT}(0)]$ is the extension by an abstract bounded CAT(0)-space. For details see [15] which also treats the case of normed spaces. Corresponding theories for general (not necessarily bounded) metric and hyperbolic spaces are studied in [8] (similar extensions by hyperbolic spaces in the sense of Gromov and by $\mathbb{R}$-trees have recently been defined in [17]). Since the results in this paper are most natural and useful in the bounded case we do not consider these latter theories here.
That our theories are 'weakly extensional' means that we only have Spector's quantifier-free extensionality rule. In particular, for the defined equality $x =_X y :\equiv (d_X(x, y) =_{\mathbb{R}} 0_{\mathbb{R}})$, we do not have

$$x =_X y \to f^{X \to X}(x) =_X f(y)$$

but only from a proof of $s =_X t$ can infer that $f(s) =_X f(t)$. As discussed in great detail in [15], this restriction is crucial for our results. In practice, we usually can prove the extensionality of f for those functions we consider, e.g. for nonexpansive functions, so that this issue only occasionally matters.

Definition 2.3 A formula F in $\mathcal{L}(\mathcal{A}^\omega[X, d])$ or $\mathcal{L}(\mathcal{A}^\omega[X, d, W])$ is called $\forall$-formula (resp. $\exists$-formula) if it has the form $F \equiv \forall \underline{a}^{\underline{\sigma}} F_{qf}(\underline{a})$ (resp. $F \equiv \exists \underline{a}^{\underline{\sigma}} F_{qf}(\underline{a})$) where F_{qf} does not contain any quantifier and the types in $\underline{\sigma}$ are of degree 1 or $(1, X)$. We call a formula a generalized $\exists$-formula, if there are no restrictions imposed on the types $\underline{\sigma}$.

[2] For DC the form with tuples is not stated in [15] but the proofs immediately work also in the presents of tuples.

Real numbers are represented as Cauchy sequences of rationals with fixed rate 2^{-n} of convergence which in turn are encoded as number theoretic functions f^1, where an equivalence relation $f =_{\mathbb{R}} g$ expresses that f^1, g^1 denote the same real numbers and $\leq_{\mathbb{R}}, <_{\mathbb{R}}, |\cdot|_{\mathbb{R}}$ express the obvious relations and operations on the level of these codes. Here $=_{\mathbb{R}}, \leq_{\mathbb{R}} \in \Pi^0_1$ whereas $<_{\mathbb{R}} \in \Sigma^0_1$. Again details can be found in [15].

3 Main results

Definition 3.1 The uniform boundedness schema $\exists$-UB^X for generalized $\exists$-formulas and bounded abstract metric spaces is defined as follows [3]

$$\exists\text{-UB}^X :\equiv \begin{cases} \forall y^{0\to\alpha}(\forall k^0, x^\alpha, \underline{z}^{\underline{\beta}} \exists n^0 A_\exists(y,k,\min_\alpha(x,yk),\underline{z},n) \to \\ \exists \chi^1 \forall k^0, x^\alpha, \underline{z}^{\underline{\beta}} \exists n \leq_0 \chi k\, A_\exists(y,k,\min_\alpha(x,yk),\underline{z},n)), \end{cases}$$

where α is of degree $(\cdot, 0)$, $\underline{\beta} = \beta_1, \ldots, \beta_m$ is a tuple of types in $\mathbf{T}^X$ of degree $(\cdot, X)$ and $A_\exists$ is a generalized $\exists$-formula which may in addition to the variables indicated may have arbitrary further parameters of arbitrary types.

Remark 3.2 If $A_\exists(y,k,x,\underline{z},n)$ is extensional in x w.r.t. $x_1 =_\alpha x_2 :\equiv \forall \underline{v}(x_1\underline{v} =_0 x_2\underline{v})$, i.e. if

$$\forall y,k,\underline{z},n,x_1,x_2(x_1 =_\alpha x_2 \wedge A_\exists(y,k,x_1,\underline{z},n) \to A_\exists(y,k,x_2,\underline{z},n)),$$

then $\exists$-UB^X can be rewritten equivalently as follows

$$\begin{cases} \forall y^{0\to\alpha}(\forall k^0 \forall x \leq_\alpha yk \forall \underline{z}^{\underline{\beta}} \exists n^0 A_\exists(y,k,x,\underline{z},n) \to \\ \exists \chi^1 \forall k^0 \forall x \leq_\alpha yk \forall \underline{z}^{\underline{\beta}} \exists n \leq_0 \chi k\, A_\exists(y,k,x,\underline{z},n)), \end{cases}$$

Definition 3.3 Let $\underline{\beta}$ be as before.

$$F^X :\equiv \forall \Phi, y \exists X \leq y \exists \underline{Z} \forall k^0, x^\alpha, \underline{z}^{\underline{\beta}} (\Phi(k, Xk, \underline{Z}k) \geq_0 \Phi(k, \min_\alpha(x,yk), \underline{z})).$$

Here X has type $0 \to \alpha$, Z_i has type $0 \to \beta_i$ and Φ has type $0 \to \alpha \to \beta_1 \to \ldots \to \beta_m \to 0$.

Lemma 3.4

$$\mathcal{A}^\omega[X,d] + F^X \vdash \exists\text{-UB}^X.$$

Analogously for $\mathcal{A}^\omega[X,d,W]$ and the other extensions we consider.

Proof: Assume

$$\forall k^0, x^\alpha, \underline{z}^{\underline{\beta}} \exists n^0 A_\exists(y,k,\min_\alpha(x,yk),\underline{z},n).$$

[3] For notational simplicity for formulate the principle only for a single variable x but we can here (and in the proofs below) also allow tuples as in the case of $\underline{z}$ which we do formulate for tuples as it is used this way in our applications. Using appropriate contractions of tuples of variables of degree $(\cdot, 0)$ into a single variable of degree $(\cdot, 0)$ one, alternatively, can also reduce the case with tuples $\underline{x}$ to the one we formulate.

By the schema of quantifier-free choice QF-AC from $\mathcal{A}^\omega[X,d]$ (which is formulated for tuples of variables) it follows that there exists a functional Φ such that

$$\forall k^0, x^\alpha, \underline{z}^\beta A_\exists(y,k,\min_\alpha(x,yk),\underline{z},\Phi kx\underline{z}).$$

Since

$$\mathcal{A}^\omega[X,d] \vdash \min_\alpha(\min_\alpha(x,yk),yk) =_\alpha \min_\alpha(x,yk)$$

we can use the quantifier-free extensionality rule QF-ER from $\mathcal{A}^\omega[X,d]$ to conclude that

$$\forall k^0, x^\alpha, \underline{z}^\beta A_\exists(y,k,\min_\alpha(x,yk),\underline{z},\Phi(k,\min_\alpha(x,yk),\underline{z})).$$

F^X applied to Φ and y yields $X(\leq y)$ and $\underline{Z}$ with

$$\forall k^0, x^\alpha, \underline{z}^\beta(\Phi(k,Xk,\underline{Z}k) \geq_0 \Phi(k,\min_\alpha(x,yk),\underline{z})).$$

Now define $\chi(k) := \Phi(k,Xk,\underline{Z}k)$. $\square$

Theorem 3.5 1) Let σ,ρ be types of degree 1 and τ be a type of degree $(1,X)$. Let $s^{\sigma\to\rho}$ be a closed term of $\mathcal{A}^\omega[X,d]$ and $B_\forall(x^\sigma,y^\rho,z^\tau,u^0)$ $(C_\exists(x^\sigma,y^\rho,z^\tau,v^0))$ be a $\forall$-formula containing only x,y,z,u free (resp. an $\exists$-formula containing only x,y,z,v free).
If

$$\forall x^\sigma \forall y \leq_\rho s(x)\forall z^\tau(\forall u^0 B_\forall(x,y,z,u) \to \exists v^0 C_\exists(x,y,z,v))$$

is provable in $\mathcal{A}^\omega[X,d] + \exists\text{-UB}^X$, then one can extract a computable functional $\Phi : \mathbb{N}^{\mathbb{N}\times\ldots\times\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}^{\mathbb{N}\times\ldots\times\mathbb{N}}$ and all $b \in \mathbb{N}$

$$\forall y \leq_\rho s(x)\forall z^\tau[\forall u \leq \Phi(x,b)\, B_\forall(x,y,z,u) \to \exists v \leq \Phi(x,b)\, C_\exists(x,y,z,v)]$$

holds[4] in any (non-empty) b-bounded metric space (X,d) (where b_X is to be interpreted by the integer upper bound b on d).[5]
The computational complexity of Φ can be estimated in terms of the strength of the $\mathcal{A}^\omega$-principle instances actually used in the proof (see remark 3.6 below).

2) If the premise is proved in '$\mathcal{A}^\omega[X,d,W]+\exists\text{-UB}^X$' instead of '$\mathcal{A}^\omega[X,d]+\exists\text{-UB}^X$', then the conclusion holds in all (non-empty) b-bounded hyperbolic spaces.

3) If the premise is proved in '$\mathcal{A}^\omega[X,d,W,\mathrm{CAT}(0)] + \exists\text{-UB}^X$' instead of '$\mathcal{A}^\omega[X,d] + \exists\text{-UB}^X$', then the conclusion holds in all (non-empty) b-bounded CAT(0)-spaces.

Instead of single variables x,y,z,u,v we may also have finite tuples of variables $\underline{x},\underline{y},\underline{z},\underline{u},\underline{v}$ as long as the elements of the respective tuples satisfy the same type restrictions as x,y,z,u,v. Moreover, instead of a single premise of the form '$\forall u^0 B_\forall(x,y,z,u)$' we may have a finite conjunction of such premises.

[4] See [15] for the precise definition of 'holds'.
[5] Here b_X is the constant of type 0 from $\mathcal{A}^\omega[X,d]$ representing an upper bound on d_X.

Remark 3.6 1) The proof of theorem 3.5 which we will give below is based on the proof of theorem 3.7 in [15] and will actually provide an extraction algorithm for Φ. The functional Φ is given by a closed term of WE-PA$^\omega$+BR where BR refers to Spector's schema of bar recursion ([23]), i.e. Φ is a so-called bar recursive functional. However, for concrete proofs usually only small fragments of $\mathcal{A}^\omega[X,d,W]$ (corresponding to fragments of $\mathcal{A}^\omega$ such as WE-PA$^\omega$+QF-AC+WKL) will be needed to formalize the proof so that Φ will be of much lower complexity.

2) Without the addition of the non-standard axiom $\exists$-UB^X the theorem is proved in [15] and again in [8] as a corollary to a more refined metatheorem.

Remark 3.7 The most important aspects of theorem 3.5 are that the bound $\Phi(x,b)$ does not depend on y,z nor does it depend on X,d or W and that the conclusion is true in all b-bounded metric spaces (X,d), hyperbolic spaces (X,d,W) and CAT(0)-spaces, respectively, although the axiom $\exists$-UB^X is not (see below).

Proof of theorem 3.5: 1. By the previous lemma, the assumption implies that $\mathcal{A}^\omega[X,d]+F^X$ proves that

$$\forall x^\sigma \forall y \leq_\rho s(x) \forall z^\tau (\forall u^0 B_\forall(x,y,z,u) \to \exists v^0 C_\exists(x,y,z,v)).$$

As in [15] one shows that $\mathcal{A}^\omega[X,d]+F^X$ has a Gödel functional interpretation in $\mathcal{A}^\omega[X,d,\mathcal{X},\underline{\mathcal{Z}}]^- + \tilde{F}^X$+(BR), where [6]

$$\mathcal{A}^\omega[X,d,\mathcal{X},\underline{\mathcal{Z}}]^- := \mathcal{A}^\omega[X,d,\mathcal{X},\underline{\mathcal{Z}}] \setminus \{\text{ QF-AC }\}$$

and

$$\tilde{F}^X :\equiv \mathcal{X} \leq \lambda\Phi,y.y \wedge \forall \Phi,y,k,x,\underline{z}^{\underline{\beta}}(\Phi(k,\mathcal{X}\Phi yk,\underline{\mathcal{Z}}\Phi yk) \geq_0 \Phi(k,\min\nolimits_\alpha(x,yk),\underline{z}).$$

Here (BR) is the schema of (simultaneous) bar recursion of Spector (see [23,3,18]) extended to the types $\mathbf{T}^X$ (see [15]) and $\mathcal{A}^\omega[X,d,\mathcal{X},\underline{\mathcal{Z}}]$ results from $\mathcal{A}^\omega[X,d]$ by adding new constants $\mathcal{X}$ and $\underline{\mathcal{Z}}$ of type $(0 \to \alpha \to \beta_1 \to \ldots \to \beta_m \to 0) \to (0 \to \alpha) \to 0 \to \alpha$ resp. $(0 \to \alpha \to \beta_1 \to \ldots \to \beta_m \to 0) \to (0 \to \alpha) \to 0 \to \beta_i$ to the language.
In addition to the proof given in [15] we only have to consider the functional interpretation $((F^X)')^D$ of the negative translation $(F^X)'$ of F^X : clearly $(F^X)'$ is intuitionistically implied by F^X so that it suffices to solve the functional interpretation $(F^X)^D$ of F^X. However, $(F^X)^D$ precisely asks for functionals $\mathcal{X},\underline{\mathcal{Z}}$ satisfying

$$\forall \Phi,y,k^0,x^\alpha,\underline{z}^{\underline{\beta}}(\mathcal{X}\Phi y \leq y \wedge (\Phi(k,\mathcal{X}\Phi yk,\underline{\mathcal{Z}}\Phi yk) \geq_0 \Phi(k,\min\nolimits_\alpha(x,yk),\underline{z})).$$

[6] Even the axiom of dependent choice can be dropped as it disappears during the interpretation. But this is not needed here.

But this is just what we provided for in $\mathcal{A}^\omega[X,d,\mathcal{X},\underline{\mathcal{Z}}]+\tilde{F}^X$.
The next step in the proof of theorem 3.7 in [15] consists in establishing that the model $\mathcal{M}^{\omega,X}$ of all strongly majorizable functionals over $\mathbb{N}$ and an arbitrary nonempty bounded metric space (X,d) is a model of $\mathcal{A}^\omega[X,d]^-$+(BR) and, moreover, that for any closed term t of $\mathcal{A}^\omega[X,d]^-$+(BR) one can construct a closed term t^* of $\mathcal{A}^\omega[X,d]^-$+(BR) which does not contain d_X such that

$$\mathcal{M}^{\omega,X} \models t^* \text{ s-maj } t.$$

We now extend this by showing that

$$\mathcal{M}^{\omega,X} \models \mathcal{A}^\omega[X,d,\mathcal{X},\underline{\mathcal{Z}}]^- + \tilde{F}^X + \text{(BR)}$$

for a suitable interpretation of the new constants $\mathcal{X}$ and $\underline{\mathcal{Z}}$ and that for any closed term t of $\mathcal{A}^\omega[X,d,\mathcal{X},\underline{\mathcal{Z}}]^-$+(BR) we can construct a closed term t^* of $\mathcal{A}^\omega[X,d]^-$+(BR) which does not contain d_X such that

$$\mathcal{M}^{\omega,X} \models t^* \text{ s-maj } t.$$

Note that t^* must not contain any of the constants $\mathcal{X},\underline{\mathcal{Z}}$:
We reason in $\mathcal{M}^{\omega,X}$. Let Φ,y,k be in $\mathcal{M}^{\omega,X}$ with types as above and let Φ^*,y^* be strong majorants for Φ,y in $\mathcal{M}^{\omega,X}$. Since $\min_\alpha(x,yk)\le_\alpha yk$ and $\underline{\beta}$ are types of degree $(\cdot,X)$ it follows (using the trivial definition of s-maj$_X$) that

$$y^*k \text{ s-maj } \min_\alpha(x,yk)\wedge z_i^* := \lambda\underline{v}.0_X \text{ s-maj } z_i$$

for all $k\in\mathbb{N}$ and all $x,\underline{z}$ in $\mathcal{M}^{\omega,X}$ of types α and $\underline{\beta}$ and suitable tuples of variables $\underline{v}$. Hence

$$\forall x\in M_\alpha^{\omega,X},\underline{z}\in M_{\underline{\beta}}^{\omega,X}(\Phi^*(k,y^*k,\underline{z}^*)\ge_0 \Phi(k,\min_\alpha(x,yk),\underline{z})).$$

Thus

$$Max_{\Phi,y,k} := \max\{\Phi(k,\min_\alpha(x,yk),\underline{z}) : x\in M_\alpha^{\omega,X}\wedge \underline{z}\in M_{\underline{\beta}}^{\omega,X}\}$$

exists (not that $M_\rho^{\omega,X}\neq\emptyset$ for all $\rho\in\mathbf{T}^X$) and hence

$$(+)\ \forall\Phi,y,k\in\mathcal{M}^{\omega,X}\exists x,\underline{z}\in\mathcal{M}^{\omega,X}(x\le_\alpha yk\wedge\Phi(k,x,\underline{z})=_0 Max_{\Phi,y,k}).$$

By the axiom of choice applied to (+) we obtain functionals $\Xi\le\lambda\Phi,y.y$ and $\underline{\Theta}$ such that $x:=\Xi\Phi yk$ and $\underline{z}:=\underline{\Theta}\Phi yk$ satisfy (+). We now put

$$[\mathcal{X}]_{\mathcal{M}^{\omega,X}} := \Xi \wedge [\underline{\mathcal{Z}}]_{\mathcal{M}^{\omega,X}} := \underline{\Theta}.$$

In order to show that $\Xi,\underline{\Theta}\in\mathcal{M}^{\omega,X}$ we construct closed terms $\mathcal{X}^*$ and $\underline{\mathcal{Z}}^*$ such that

$$\mathcal{M}^{\omega,X}\models \mathcal{X}^* \text{ s-maj } \mathcal{X}\wedge\underline{\mathcal{Z}}^* \text{ s-maj } \underline{\mathcal{Z}}.$$

The terms

$$\mathcal{X}^* := \lambda\Phi,y.y,\quad \mathcal{Z}_i^* := \lambda\underline{v}.0_X$$

do the job (using that $\mathcal{M}^{\omega,X}\models\mathcal{X}\le\lambda\Phi,y.y$) for a suitable tuple $\underline{v}$ of variables, where the length of the tuple and the types of its components only depend on β_i. It is clear that with this interpretation of $\mathcal{X},\underline{\mathcal{Z}}$ in $\mathcal{M}^{\omega,X}$ the axiom $\tilde{F}^X$ is

satisfied.
The construction of t^* from t now proceeds as in [15] with the additional clauses that all occurrences of $\mathcal{X}, \underline{\mathcal{Z}}$ are replaced by $\mathcal{X}^*, \underline{\mathcal{Z}}^*$. The rest of the proof is exactly as in [15].
2. and 3. are proved analogously. $\square$

Definition 3.8 The class $\mathcal{K}$ of formulas consists of all formulas F that are logically equivalent to a prenex normal form $F' \equiv \exists x_1^{\rho_1} \forall y_1^{\tau_1} \ldots \exists x_n^{\rho_n} \forall y_n^{\tau_n} F_\exists(\underline{x}, \underline{y})$ where $F_\exists$ is an $\exists$-formula, the types ρ_i are of degree 0 and the types τ_i are of degree 1 or $(1, X)$. If $\tau_i, \ldots, \tau_n$ are of degree $(1, X)$, then ρ_i might even be of degree $(\leq)1$ or $(0, X)$.

Corollary 3.9 1) Let A be a sentence in $\mathcal{K}$. If

$$\mathcal{A}^\omega[X, d] + \exists\text{-UB}^X \vdash A,$$

then A holds in any (non-empty) b-bounded metric space (X, d).
If A does not contain b_X, then it holds in any (non-empty) bounded metric space.

2) If the premise is proved in '$\mathcal{A}^\omega[X, d, W] + \exists\text{-UB}^X$' instead of '$\mathcal{A}^\omega[X, d] + \exists\text{-UB}^X$', then the conclusion holds in all (non-empty) b-bounded hyperbolic spaces.

3) If the premise is proved in '$\mathcal{A}^\omega[X, d, W, \mathrm{CAT}(0)] + \exists\text{-UB}^X$' instead of '$\mathcal{A}^\omega[X, d] + \exists\text{-UB}^X$', then the conclusion holds in all (non-empty) b-bounded CAT(0)-spaces.

Proof: [7] Let A be in prenex normal form of the form guaranteed by $A \in \mathcal{K}$. Consider the Herbrand normal form

$$A^H :\equiv \forall Y_1, \ldots, Y_n \exists x_1, \ldots, x_n \, A_\exists(x_1, \ldots, x_n, Y_1 x_1, \ldots, Y_n x_1 \ldots x_n)$$

of A. Since $A \to A^H$ holds by logic, the assumption implies that

$$\mathcal{A}^\omega[X, d] + \exists\text{-UB}^X \vdash A^H.$$

Since the types of $\underline{Y}$ are of degree 1 or $(1, X)$ and those of $\underline{x}$ of degree 1 or $(0, X)$ (and hence a-fortiori of degree 1 or $(1, X)$ so that
$\exists x_1, \ldots, x_n \, A_\exists(x_1, \ldots, x_n, Y_1 x_1, \ldots, Y_n x_1 \ldots x_n$ is an $\exists$-formula) we can apply theorem 3.5 to conclude that A^H holds in any (non-empty) b-bounded metric space (X, d). By logic and the axiom of choice A^H implies A so that the corollary follows.
2) and 3) are proved analogously. $\square$

4 Applications of $\exists$-UB^X

In this sections we focus on applications of $\exists$-UB^X involving the types $\underline{\beta}$. Of course, since $\exists$-UB^X entails Σ_1^0-UB, it also covers all the applications of the

[7] A similar argument is used already in [8].

latter (see [13]).

4.1 Application 1:

We now show that $\exists$-UB^X strengthens the assumption of separability of the bounded metric space (X,d) to total boundedness. This puts into focus the phenomenon implicit already in the counterexample presented in [15] (p.91) to the possibility of a metatheorem corresponding to theorem 3.5 for separable spaces.

Definition 4.1 Let (X,d) be a totally bounded metric space. A function $\alpha : \mathbb{N} \to \mathbb{N}$ satisfying

$$\exists (a_n)_n \text{ in } X \forall k \in \mathbb{N} \forall x \in X \exists n \leq \alpha(k)(d(x,a_n) < 2^{-k})$$

is called a modulus to total boundedness.

Proposition 4.2 $\mathcal{A}^\omega[X,d] + \exists$-UB^X *proves the following:*
'If (X,d) is separable, then (X,d) is totally bounded and has a modulus of total boundedness α'.
More precisely

$$\mathcal{A}^\omega[X,d] + \exists\text{-UB}^X \vdash \forall f^{0\to X}(\forall k^0, x^X \exists n^0(d_X(f(n),x) <_{\mathbb{R}} 2^{-k}) \to \exists \alpha^1 \forall k^0, x^X \exists n \leq \alpha(k)(d_X(f(n),x) <_{\mathbb{R}} 2^{-k})).$$

Proof: $\exists$-UB^X applied to

$$\forall k^0, x^X \exists n^0 (d_X(f(n),x) <_{\mathbb{R}} 2^{-k})$$

yields (noticing that '$d_X(f(n),x) <_{\mathbb{R}} 2^{-k}$' is an $\exists$-formula) that

$$\exists \alpha^1 \forall k^0, x^X \exists n \leq \alpha(k)(d_X(f(n),x) <_{\mathbb{R}} 2^{-k}).$$

$\square$

4.2 Application 2:

The next proposition shows that $\exists$-UB^X implies that every $f^{X\to X}$ that represents a function : $X \to X$, i.e. that respects $=_X$, is uniformly continuous with a modulus of uniform continuity ω. Moreover from the assumption that all $f^{X\to X}$ represent functions, $\exists$-UB^X allows one to derive that all functions $f : X \to X$ have a common modulus of uniform continuity. This corresponds to the counterexample from [15] (p.115) to the possibility to add full extensionality in theorem 3.5: If full extensionality is used in a proof it has to **follow** as a consequence of uniform equi-continuity in order to permit the extraction of uniform bounds. This is the case e.g. for nonexpansive functions but (as discussed in [15,8]) not for directionally nonexpansive or weakly quasi-nonexpansive functions where only the use of the quantifier-rule of extensionality is allowed.

Proposition 4.3 *Let*

$$Ext(f^{X\to X}) :\equiv \forall x^X, y^X (x =_X y \to f(x) =_X f(y)).$$

1) $\mathcal{A}^\omega[X,d] + \exists\text{-UB}^X$ *proves that*

$$\forall f^{X\to X}(Ext(f) \to$$
$$\exists \omega^1 \forall k^0, x^X, y^X (d_X(x,y) <_{\mathbb{R}} 2^{-\omega(k)} \to d_X(f(x),f(y)) <_{\mathbb{R}} 2^{-k})).$$

2) $\mathcal{A}^\omega[X,d] + \exists\text{-UB}^X$ *proves that*

$$\forall f^{X\to X}(Ext(f)) \to$$
$$\exists \omega^1 \forall f^{X\to X}, k^0, x^X, y^X (d_X(x,y) <_{\mathbb{R}} 2^{-\omega(k)} \to d_X(f(x),f(y)) <_{\mathbb{R}} 2^{-k}).$$

Proof: 1) By the definition of $=_X$, the assumption $Ext(f)$ can be written as

$$\forall x^X, y^X (\forall n^0 (d_X(x,y) \le_{\mathbb{R}} 2^{-n}) \to \forall k^0 (d_X(f(x),f(y)) <_{\mathbb{R}} 2^{-k}))$$

and hence as

$$(+)\ \forall x^X, y^X \forall k^0 \exists n^0 (d_X(x,y) \le_{\mathbb{R}} 2^{-n} \to d_X(f(x),f(y)) <_{\mathbb{R}} 2^{-k}),$$

where

$$d_X(x,y) \le_{\mathbb{R}} 2^{-n} \to d_X(f(x),f(y)) <_{\mathbb{R}} 2^{-k}$$

is (logically equivalent to) an $\exists$-formula. Hence $\exists\text{-UB}^X$ applied to (+) yields

$$\exists \omega^1 \forall k^0, x^X, y^X (d_X(x,y) \le_{\mathbb{R}} 2^{-\omega(k)} \to d_X(f(x),f(y)) <_{\mathbb{R}} 2^{-k})$$

which establishes the claim.
2) is proved analogously by applying $\exists\text{-UB}^X$ to

$$(++)\ \forall f^{X\to X} \forall x^X, y^X \forall k^0 \exists n^0 (d_X(x,y) \le_{\mathbb{R}} 2^{-n} \to d_X(f(x),f(y)) <_{\mathbb{R}} 2^{-k})$$

noticing that also the type $X \to X$ is admissible as a type β in $\exists\text{-UB}^X$. $\square$

4.3 Application 3:

The next application shows that $\exists\text{-UB}^X$ extends the usual WKL-applications for compact spaces and continuous functions to bounded spaces and arbitrary functions.

Proposition 4.4 *Let* β *be of degree* $(\cdot, X)$*. Then* $\mathcal{A}^\omega[X,d] + \exists\text{-UB}^X$ *proves the following*

$$\forall \Phi^{\beta\to 1}(\forall k^0 \exists y^\beta (|\Phi(y)|_{\mathbb{R}} <_{\mathbb{R}} 2^{-k}) \to \exists y^\beta (\Phi(y) =_{\mathbb{R}} 0)).$$

This also holds for tuples of variables $\underline{y}^{\underline{\beta}}$ *as long as the types* $\underline{\beta}$ *are all of degree* $(\cdot, X)$.

Proof: Suppose that

$$\forall y^\beta (\Phi(y) \ne_{\mathbb{R}} 0).$$

Then

$$\forall y^\beta \exists k^0 (|\Phi(y)|_{\mathbb{R}} >_{\mathbb{R}} 2^{-k})$$

and hence by $\exists$-UB^X

$$\exists k^0 \forall y^\beta (|\Phi(y)|_{\mathbb{R}} >_{\mathbb{R}} 2^{-k})$$

contradicting the assumption. □

4.4 Application 4:

In this application we show that $\exists$-UB^X allows one to make free use of fixed points of nonexpansive mappings in proofs (and still obtain correct $\mathcal{K}$-conclusions) despite the fact that such fixed points in general do not exist (not even for nonexpansive selfmappings of bounded, closed, convex subsets of Banach spaces such as c_0, see [22]).

Proposition 4.5 *$\mathcal{A}^\omega[X,d,W] + \exists$-UB^X proves the following*

$$\forall f^{X\to X}(f \text{ nonexpansive } \to \exists x^X (f(x) =_X x)).$$

Proof: Since $f(x) =_X x \leftrightarrow d_X(x, f(x)) =_{\mathbb{R}} 0$ we obtain from the previous application applied to $\Phi(x) := d_X(x, f(x))$ that it suffices to show

$$\forall k^0 \exists x^X (d_X(x, f(x)) <_{\mathbb{R}} 2^{-k}).$$

This, however, follows from theorem 1 in [9] whose proof can be formalized in $\mathcal{A}^\omega[X,d,W]$. □

Remark 4.6 The existence of approximate fixed points of nonexpansive mappings between bounded hyperbolic spaces used in the proof above rests strongly on the presence of the hyperbolic structure provided by W and is false for general bounded metric spaces: consider $\mathbb{R}$ equipped with the truncated metric $D(x,y) := \min(|x-y|, 1)$ and $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := x+1$. f is a nonexpansive (even isometric) selfmapping of the bounded metric space $(\mathbb{R}, D)$ but has no ε-fixed points for $0 < \varepsilon < 1$.

4.5 Application 5:

There are numerous fixed point theorems for various classes of functions of so-called contractive type (see [4,19,20,21]). Often compactness assumptions are used to ensure certain uniform versions of contractivity and the assumption of compactness can be replaced by boundedness if the functions are assumed to satisfy the uniform contractivity notions. In the cause of proof mining, the need to uniformize contractivity conditions on f has turned out to be crucial as well (see e.g. [7,6,1,2]). $\exists$-UB^X is a general tool for producing such uniformizations:

Definition 4.7 Let (X,d) be a metric space and $f : X \to X$ a selfmapping of X.

1) f is called contractive (see [4]) if
$$\forall x, y \in X(x \neq y \to d(f(x), f(y)) < d(x, y)).$$

2) f is called uniformly contractive with modulus $\alpha : \mathbb{N} \to \mathbb{N}$ (see [19]) if
$$\forall k \in \mathbb{N} \forall x, y \in X(d(x, y) > 2^{-k} \to d(f(x), f(y)) < (1 - 2^{-\alpha(k)})\, d(x, y)).$$

Proposition 4.8 *$\mathcal{A}^\omega[X, d] + \exists$-$\mathrm{UB}^X$ proves the following: 'every contractive mapping $f : X \to X$ is uniformly contractive with some modulus α'.*

Proof: Assume that
$$\forall x^X, y^X (x \neq_X y \to d_X(f(x), f(y)) <_{\mathbb{R}} d_X(x, y)).$$
Then
$$\forall x^X, y^X, k^0 \exists n^0 (d_X(x, y) \geq_{\mathbb{R}} 2^{-k} \to d_X(f(x), f(y)) <_{\mathbb{R}} (1 - 2^{-n}) d_X(x, y)),$$
where
$$d_X(x, y) \geq_{\mathbb{R}} 2^{-k} \to d_X(f(x), f(y)) <_{\mathbb{R}} (1 - 2^{-n}) d_X(x, y)$$
is an $\exists$-formula. Hence $\exists$-UB^X yields
$$\exists \alpha^1 \forall k^0, x^X, y^X (d_X(x, y) \geq_{\mathbb{R}} 2^{-k} \to d_X(f(x), f(y)) <_{\mathbb{R}} (1 - 2^{-\alpha(k)}) d_X(x, y)).$$
$\square$

In a similar way, $\exists$-UB^X implies corresponding uniform versions of many other more liberal notions of contractivity (see [20,21] and [1,2]).

References

[1] Briseid, E.M., Proof mining applied to fixed point theorems for mappings of contractive type. Master Thesis, Oslo, 70pp., 2005.

[2] Briseid, E.M., Fixed points of generalized contractive mappings. Preprint 18 pp. (2006), submitted.

[3] Diller, J, Vogel, H., Intensionale Funktionalinterpretation der Analysis. In: Diller, J., Müller (eds.), Proof Theory Symposium, Kiel 1974. Springer LNM **500**, pp. 56-72 (1975).

[4] Edelstein, M., On fixed and periodic points under contractive mappings. J. London Math. Soc. **37**, pp. 74–79 (1962).

[5] Ferreira, F., Oliva, P., Bounded functional interpretation. Ann. Pure Appl. Logic **135**, pp. 73-112 (2005).

[6] Gerhardy, P., A quantitative version of Kirk's fixed point theorem for asymptotic contractions. J. Math. Anal. Appl. **316**, pp. 339-345 (2006).

[7] Gerhardy, P., Kohlenbach, U., Strongly uniform bounds from semi-constructive proofs. Ann. Pure Appl. Logic vol. 141, pp. 89-107 (2006).

[8] Gerhardy, P., Kohlenbach, U., General logical metatheorems for functional analysis. BRICS Research Report RS-05-21, 65pp., 2005 (submitted).

[9] Goebel, K., Kirk, W.A., Iteration processes for nonexpansive mappings. In: Singh, S.P., Thomeier, S., Watson, B., eds., Topological Methods in Nonlinear Functional Analysis. Contemporary Mathematics **21**, AMS, pp. 115-123 (1983).

[10] Kirk, W.A., Sims, B. (eds.), Handbook of Metric Fixed Point Theory. Kluwer Academic Publishers, Dordrecht, Boston, London, xi+703pp. (2001).

[11] Kohlenbach, U., Analysing proofs in analysis. In: W. Hodges, M. Hyland, C. Steinhorn, J. Truss, editors, *Logic: from Foundations to Applications. European Logic Colloquium* (Keele, 1993), pp. 225–260, Oxford University Press (1996).

[12] Kohlenbach, U., Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals. Arch. Math. Logic **36**, pp. 31–71 (1996).

[13] Kohlenbach, U., The use of a logical principle of uniform boundedness in analysis. In: Cantini, A., Casari, E., Minari, P. (eds.), Logic and Foundations of Mathematics. Synthese Library **280**, pp. 93-106. Kluwer Academic Publishers (1999).

[14] Kohlenbach, U., Foundational and mathematical uses of higher types. In: Sieg, W., Sommer, R., Talcott, C. (eds.), Reflections on the foundations of mathematics. Essays in honor of Solomon Feferman, Lecture Notes in Logic **15**, A.K. Peters, pp. 92-120 (2002).

[15] Kohlenbach, U., Some logical metatheorems with applications in functional analysis. Trans. Amer. Math. Soc. **357**, no. 1, pp. 89-128 (2005) [For some minor errata see the end of [7]]

[16] Kohlenbach, U., Proof Interpretations and the Computational Content of Proofs. Draft February 2006. ii+412pp. (electronically available at: http://www.mathematik.tu-darmstadt.de/ kohlenbach).

[17] Leuştean, L., Proof mining in $\mathbb{R}$-trees and hyperbolic spaces. To appear in: this volume.

[18] Luckhardt, H., Extensional Gödel Functional Interpretation. Springer Lecture Notes in Mathematics **306**, 1973.

[19] Rakotch, E., A note on contractive mappings. Proc. Amer. Math. Soc. **13**, pp. 459–465 (1962).

[20] Rhoades, B.E., A comparison of various definitions of contractive mappings. Trans. Amer. Math. Soc. **226**, pp. 257–290 (1977)

[21] Rhoades, B.E., Contractive definitions. In: Rassias, Th.,M. editor, Nonlinear analysis, pp. 513–526. World Sci. Publishing, Singapore, 1987.

[22] Sims, B., Examples of fixed point free mappings. In: [10], pp. 35-48 (2001).

[23] Spector, C., Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics. In: Recursive function theory, Proceedings of Symposia in Pure Mathematics, vol. 5 (J.C.E. Dekker (ed.)), AMS, Providence, R.I., pp. 1–27 (1962).

[24] Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes in Mathematics **344** (1973).