

Pricing of high-dimensional American options by neural networks

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December 21, 2006

Abstract

Pricing of American options in discrete time is considered, where the option is allowed to be based on several underlyings. It is assumed that the price processes of the underlyings are given Markov processes. We use the Monte Carlo approach to generate artificial sample paths of these price processes, and then we use the least squares neural networks regression estimates to estimate from this data the so-called continuation values, which are defined as mean values of the American options for given values of the underlyings at time t subject to the constraint that the options are not exercised at time t . Results concerning consistency and rate of convergence of the estimates are presented, and the pricing of American options is illustrated by simulated data.

AMS classification: Primary 91B28, 60G40, 93E20; secondary 65C05, 93E24, 62G05.

Key words and phrases: American options, consistency, neural networks, nonparametric regression, optimal stopping, rate of convergence, regression based Monte Carlo methods.

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1 Introduction

In this article we consider American options in discrete time. The price V_0 of such options can be defined as a solution of an optimal stopping problem

$$V_0 = \sup_{\tau \in \mathcal{T}(0, \dots, T)} \mathbf{E} \{f_\tau(X_\tau)\}. \quad (1)$$

Here f_t is the (discounted) payoff function, X_0, X_1, \dots, X_T is the underlying stochastic process describing e.g. the prices of the underlyings and the financial environment (like interest rates, etc.) and $\mathcal{T}(0, \dots, T)$ is the class of all $\{0, \dots, T\}$ -valued stopping times, i.e., $\tau \in \mathcal{T}(0, \dots, T)$ is a measurable function of X_0, \dots, X_T satisfying

$$\{\tau = \alpha\} \in \mathcal{F}(X_0, \dots, X_\alpha) \quad \text{for all } \alpha \in \{0, \dots, T\}.$$

As a simple example consider pricing of an American put option with strike K on the arithmetic mean of several correlated underlyings, where the stock values are modelled via Black-Scholes theory by

$$X_{i,t} = x_{i,0} \cdot e^{r \cdot t} \cdot e^{\sum_{j=1}^m (\sigma_{i,j} \cdot W_j(t) - \frac{1}{2} \sigma_{i,j}^2 t)} \quad (i = 1, \dots, m). \quad (2)$$

Here $r > 0$ is the (given) riskless interest rate, $\sigma_i = (\sigma_{i,1}, \dots, \sigma_{i,m})^T$ is the (given) volatility of the i -th stock, $x_{i,0}$ is the initial stock price of the i -th stock, and $\{W_j(t) : t \in \mathbb{R}_+\}$ ($j = 1, \dots, m$) are independent Wiener processes.

If we sell the option at time $t > 0$ and the stock prices are at this point $x = (x_1, \dots, x_m)$ (i.e., the arithmetic mean of the stock prices is $\frac{1}{m} \sum_{j=1}^m x_j$), we get the payoff

$$\max \left\{ K - \frac{1}{m} \sum_{j=1}^m x_j, 0 \right\},$$

and if we discount this payoff towards time zero, we get the discounted payoff function

$$f_t(x_1, \dots, x_m) = e^{-r \cdot t} \cdot \max \left\{ K - \frac{1}{m} \sum_{j=1}^m x_j, 0 \right\}. \quad (3)$$

But even if all the parameters are known (i.e., if $x_{i,0}$ ($i = 1, \dots, m$) and K are given and if we estimate the volatilities σ_i ($i = 1, \dots, m$) and the riskless interest rate from observed data from the past), it is not obvious how we can compute the price

$$V_0 = \sup_{\tau \in \mathcal{T}(0, \dots, T)} \mathbf{E} \left\{ e^{-r \cdot \tau} \cdot \max \left\{ K - \frac{1}{m} \sum_{i=1}^m X_{i,\tau}, 0 \right\} \right\}$$

of the corresponding American option.

In the above Black-Scholes model we can reformulate the whole problem as a free boundary problem for partial differential equations (cf., e.g., Chapter 8 in Elliott and Kopp (1999)), but the numerical solution of this free boundary problem gets very complicated if the number m of underlyings gets large. In addition, for $m \leq 2$ binomial trees (cf., e.g., Chapter 1 in Elliott and Kopp (1999)) are able to produce very good estimates of V_0 , but for $m > 3$ it is with this method basically impossible to model the correlation structure of the stocks correctly.

The purpose of this article is to develop an Monte Carlo algorithm which is able to compute an approximation of the price (1) even in case that the option is based on a large number of correlated stocks, that the stock prices are not modelled by a simple Black-Scholes model as in (2) and that the payoff function is not as simple as in (3). In particular the method developed in this article is also applicable in case that the process $X_{i,t}$ are adjusted to observed data by time series estimation as described, e.g., in Franke and Diagne (2002).

In the sequel we assume that X_0, X_1, \dots, X_T is a \mathbb{R}^d -valued Markov process recording all necessary information about financial variables including prices of the underlying assets as well as additional risk factors driving stochastic volatility or stochastic interest rates. Neither the Markov property nor the form of the payoff as a function of the state X_t is restrictive and can always be achieved by including supplementary variables.

The computation of (1) can be done by determination of an optimal stopping rule $\tau^* \in \mathcal{T}(0, \dots, T)$ satisfying

$$V_0 = \mathbf{E}\{f_{\tau^*}(X_{\tau^*})\}. \quad (4)$$

Let

$$q_t(x) = \sup_{\tau \in \mathcal{T}(t+1, \dots, T)} \mathbf{E}\{f_{\tau}(X_{\tau}) | X_t = x\} \quad (5)$$

be the so-called continuation value describing the value of the option at time t given $X_t = x$ and subject to the constraint of holding the option at time t rather than exercising it. Here $\mathcal{T}(t+1, \dots, T)$ is the class of all $\{t+1, \dots, T\}$ -valued stopping times. It can be shown that

$$\tau^* = \inf\{s \geq 0 : q_s(X_s) \leq f_s(X_s)\} \quad (6)$$

satisfies (4), i.e., τ^* is an optimal stopping time (cf., e.g., Chow, Robbins and Siegmund (1971) or Shiriyayev (1978)). Therefore it suffices to compute the continuation values (5) in order to solve the optimal stopping problem (1).

The continuation values satisfy the dynamic programming equations

$$\begin{aligned} q_T(x) &= 0, \\ q_t(x) &= \mathbf{E} \{ \max \{ f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1}) \} | X_t = x \} \quad (t = 0, 1, \dots, T-1). \end{aligned} \quad (7)$$

Indeed, by analogy to (6) we have

$$q_t(x) = \mathbf{E} \{ f_{\tau_t^*}(X_{\tau_t^*}) | X_t = x \} \quad \text{where} \quad \tau_t^* = \inf \{ s \geq t+1 | q_s(X_s) \leq f_s(X_s) \},$$

hence by using the Markov property of $\{X_s\}_{s=0, \dots, T}$ we get

$$\begin{aligned} q_t(X_t) &= \mathbf{E} \left\{ f_{t+1}(X_{t+1}) \cdot I_{\{q_{t+1}(X_{t+1}) \leq f_{t+1}(X_{t+1})\}} + f_{\tau_{t+1}^*}(X_{\tau_{t+1}^*}) \cdot I_{\{q_{t+1}(X_{t+1}) > f_{t+1}(X_{t+1})\}} | X_t \right\} \\ &= \mathbf{E} \{ \mathbf{E} \{ \dots | X_0, \dots, X_{t+1} \} | X_0, \dots, X_t \} \\ &= \mathbf{E} \{ \max \{ f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1}) \} | X_t \}. \end{aligned}$$

Unfortunately, the conditional expectation in (7) in general cannot be computed in applications. The basic idea of regression-based Monte Carlo methods for pricing American options is to apply recursively regression estimates to artificially created samples of

$$(X_t, \max \{ f_{t+1}(X_{t+1}), \hat{q}_{t+1}(X_{t+1}) \})$$

(so-called Monte Carlo samples) to construct estimates \hat{q}_t of q_t . In connection with linear regression this was proposed in Tsitsiklis and Van Roy (1999), and, based on a different regression estimation than (7), in Longstaff and Schwartz (2001). Nonparametric least squares regression estimates have been applied and investigated in this context in Egloff (2005) and Egloff, Kohler and Todorovic (2006), smoothing spline regression estimates have been analyzed in this context in Kohler (2006b), recursive kernel regression estimates have been considered in Barty et al. (2006).

In this article we propose to use least squares neural networks regression estimates in order to compute the conditional expectations in (7), which is particularly promising for options based on several underlyings, where high-dimensional regression problems have

to be solved in order to compute approximations of the continuation values. Due to the well-known curse of dimensionality it is difficult to choose here a reasonable nonparametric regression estimate, and neural networks belong (together with regression trees (cf., e.g., Breiman et al. (1984)) or interaction models (cf., e.g., Stone (1994))) to standard estimates in this field.

Below we define least squares neural networks regression estimates of the continuation values where all parameters of the estimates are chosen using the given data only. We will show that these estimates are universally consistent in the sense that their L_2 errors converge to zero in probability for all distributions. Furthermore, under regularity conditions on the smoothness of the continuation values we will analyze the rate of convergence of the estimates. Finally, we will illustrate the estimates by applying them to simulated data.

The precise definition of the estimates and the main theoretical results concerning consistency and rate of convergence of the estimate are given in Sections 2 and 3, respectively. The application of the estimates to simulated data will be described in Section 4, and the proofs will be given in Section 5.

2 Definition of the estimate

Let $\sigma : \mathbb{R} \rightarrow [0, 1]$ be a sigmoid function, i.e., assume that σ is monotonically increasing and satisfies

$$\sigma(x) \rightarrow 0 \quad (x \rightarrow -\infty) \quad \text{and} \quad \sigma(x) \rightarrow 1 \quad (x \rightarrow \infty).$$

An example of such a sigmoid function is the logistic squasher defined by $\sigma(x) = \frac{1}{1+e^{-x}}$ ($x \in \mathbb{R}$). In the sequel we estimate the continuation values by neural networks with $k \in \mathbb{N}$ hidden neurons and a sigmoid function σ . We will use the principle of least squares to fit such a function to the data, and for technical reasons we will restrict the sum of the absolute values of the output weights. The choice of number k of hidden neurons will be data-driven by using sample splitting.

Let $\beta_n > 0$ (which we will choose later such that $\beta_n \rightarrow \infty$ ($n \rightarrow \infty$)) and let $\mathcal{F}_k(\beta_n)$

be a class of neural networks defined by

$$\mathcal{F}_k(\beta_n) = \left\{ \sum_{i=1}^k c_i \cdot \sigma(a_i^T x + b_i) + c_0 \quad : \quad a_i \in \mathbb{R}^d, b_i \in \mathbb{R}, \sum_{i=0}^k |c_i| \leq \beta_n \right\} \quad (8)$$

where σ is the sigmoid function from above.

In the sequel we describe an algorithm to estimate the continuation values q_t recursively. To do this we generate artificial independent Markov processes $\{X_{i,t}^{(l)}\}_{t=0,\dots,T}$ ($l = 0, 1, \dots, T-1$, $i = 1, 2, \dots, n$) which are identically distributed as $\{X_t\}_{t=0,\dots,T}$. Then we use these so-called Monte Carlo samples to generate recursively data to estimate q_t by using the regression representation given in (7).

We start with

$$\hat{q}_{n,T}(x) = 0 \quad (x \in \mathbb{R}^d).$$

Fix $t \in \{0, 1, \dots, T-1\}$. Given an estimate $\hat{q}_{n,t+1}$ of q_{t+1} , we estimate

$$q_t(x) = \mathbf{E}\{\max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\} | X_t = x\}$$

by applying a neural networks regression estimate to an 'approximative' sample of

$$(X_t, \max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\}).$$

With the notation

$$\hat{Y}_{i,t}^{(t)} = \max\{f_{t+1}(X_{i,t+1}^{(t)}), \hat{q}_{n,t+1}(X_{i,t+1}^{(t)})\}$$

(where we have suppressed the dependency of $\hat{Y}_{i,t}^{(t)}$ on n) this 'approximative' sample is given by

$$\left\{ \left(X_{i,t}^{(t)}, \hat{Y}_{i,t}^{(t)} \right) : i = 1, \dots, n \right\}. \quad (9)$$

Observe that this sample depends on the t -th sample of $\{X_s\}_{s=0,\dots,T}$ and $\hat{q}_{n,t+1}$, i.e., for each time step t we use a new sample of the stochastic process $\{X_s\}_{s=0,\dots,T}$ in order to define our data (9).

To choose parameter k of the neural networks regression estimate fully automatically we use splitting of the sample. Thus we subdivide (9) in a learning sample of size $n_l = \lceil n/2 \rceil$ and a testing sample of size $n_t = n - n_l$ and define for a given $k \in \mathcal{P}_n = \{1, \dots, n\}$ a regression estimate of q_t by

$$\hat{q}_{n_l,t}^k(\cdot) = \arg \min_{f \in \mathcal{F}_k(\beta_n)} \left(\frac{1}{n_l} \sum_{i=1}^{n_l} |f(X_{i,t}^{(t)}) - \hat{Y}_{i,t}^{(t)}|^2 \right), \quad (10)$$

where $z = \arg \min_{x \in \mathcal{D}} f(x)$ is an abbreviation for $z \in \mathcal{D}$ and $f(z) = \min_{x \in \mathcal{D}} f(x)$. Here we assume for simplicity that the above minima exist, however we do not require them to be unique.

Then we minimize the empirical L_2 risk on the testing sample in order to choose the value of parameter k . So we choose

$$\hat{k} = \arg \min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n_l,t}^k(X_{i,t}^{(t)}) - \hat{Y}_{i,t}^{(t)}|^2 \quad (11)$$

and define our final neural networks regression estimate of q_t by

$$\hat{q}_{n,t}(x) = \hat{q}_{n_l,t}^{\hat{k}}(x) \quad (x \in \mathbb{R}^d). \quad (12)$$

3 Theoretical results

We say that $a_n = O_{\mathbf{P}}(b_n)$ if $\limsup_{n \rightarrow \infty} \mathbf{P}(a_n > c \cdot b_n) = 0$ for some finite constant c . Our main theoretical result is the following theorem.

Theorem 1 *Let $L > 0$. Assume that X_0, X_1, \dots, X_T is a \mathbb{R}^d -valued Markov process and that the discounted payoff function f_t is bounded in absolute value by L . Define the estimate $\hat{q}_{n,t}$ by (10), (11) and (12) for some $\beta_n > 0$. Let $k_n \in \mathcal{P}_n$ and assume that k_n, β_n satisfy*

$$\beta_n \rightarrow \infty \quad (n \rightarrow \infty), \quad k_n \rightarrow \infty \quad (n \rightarrow \infty), \quad \frac{\beta_n^4 \cdot k_n \cdot \log n}{n} \rightarrow 0 \quad (n \rightarrow \infty).$$

Then

$$\begin{aligned} & \int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\ &= O_{\mathbf{P}} \left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s \in \{t, t+1, \dots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \right) \end{aligned}$$

for all $t \in \{0, 1, \dots, T\}$.

As a first consequence we get consistency of the estimate.

Corollary 1 *Let $L > 0$. Assume that X_0, X_1, \dots, X_T is a \mathbb{R}^d -valued Markov process and that the discounted payoff function f_t is bounded in absolute value by L , i.e.,*

$$|f_t(x)| \leq L \text{ for } x \in \mathbb{R}^d \text{ and } t \in \{0, 1, \dots, T\}. \quad (13)$$

Define the estimate $\hat{q}_{n,t}$ by (10),(11) and (12). Let $\beta_n > 0$ and assume that β_n satisfies

$$\beta_n \rightarrow \infty \quad (n \rightarrow \infty), \quad \frac{\beta_n^4 \cdot \log n}{n} \rightarrow 0 \quad (n \rightarrow \infty).$$

Then

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \rightarrow 0 \quad \text{in probability}$$

for all $t \in \{0, 1, \dots, T\}$.

Proof of Corollary 1. Because of the conditions of Corollary 1 we can choose $k_n \in \mathcal{P}_n$ such that $k_n \rightarrow \infty$ ($n \rightarrow \infty$) and $\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} \rightarrow 0$ ($n \rightarrow \infty$). By Theorem 1 we get

$$\begin{aligned} & \int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\ &= O_{\mathbf{P}} \left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s \in \{t, t+1, \dots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \right) \end{aligned}$$

for all $t \in \{0, 1, \dots, T\}$. Condition (13) implies that q_t is bounded, hence we get by Lemma 16.2 in Györfi et al. (2002)

$$\max_{s \in \{t, t+1, \dots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \rightarrow 0 \quad (n \rightarrow \infty).$$

□

The above corollary shows that the L_2 error of our estimate converges to zero in probability for sample size of the Monte Carlo sample tending to infinity. In view of an application with necessarily finite sample size it would be nice to know how quickly the error converges to zero for sample size tending to infinity. It is well-known in nonparametric regression that assumptions on the underlying distribution, in particular on the smoothness of the regression function, are necessary in order to be able to derive non-trivial rates of convergence results (see, e.g., Cover (1968), Devroye (1982) or Chapter 3 in Györfi et al. (2002)). For our neural networks estimate we restrict the smoothness of the continuation values by imposing constraints on their Fourier transformation (see below).

In addition we assume that the stochastic process is bounded. Usually in modelling of financial processes one models them by unbounded processes. In this case we choose a large value $A > 0$ and replace X_t by its bounded approximation

$$X_t^A = X_{\min\{t, \tau_A\}} \quad \text{where} \quad \tau_A = \inf\{s \geq 0 : X_s \notin [-A, A]^d\}.$$

(Here we assume for simplicity that the stochastic process has continuous paths in order to be able to neglect an additional truncation of X_t^A). This boundedness assumption enables us to estimate the price of the American option from samples of polynomial size in the number of free parameters, in contrast to Monte Carlo estimation from standard (unbounded) Black-Scholes models, where Glasserman and Yu (2004) showed that samples of exponential size in the number of free parameters are needed.

Next we analyze the rate of convergence of the estimate. To this end we need to introduce the class of functions having Fourier transform with the first absolute moment finite. The Fourier transform \tilde{F} of a function $f \in L_1(\mathbb{R}^d)$ is defined by

$$\tilde{F}(\omega) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\omega^T x} f(x) dx \quad (\omega \in \mathbb{R}^d).$$

If $\tilde{F} \in L_1(\mathbb{R}^d)$ then the inverse formula

$$f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\omega^T x} \tilde{F}(\omega) d\omega \quad (14)$$

holds almost everywhere with respect to the Lebesgue measure. Let $0 < C < \infty$ and consider the class of functions \mathcal{F}_C for which (14) holds on \mathbb{R}^d and, in addition,

$$\int_{\mathbb{R}^d} \|\omega\| F(\omega) d\omega \leq C. \quad (15)$$

A class of functions satisfying (15) is a subclass of functions with Fourier transform having first absolute moment finite, i.e., $\int_{\mathbb{R}^d} \|\omega\| F(\omega) d\omega < \infty$ (these functions are continuously differentiable on \mathbb{R}^d). The next corollary provides the rate of convergence of the estimate.

Corollary 2 *Let $L > 0$. Assume that X_0, X_1, \dots, X_T is a \mathbb{R}^d -valued Markov process, $X_t \in [-A, A]^d$ almost surely for some $A > 0$ and all $t \in \{0, 1, \dots, T\}$, that the discounted payoff function f_t is bounded in absolute value by L , i.e.,*

$$|f_t(x)| \leq L \text{ for } x \in \mathbb{R}^d \text{ and } t \in \{0, 1, \dots, T\},$$

and that the Fourier transform \tilde{Q}_t of q_t satisfies (14) and (15) for all $x \in \mathbb{R}^d$ and all $t \in \{0, \dots, T\}$. Let $\beta_n = \text{const} \cdot \log n$ and define the estimate $\hat{q}_{n,t}$ by (10), (11) and (12).

Then

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}} \left(\left(\frac{\log^5 n}{n} \right)^{1/2} \right)$$

for all $t \in \{0, 1, \dots, T\}$.

Proof of Corollary 2. Set $k_n = \left(\frac{n}{\log^5 n}\right)^{1/2}$. From Lemma 16.8 in Györfi et al. (2002) we have for n sufficiently large

$$\max_{s \in \{t, t+1, \dots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \leq \frac{(2\sqrt{d}AC)^2}{k_n}.$$

Then Theorem 1 implies

$$\begin{aligned} & \int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\ &= O_{\mathbf{P}} \left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s \in \{t, t+1, \dots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \right) \\ &= O_{\mathbf{P}} \left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \frac{(2\sqrt{d}AC)^2}{k_n} \right) \\ &= O_{\mathbf{P}} \left(\sqrt{\frac{\log^5 n}{n}} \right) \end{aligned}$$

for all $t \in \{0, 1, \dots, T\}$. □

Remark. Assume $X_0 = x_0$ a.s. for some $x_0 \in \mathbb{R}$. We can estimate the price

$$V_0 = \max\{f_0(x_0), q_0(x_0)\}$$

(cf., (1) and (5)) of the American option by

$$\hat{V}_0 = \max\{f_0(x_0), \hat{q}_{n,0}(x_0)\}.$$

Since the distribution of X_0 is concentrated on x_0 , under the assumptions of Corollary 2 we have the following error bound:

$$\begin{aligned} |\hat{V}_0 - V_0|^2 &= |\max\{f_0(x_0), \hat{q}_{n,0}(x_0)\} - \max\{f_0(x_0), q_0(x_0)\}|^2 \\ &\leq |\hat{q}_{n,0}(x_0) - q_0(x_0)|^2 \\ &= O_{\mathbf{P}} \left(\left(\frac{\log^5 n}{n} \right)^{1/2} \right). \end{aligned}$$

4 Application to simulated data

In this section, we illustrate the finite sample behavior of our algorithm by comparing it with the Tsitsiklis–Van Roy algorithm and Longstaff–Schwartz algorithm proposed by Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001), respectively.

We simulate the paths of the underlying stocks with a simple Black-Scholes model. The time to maturity is assumed to be one year. We discretize the time interval $[0, 1]$ by dividing it into m equidistant time steps with $t_0 = 0 < t_1 < \dots < t_m = 1$. In the first two examples we consider an option on a single stock. The prices of the underlying stock at time points t_j ($j = 0, \dots, m$) are then given by

$$X_{i,t_j} = x_0 \cdot \exp\left((r - 1/2 \cdot \sigma^2) \cdot t_j + \sigma \cdot W_{t_j}\right) \quad (i = 1, \dots, n, j = 0, \dots, m).$$

We choose $x_0 = 100$, $r = 0.05$, $m = 12$ and discount factors e^{-rt_j} for $j = 0, \dots, m$. For our algorithm we use sample size of 2000 while for the other algorithms sample size of 10000.

For our algorithm we set the number of learning and training samples to $n_l = n_t = 1000$. To simplify the implementation we select the k hidden neurons by sample splitting (as described in Section 2) from the set $\{2^0, 2^1, \dots, 2^5\}$. The neural networks least squares estimate is computed approximately by backpropagation (i.e., by gradient descent). For the Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms we use polynomials of degree 3 in the one-dimensional case and degree 1 in the high-dimensional case, since these choices yield the best results.

We apply all three algorithms to 100 independently generated sets of paths. We would like to stress that all three algorithms provide lower bounds to the optimal stopping value. Since we evaluate the approximative optimal stopping rule on newly generated data, a higher MCE indicates a better performance of the algorithm. We compare the algorithms using boxplots. Observe that the higher the boxplot of the MCE the better the performance of the corresponding algorithm.

In our first example we analyze a standard put-payoff with exercise price 90 as illustrated in Figure 1, and simulate the paths of the underlying stock with a volatility of $\sigma = 0.25$. As we can see from Figure 2, our algorithm is slightly better than the Longstaff–Schwartz algorithm and comparable to the algorithm of Tsitsiklis–Van Roy. This is not surprising, since it is well known that for simple payoff functions the Longstaff–Schwartz as well as the Tsitsiklis–Van Roy algorithms perform very well.

In our second example we make the pricing problem more difficult. We consider $m = 48$ time steps, a strangle spread payoff with strikes 50, 90, 110 and 150 as illustrated in Figure 3, and a large volatility of $\sigma = 0.5$. Figure 4 shows that our algorithm provides a higher

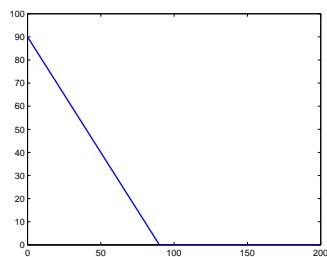


Figure 1: Put-payoff with exercise price 90.

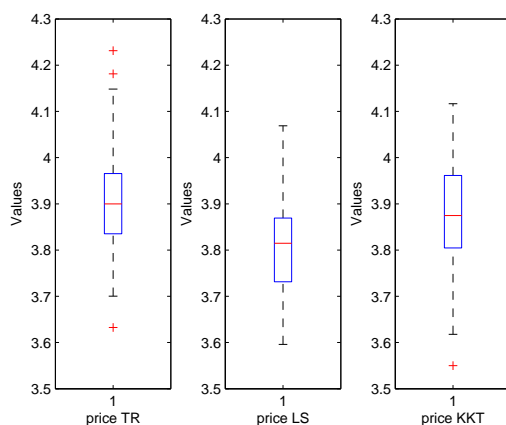


Figure 2: Boxplots for the realized option prices for the put-payoff of the Tsitsiklis–Van Roy (price TR), Longstaff–Schwartz (price LS) algorithms and our algorithm (price KKT). In the boxplot the box stretches from the 25th percentile to the 75th percentile and the median is shown as a line across the box.

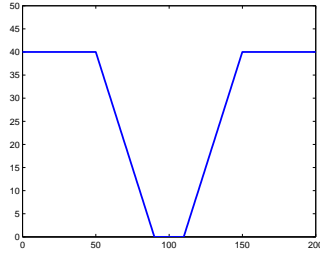


Figure 3: Strangle spread payoff with strike prices 50, 90, 110 and 150.

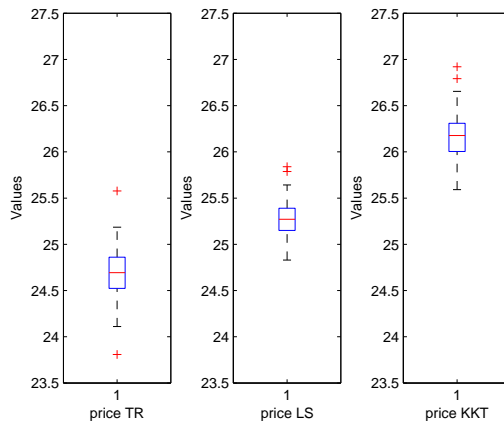


Figure 4: Realized option prices for the strangle spread-payoff of the Tsitsiklis–Van Roy (price TR), Longstaff–Schwartz (price LS) and our algorithm (price KKT) in a 1-dimensional case.

MCE of the option price than Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms.

Finally, in our third example we consider the high-dimensional case and use for the pricing problem a strangle spread function with strikes 75, 90, 110 and 125 for the average of five correlated stock prices. The stocks are ADECCO R, BALOISE R, CIBA, CLARIANT and CREDIT SUISSE R. The stock prices were observed from Nov. 10, 2000 until Oct. 3, 2003 on weekdays when the stock market was open for the total of 756 days. We

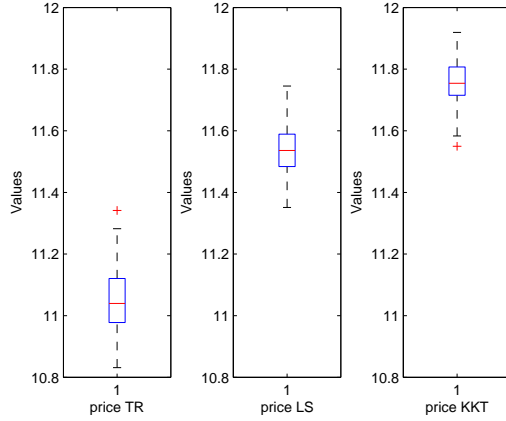


Figure 5: Realized option prices for the strangle spread-payoff of the Tsitsiklis–Van Roy (price TR), Longstaff–Schwartz (price LS) and our algorithm (price KKT) in a 5-dimensional case.

estimate the volatility from data observed in the past by the historical volatility

$$\sigma = \begin{pmatrix} 0.3024 & 0.1354 & 0.0722 & 0.1367 & 0.1641 \\ 0.1354 & 0.2270 & 0.0613 & 0.1264 & 0.1610 \\ 0.0722 & 0.0613 & 0.0717 & 0.0884 & 0.0699 \\ 0.1367 & 0.1264 & 0.0884 & 0.2937 & 0.1394 \\ 0.1641 & 0.1610 & 0.0699 & 0.1394 & 0.2535 \end{pmatrix}.$$

Again we used $x_0 = 100$, $r = 0.05$ and $m = 48$. As we can see in Figure 5, our algorithm is superior to Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms, since the higher boxplot of the MCE again indicates better performance.

5 Proofs

5.1 Auxiliary results

In the sequel we formulate auxiliary results which will be needed in the derivation of the rate of convergence. We start by defining so-called covering numbers:

Let $x_1, \dots, x_n \in \mathbb{R}^d$ and set $x_1^n = (x_1, \dots, x_n)$. Define the distance $d_2(f, g)$ between

$f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$d_2(f, g) = \left(\frac{1}{n} \sum_{i=1}^n |f(x_i) - g(x_i)|^2 \right)^{1/2}.$$

An ϵ -cover of \mathcal{F} (w.r.t. the distance d_2) is a set of functions $f_1, \dots, f_\kappa : \mathbb{R}^d \rightarrow \mathbb{R}$ with the property

$$\min_{1 \leq j \leq \kappa} d_2(f, f_j) < \epsilon \quad \text{for all } f \in \mathcal{F}.$$

Let $\mathcal{N}_2(\epsilon, \mathcal{F}, x_1^n)$ denote the size κ of the smallest ϵ -cover of \mathcal{F} w.r.t. the distance d_2 , and set $\mathcal{N}_2(\epsilon, \mathcal{F}, x_1^n) = \infty$ if there does not exist any ϵ -cover of \mathcal{F} of a finite size. $\mathcal{N}_2(\epsilon, \mathcal{F}, x_1^n)$ is called L_2 - ϵ -covering number of \mathcal{F} on x_1^n .

In the appendix we will prove the following bound on the covering number of $\mathcal{F}_k(\beta_n)$, where $\mathcal{F}_k(\beta_n)$ is defined by (8).

Lemma 1 *Let $\mathcal{F}_k(\beta_n)$ be defined by (8), let $\epsilon > 0$ and let $x_1^n \in (\mathbb{R}^d)^n$. Then*

$$\mathcal{N}_2(\epsilon, \mathcal{F}_k(\beta_n), x_1^n) \leq \left(\frac{12e\beta_n(k+1)}{\epsilon} \right)^{(4d+9)k+1}. \quad (16)$$

In the proof we will use results concerning regression estimation in case of additional measurement errors in the dependent variable, which we describe in the sequel.

Let $(X, Y), (X_1, Y_1), \dots$ be independent and identically distributed $\mathbb{R}^d \times \mathbb{R}$ valued random variables with $\mathbf{E}Y^2 < \infty$. Let $m(x) = \mathbf{E}\{Y|X = x\}$ be the corresponding regression function. Assume that we want to estimate m from observed data, but instead of a sample

$$\mathcal{D}_n = \{(X_1, Y_1), \dots, (X_n, Y_n)\}$$

of (X, Y) we have only available a set of data

$$\bar{\mathcal{D}}_n = \{(X_1, \hat{Y}_{1,n}), \dots, (X_n, \hat{Y}_{n,n})\}$$

where the only assumption on $\hat{Y}_{1,n}, \dots, \hat{Y}_{n,n}$ is that the measurement error

$$\frac{1}{n} \sum_{i=1}^n |Y_i - \hat{Y}_{i,n}|^2 \quad (17)$$

is small. In particular we do not assume that the random variables in $\bar{\mathcal{D}}_n$ are independent or identically distributed. In the sequel we are interested in the influence of the measurement error (17) on the L_2 error of a regression estimate applied to the data $\bar{\mathcal{D}}_n$.

As we do not assume anything on the difference between the true y -values Y_i and the observed values $\hat{Y}_{i,n}$ besides the assumption that (17) is small, it is clear that there is no chance to get rid of this measurement error completely. But a natural conjecture is that a small measurement error (17) does only slightly influence the L_2 error of suitably defined regression estimates. That this conjecture is indeed true was proven for the least squares estimates in Kohler (2006a). Next we describe the part of this result, which will be needed in the proof of our main result.

Assume

$$Y_i, \hat{Y}_{i,n} \in [-L, L] \text{ a.s.}$$

($i = 1, \dots, n$) and define the estimate m_n by

$$m_n(\cdot) = \arg \min_{f \in \mathcal{F}_n} \left(\frac{1}{n} \sum_{i=1}^n |f(X_i) - \hat{Y}_{i,n}|^2 \right),$$

where \mathcal{F}_n is a set of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Then the following result holds.

Lemma 2 *Assume that $Y - m(X)$ is sub-Gaussian in the sense that*

$$C^2 \mathbf{E} \left\{ e^{(Y-m(X))^2/C^2} - 1 | X \right\} \leq \sigma_0^2 \quad \text{almost surely} \quad (18)$$

for some $C, \sigma_0 > 0$. Let $\beta_n, L \geq 1$ and assume that the regression function is bounded in absolute value by L and that β_n satisfies $\beta_n \rightarrow \infty$ ($n \rightarrow \infty$). Let \mathcal{F}_n be a set of functions $f : \mathbb{R}^d \rightarrow [-\beta_n, \beta_n]$ and define the estimate m_n as above. Then there exist constants $c_1, c_2, c_3 > 0$ depending only on σ_0 and C such that for any δ_n which satisfies

$$\delta_n \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{and} \quad \frac{n \cdot \delta_n}{\beta_n^2} \rightarrow \infty \quad (n \rightarrow \infty)$$

and

$$c_1 \frac{\sqrt{n} \delta}{\beta_n^2} \geq \int_{(c_2 \delta)/\beta_n^2}^{\sqrt{\delta}} \left(\log \mathcal{N}_2 \left(\frac{u}{4\beta_n}, \left\{ f - g : f \in \mathcal{F}_n, \frac{1}{n} \sum_{i=1}^n |f(x_i) - g(x_i)|^2 \leq \frac{\delta}{\beta_n^2} \right\}, x_1^n \right) \right)^{1/2} du$$

for all $\delta \geq \delta_n$, all $x_1, \dots, x_n \in \mathbb{R}^d$ and all $g \in \mathcal{F}_n \cup \{m\}$ we have

$$\mathbf{P} \left\{ \int |m_n(x) - m(x)|^2 \mu(dx) > c_3 \left(\frac{1}{n} \sum_{i=1}^n |Y_i - \hat{Y}_{i,n}|^2 + \delta_n + \inf_{f \in \mathcal{F}_n} \int |f(x) - m(x)|^2 \mu(dx) \right) \right\} \rightarrow 0$$

for $n \rightarrow \infty$.

Proof. See proof of Theorem 1 in Kohler (2006a) and observe that we can assume $\beta_n \geq L$ (since $\beta_n \rightarrow \infty$ for $n \rightarrow \infty$). \square

The above lemma enables us to analyze the rate of convergence of the estimate for fixed function space. Next we explain how we can use the data to choose an appropriate function space from a finite collection

$$\{\mathcal{F}_{n,k} : k \in \mathcal{P}_n\}$$

of function spaces. To do this we split the sample into a learning sample

$$\hat{\mathcal{D}}_{n_l} = \{(X_1, \hat{Y}_{1,n}), \dots, (X_{n_l}, \hat{Y}_{n_l,n})\}$$

of size $n_l = \lceil n/2 \rceil$ and a testing sample

$$\{(X_{n_l+1}, \hat{Y}_{n_l+1,n}), \dots, (X_n, \hat{Y}_{n,n})\}$$

of size $n_t = n - n_l$. For fixed $k \in \mathcal{P}_n$ we use the learning sample to define a estimate $m_{n_l}^k$ by

$$m_{n_l}^k(\cdot) = \arg \min_{f \in \mathcal{F}_{n_l,k}} \left(\frac{1}{n_l} \sum_{i=1}^{n_l} |f(X_i) - \hat{Y}_{i,n}|^2 \right)$$

Next we choose $\hat{k} \in \mathcal{P}_n$ by minimizing the empirical L_2 risk on the testing sample, i.e., we set

$$m_n(x) = m_{n_l}^{\hat{k}}(x) \quad (x \in \mathbb{R}^d),$$

where

$$\hat{k} = \arg \min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^n |m_{n_l}^k(X_i) - \hat{Y}_{i,n}|^2.$$

Then the following result holds.

Lemma 3 *Assume that $Y - m(X)$ is sub-Gaussian in the sense that (18) holds for some $C, \sigma > 0$ and assume $|\mathcal{P}_n| \rightarrow \infty$ ($n \rightarrow \infty$). Assume furthermore that conditioned on X_1, \dots, X_n the data sets*

$$\hat{\mathcal{D}}_{n_l} \quad \text{and} \quad \{Y_{n_l+1}, \dots, Y_n\}$$

are independent. Let for each $k \in \mathcal{P}_n$ a set $\mathcal{F}_{n,k}$ of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be given and let the estimate m_n be defined as above. Then

$$\begin{aligned} & \frac{1}{n_t} \sum_{i=n_l+1}^n |m_n(X_i) - m(X_i)|^2 \\ &= O_{\mathbf{P}} \left(\frac{\log |\mathcal{P}_n|}{n_t} + \frac{1}{n_t} \sum_{i=n_l+1}^n |Y_i - \hat{Y}_{i,n}|^2 + \min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^n |m_{n_l}^k(X_i) - m(X_i)|^2 \right). \end{aligned}$$

Proof. The results follows by applying Lemma 2 in Kohler (2006a) conditioned on $\hat{\mathcal{D}}_{n_l}$ and X_1, \dots, X_n and with

$$\mathcal{F}_n = \{m_{n_l}^k : k \in \mathcal{P}_n\}.$$

Here we bound the covering number by the finite cardinality $|\mathcal{P}_n|$ of the set of estimates. \square

5.2 Proof of Theorem 1

Before we start with the proof, observe that the boundedness of the discounted payoff function f_t by L implies $|q_t(x)| \leq L$ for $x \in \mathbb{R}^d$. W.l.o.g. assume $\beta_n \geq L$ (since $\beta_n \rightarrow \infty$ for n tending to infinity).

In the sequel we will show

$$\begin{aligned} & \int |\hat{q}_{n,s}(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx) \\ &= O_{\mathbf{P}} \left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{t \in \{s, s+1, \dots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \right) \quad (19) \end{aligned}$$

for all $s \in \{0, 1, \dots, T\}$.

For $s = T$ we have $\hat{q}_{n,T}(x) = 0 = q_T(x)$, so the assertion is trivial. So let $t < T$ and assume that the assertion holds for $s \in \{t+1, \dots, T\}$. By induction it suffices to show (19) for $s = t$, which we will show in the sequel in seven steps.

In the first step of the proof we show

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}} \left(\frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n,t}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 + \frac{\beta_n^4 \cdot \log |\mathcal{P}_n|}{n_t} \right).$$

Let $\mathcal{D}_{n,t}$ be the set of all $X_{j,s}^{(r)}$ with either $r \geq t+1, s \in \{0, \dots, T\}$ and $j \in \{1, \dots, n\}$ or $r = t, s \in \{0, \dots, T\}$ and $j \in \{1, \dots, n_l\}$. Conditioned on $\mathcal{D}_{n,t}$,

$$\{\hat{q}_{n_l,t}^k : k \in \mathcal{P}_n\}$$

consists of $|\mathcal{P}_n|$ different functions. Furthermore, because of boundedness of $\hat{q}_{n_l,t}^k$ and q_t by β_n we have

$$\begin{aligned} \sigma_k^2 &:= \mathbf{Var}\{|\hat{q}_{n_l,t}^k(X_{n_l+1,t}^{(t)}) - q_t(X_{n_l+1,t}^{(t)})|^2 | \mathcal{D}_{n,t}\} \\ &\leq \mathbf{E}\{|\hat{q}_{n_l,t}^k(X_{n_l+1,t}^{(t)}) - q_t(X_{n_l+1,t}^{(t)})|^4 | \mathcal{D}_{n,t}\} \\ &\leq 4\beta_n^2 \int |\hat{q}_{n_l,t}^k(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx). \end{aligned}$$

Using this and the Bernstein inequality (cf., e.g., Lemma A.2 in Györfi et al. (2002)) we get using the notation $\epsilon_n = c_4 \cdot (\beta_n^4 \log |\mathcal{P}_n|)/n_t$:

$$\begin{aligned} &\mathbf{P}\{\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) > (4\beta_n^2 + 1) \cdot \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n,t}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 + \epsilon_n \mid \mathcal{D}_{n,t}\} \\ &\leq |\mathcal{P}_n| \cdot \max_{k \in \mathcal{P}_n} \mathbf{P}\{\int |\hat{q}_{n_l,t}^k(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\ &\quad > (4\beta_n^2 + 1) \cdot \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n_l,t}^k(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 + \epsilon_n \mid \mathcal{D}_{n,t}\} \\ &\leq |\mathcal{P}_n| \cdot \max_{k \in \mathcal{P}_n} \mathbf{P}\{\int |\hat{q}_{n_l,t}^k(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) + 4\beta_n^2 \int |\hat{q}_{n_l,t}^k(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\ &\quad > (4\beta_n^2 + 1) \cdot \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n_l,t}^k(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 + \epsilon_n + \sigma_k^2 \mid \mathcal{D}_{n,t}\} \\ &\leq |\mathcal{P}_n| \cdot \max_{k \in \mathcal{P}_n} \mathbf{P}\{\int |\hat{q}_{n_l,t}^k(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) - \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n_l,t}^k(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 \\ &\quad > \frac{1}{4\beta_n^2 + 1} \cdot (\sigma_k^2 + \epsilon_n) \mid \mathcal{D}_{n,t}\} \\ &\leq |\mathcal{P}_n| \cdot \max_{k \in \mathcal{P}_n} \exp\left(\frac{-n_t \cdot \left(\frac{\sigma_k^2 + \epsilon_n}{4\beta_n^2 + 1}\right)^2}{2\sigma_k^2 + 2\frac{\sigma_k^2 + \epsilon_n}{4\beta_n^2 + 1} \cdot \frac{4\beta_n^2}{3}}\right) \\ &\leq |\mathcal{P}_n| \cdot \max_{k \in \mathcal{P}_n} \exp\left(-\frac{n_t(\sigma_k^2 + \epsilon_n)}{2(4\beta_n^2 + 1)^2 + 2(4\beta_n^2 + 1) \cdot \frac{4\beta_n^2}{3}}\right) \\ &\leq |\mathcal{P}_n| \cdot \exp\left(-\frac{c_4}{2(4\beta_n^2 + 1)^2 + 2(4\beta_n^2 + 1) \cdot \frac{4\beta_n^2}{3}} \cdot \beta_n^4 \log |\mathcal{P}_n|\right) \\ &\rightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

provided we choose c_4 sufficiently large.

In the second step of the proof we show

$$\begin{aligned} & \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n,t}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 \\ &= O_{\mathbf{P}} \left(\frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 + \frac{\log |\mathcal{P}_n|}{n_t} \right. \\ & \quad \left. + \min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^n |q_{n_l,t}^k(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 \right). \end{aligned}$$

To do this we apply Lemma 3. In the context of Lemma 3 we have

$$X_i = X_{i,t}^{(t)}, Y_i = \max\{f_{t+1}(X_{i,t+1}^{(t)}), q_{t+1}(X_{i,t+1}^{(t)})\} \text{ and } \hat{Y}_{i,n} = \max\{f_{t+1}(X_{i,t+1}^{(t)}), \hat{q}_{n,t+1}(X_{i,t+1}^{(t)})\}. \quad (20)$$

Observing

$$\frac{1}{n_t} \sum_{i=n_l+1}^n |Y_i - \hat{Y}_{i,n}|^2 \leq \frac{1}{n_t} \sum_{i=n_l+1}^n |q_{t+1}(X_{i,t+1}^{(t)}) - \hat{q}_{n,t+1}(X_{i,t+1}^{(t)})|^2$$

the assertion follows from Lemma 3 if we apply it conditioned on $\mathcal{D}_{n,t}$.

In the third step of the proof we show

$$\frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 = O_{\mathbf{P}} \left(\int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \frac{\beta_n^4 \cdot \log |\mathcal{P}_n|}{n_t} \right).$$

Using

$$\begin{aligned} & \mathbf{P} \left\{ \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 \right. \\ & \quad \left. > (4\beta_n^2 + 1) \int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \epsilon_n |\mathcal{D}_{n,t}| \right\} \\ &= \mathbf{P} \left\{ \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 - \int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) \right. \\ & \quad \left. > 4\beta_n^2 \cdot \int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \epsilon_n |\mathcal{D}_{n,t}| \right\} \end{aligned}$$

this follows as in the first step by an application of the Bernstein inequality.

In the fourth step of the proof we show

$$\min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n_l,t}^k(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 = O_{\mathbf{P}} \left(\int |\hat{q}_{n_l,t}^k(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) + \frac{\beta_n^4 \cdot \log |\mathcal{P}_n|}{n_t} \right).$$

To see this, we observe that we have as in the third step of the proof

$$\frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n_l,t}^k(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 = O_{\mathbf{P}} \left(\int |\hat{q}_{n_l,t}^k(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) + \frac{\beta_n^4 \cdot \log |\mathcal{P}_n|}{n_t} \right),$$

hence the assertion follows from

$$\min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n_l,t}^k(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 \leq \frac{1}{n_t} \sum_{i=n_l+1}^n |\hat{q}_{n_l,t}^k(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2.$$

In the fifth step of the proof we show

$$\begin{aligned} & \int |\hat{q}_{n,t}^{k_n}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\ &= O_{\mathbf{P}} \left(\frac{1}{n_l} \sum_{i=1}^{n_l} |Y_i - \hat{Y}_{i,n}|^2 + \delta_n + \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \right), \end{aligned}$$

where $\delta_n = c_6 \cdot \beta_n^4 \cdot k_n \cdot \log(n)/n$ and Y_i and $\hat{Y}_{i,n}$ are defined in (20). To do this we show that with this choice of δ_n the conditions of Lemma 2 are satisfied.

Observe that Y_1 is bounded in absolute value by L , hence (18) holds. By Lemma 1 we get for $g \in \mathcal{F}_{k_n}(\beta_n) \cup \{q_t\}$

$$\begin{aligned} & \mathcal{N}_2 \left(\frac{u}{4\beta_n}, \left\{ f - g : f \in \mathcal{F}_{k_n}(\beta_n), \frac{1}{n} \sum_{i=1}^n |f(X_i) - g(X_i)|^2 \leq \frac{\delta}{\beta_n^2} \right\}, X_1^n \right) \\ & \leq \mathcal{N}_2 \left(\frac{u}{4\beta_n}, \mathcal{F}_{k_n}(\beta_n), X_1^n \right) \\ & \leq \left(\frac{48e\beta_n^2(k_n + 1)}{u} \right)^{(4d+9)k_n+1}, \end{aligned}$$

thus

$$\begin{aligned} & \int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left\{ \log \mathcal{N}_2 \left(\frac{u}{4\beta_n}, \left\{ f - g : f \in \mathcal{F}_{k_n}(\beta_n), \frac{1}{n} \sum_{i=1}^n |f(X_i) - g(X_i)|^2 \leq \frac{\delta}{\beta_n^2} \right\}, X_1^n \right) \right\}^{1/2} du \\ & \leq \int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left\{ \log \left(\frac{48e\beta_n^2(k_n + 1)}{u} \right)^{(4d+9)k_n+1} \right\}^{1/2} du. \end{aligned}$$

Let $\delta > 1/n$. Then by bounding u from below by $(c_2\delta)/\beta_n^2$ and using constant $c_5 > 0$ we get

$$\begin{aligned} & \int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left\{ \log \left(\frac{48e\beta_n^2(k_n + 1)}{u} \right)^{(4d+9)k_n+1} \right\}^{1/2} du \\ & \leq \int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left\{ \log \left(\frac{48e\beta_n^4(k_n + 1)}{c_2\delta} \right)^{(4d+9)k_n+1} \right\}^{1/2} du \\ & \leq \int_{(c_2\delta)/\beta_n^2}^{\sqrt{\delta}} \left\{ \log \left(\frac{48e\beta_n^4(k_n + 1)n}{c_2} \right)^{(4d+9)k_n+1} \right\}^{1/2} du \\ & \leq c_5 \sqrt{\delta} \sqrt{k_n} (\log(n))^{1/2}. \end{aligned}$$

This together with

$$\begin{aligned} \frac{c_1\sqrt{n}\delta}{\beta_n^2} &\geq c_5\sqrt{\delta}\sqrt{k_n}\sqrt{\log(n)} \\ \Leftrightarrow \delta &\geq c_6\beta_n^4 k_n \frac{\log(n)}{n} \end{aligned}$$

shows that

$$\delta_n := c_6\beta_n^4 k_n \frac{\log(n)}{n}$$

satisfies the condition of Lemma 2.

In the sixth step of the proof we show

$$\frac{1}{n_l} \sum_{i=1}^{n_l} |Y_i - \hat{Y}_{i,n}|^2 = O_{\mathbf{P}} \left(\int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \frac{\beta_n^4 \cdot \log |\mathcal{P}_n|}{n_l} \right).$$

First we observe

$$\frac{1}{n_l} \sum_{i=1}^{n_l} |Y_i - \hat{Y}_{i,n}|^2 \leq \frac{1}{n_l} \sum_{i=1}^{n_l} |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2.$$

To show

$$\frac{1}{n_l} \sum_{i=1}^{n_l} |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 = O_{\mathbf{P}} \left(\int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \frac{\beta_n^4 \cdot \log |\mathcal{P}_n|}{n_l} \right)$$

we condition on all data points $X_{j,s}^{(r)}$ with $r \geq t+1, s \in \{0, \dots, T\}$ and $j \in \{1, \dots, n\}$.

Then the assertion follows by an application of Bernstein inequality as in steps 1 and 3.

In the seventh (and last) step of the proof we observe that we get by induction

$$\begin{aligned} &\int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) \\ &= O_{\mathbf{P}} \left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s \in \{t+1, \dots, T-1\}} \inf_{f \in \mathcal{F}_{k_n}(\beta_n)} \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_s}(dx) \right). \end{aligned}$$

We complete the proof by gathering the above results. \square

6 Appendix

Lemma 4 Let \mathcal{F} and \mathcal{G} be two families of real functions on \mathbb{R}^m . If $\mathcal{F} \oplus \mathcal{G}$ denotes the set of functions $\{f + g : f \in \mathcal{F}, g \in \mathcal{G}\}$, then for any $z_1^n \in \mathbb{R}^{n \cdot m}$ and $\epsilon, \delta > 0$, we have

$$\mathcal{N}_2(\epsilon + \delta, \mathcal{F} \oplus \mathcal{G}, z_1^n) \leq \mathcal{N}_2(\epsilon, \mathcal{F}, z_1^n) \mathcal{N}_2(\delta, \mathcal{G}, z_1^n).$$

Proof of Lemma 4. Let $\{f_1, \dots, f_{\mathcal{K}}\}$ and $\{g_1, \dots, g_{\Lambda}\}$ be an ϵ -cover and a δ -cover of \mathcal{F} and \mathcal{G} , respectively, on z_1^n of minimal size. Then, for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, there exist $\kappa \in \{1, \dots, \mathcal{K}\}$ and $\lambda \in \{1, \dots, \Lambda\}$ such that

$$\left(\frac{1}{n} \sum_{i=1}^n |f(z_i) - f_{\kappa}(z_i)|^2 \right)^{1/2} < \epsilon$$

and

$$\left(\frac{1}{n} \sum_{i=1}^n |g(z_i) - g_{\lambda}(z_i)|^2 \right)^{1/2} < \delta.$$

By the triangle inequality for norms we have

$$\begin{aligned} & \left(\frac{1}{n} \sum_{i=1}^n |f(z_i) + g(z_i) - (f_{\kappa}(z_i) + g_{\lambda}(z_i))|^2 \right)^{1/2} \\ & \leq \left(\frac{1}{n} \sum_{i=1}^n |f(z_i) - f_{\kappa}(z_i)|^2 \right)^{1/2} + \left(\frac{1}{n} \sum_{i=1}^n |g(z_i) - g_{\lambda}(z_i)|^2 \right)^{1/2} \\ & \leq \epsilon + \delta \end{aligned}$$

which proves that $\{f_{\kappa} + g_{\lambda} : 1 \leq \kappa \leq \mathcal{K}, 1 \leq \lambda \leq \Lambda\}$ is an $(\epsilon + \delta)$ -cover of $\mathcal{F} \oplus \mathcal{G}$ on z_1^n . \square

Lemma 5 *Let \mathcal{F} and \mathcal{G} be two families of real functions on \mathbb{R}^m such that $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all $x \in \mathbb{R}^m$, $f \in \mathcal{F}$, $g \in \mathcal{G}$. If $\mathcal{F} \odot \mathcal{G}$ denotes the set of functions $\{f \cdot g : f \in \mathcal{F}, g \in \mathcal{G}\}$ then, for any $z_1^n \in \mathbb{R}^{n \cdot m}$ and $\epsilon, \delta > 0$ we have*

$$\mathcal{N}_2(\epsilon + \delta, \mathcal{F} \odot \mathcal{G}, z_1^n) \leq \mathcal{N}_2(\epsilon/M_2, \mathcal{F}, z_1^n) \mathcal{N}_2(\delta/M_1, \mathcal{G}, z_1^n).$$

Proof of Lemma 5. Let $\{f_1, \dots, f_{\mathcal{K}}\}$ and $\{g_1, \dots, g_{\Lambda}\}$ be an ϵ/M_2 -cover and a δ/M_1 -cover of \mathcal{F} and \mathcal{G} , respectively, on z_1^n of minimal size. By the boundedness of f and g we can assume w.l.o.g. $|f_{\kappa}(z)| \leq M_1$, $|g_{\lambda}(z)| \leq M_2$. For every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, there exist $\kappa \in \{1, \dots, \mathcal{K}\}$ and $\lambda \in \{1, \dots, \Lambda\}$ such that

$$\left(\frac{1}{n} \sum_{i=1}^n |f(z_i) - f_{\kappa}(z_i)|^2 \right)^{1/2} < \frac{\epsilon}{M_2}$$

and

$$\left(\frac{1}{n} \sum_{i=1}^n |g(z_i) - g_{\lambda}(z_i)|^2 \right)^{1/2} < \frac{\delta}{M_1}.$$

We have, by the triangle inequality of norms

$$\begin{aligned}
& \left(\frac{1}{n} \sum_{i=1}^n |f(z_i) \cdot g(z_i) - f_{\kappa}(z_i) \cdot g_{\lambda}(z_i)|^2 \right)^{1/2} \\
&= \left(\frac{1}{n} \sum_{i=1}^n |f(z_i) \cdot (g_{\lambda}(z_i) + g(z_i) - g_{\lambda}(z_i)) - f_{\kappa}(z_i) \cdot g_{\lambda}(z_i)|^2 \right)^{1/2} \\
&\leq \left(\frac{1}{n} \sum_{i=1}^n |g_{\lambda}(z_i) \cdot (f(z_i) - f_{\kappa}(z_i))|^2 \right)^{1/2} + \left(\frac{1}{n} \sum_{i=1}^n |f(z_i) \cdot (g(z_i) - g_{\lambda}(z_i))|^2 \right)^{1/2} \\
&\leq M_2 \cdot \left(\frac{1}{n} \sum_{i=1}^n |f(z_i) - f_{\kappa}(z_i)|^2 \right)^{1/2} + M_1 \cdot \left(\frac{1}{n} \sum_{i=1}^n |g(z_i) - g_{\lambda}(z_i)|^2 \right)^{1/2} \\
&\leq \epsilon + \delta
\end{aligned}$$

which implies that $\{f_{\kappa} \cdot g_{\lambda} : 1 \leq \kappa \leq \mathcal{K}, 1 \leq \lambda \leq \Lambda\}$ is an $(\epsilon + \delta)$ -cover of $\mathcal{F} \odot \mathcal{G}$ on z_1^n . \square

Proof of Lemma 1: Define the following classes of functions:

$$\begin{aligned}
\mathcal{G}_1 &= \{a^T x + b : a \in \mathbb{R}^d, b \in \mathbb{R}\}, \\
\mathcal{G}_2 &= \{\sigma(a^T x + b) : a \in \mathbb{R}^d, b \in \mathbb{R}\}, \\
\mathcal{G}_3 &= \{c \cdot \sigma(a^T x + b) : a \in \mathbb{R}^d, b \in \mathbb{R}, c \in [-\beta_n, \beta_n]\},
\end{aligned}$$

where $\sigma : \mathbb{R} \rightarrow [0, 1]$ is a sigmoid function (i.e. σ is a nondecreasing function with the property $\lim_{x \rightarrow -\infty} \sigma(x) = 0$ and $\lim_{x \rightarrow \infty} \sigma(x) = 1$) and $\beta_n > 0$.

\mathcal{G}_1 is a linear vector space of dimension $d + 1$, thus Theorem 9.5 in Györfi et al. (2002) implies

$$V_{\mathcal{G}_1^+} \leq d + 2,$$

where \mathcal{G}^+ denotes the set

$$\mathcal{G}^+ = \{(z, t) \in \mathbb{R}^d \times \mathbb{R}, t \leq g(z)\}; g \in \mathcal{G}\}$$

for all subgraphs of functions of \mathcal{G} and $V_{\mathcal{G}^+}$ is the so-called VC-dimension of \mathcal{G}^+ (see Györfi et al. (2002), Definition 9.6).

Since σ is a nondecreasing function, Lemma 16.3 in Györfi et al. (2002) yields

$$V_{\mathcal{G}_2^+} \leq d + 2.$$

Thus, by Theorem 9.4 in Györfi et al. (2002), we have for $0 < \epsilon < 1/4$

$$\begin{aligned} \mathcal{N}_2(\epsilon, \mathcal{G}_2, x_1^n) &\leq \mathcal{M}_2(\epsilon, \mathcal{G}_2, x_1^n) \\ &\leq 3 \left(\frac{2e}{\epsilon^2} \log \frac{3e}{\epsilon^2} \right)^{d+2} \\ &\leq 3 \left(\frac{3e}{\epsilon^2} \right)^{2d+4}. \end{aligned}$$

By Lemma 5 we have for $0 < \epsilon/2\beta_n < 1/4$ or equivalently $0 < \epsilon < \beta_n/2$

$$\begin{aligned} \mathcal{N}_2(\epsilon, \mathcal{G}_3, x_1^n) &\leq \mathcal{N}_2\left(\frac{\epsilon}{2}, \{c : |c| \leq \beta_n\}, x_1^n\right) \mathcal{N}_2\left(\frac{\epsilon}{2\beta_n}, \mathcal{G}_2, x_1^n\right) \\ &\leq \frac{2\beta_n}{(\epsilon/2)} \cdot 3 \left(\frac{3e}{(\epsilon/(2\beta_n))^2} \right)^{2d+4} \\ &\leq \left(\frac{12e\beta_n}{\epsilon} \right)^{4d+9}. \end{aligned}$$

By applying Lemma 4 we obtain for $0 < \epsilon < (k+1) \cdot \beta_n/2$

$$\begin{aligned} \mathcal{N}_2(\epsilon, \mathcal{F}_k(\beta_n), x_1^n) &\leq \mathcal{N}_2\left(\frac{\epsilon}{k+1}, \{c_0 : |c_0| \leq \beta_n\}, x_1^n\right) \left(\mathcal{N}_2\left(\frac{\epsilon}{k+1}, \mathcal{G}_3, x_1^n\right) \right)^k \\ &\leq \frac{2\beta_n(k+1)}{\epsilon} \left(\frac{12e\beta_n(k+1)}{\epsilon} \right)^{(4d+9)k} \\ &\leq \left(\frac{12e\beta_n(k+1)}{\epsilon} \right)^{(4d+9)k+1}. \end{aligned} \tag{21}$$

By boundedness of $\mathcal{F}_n(\beta_n)$, the proof is trivial for $\epsilon \geq (k+1) \cdot \beta_n/2 \geq \beta_n$, which completes the proof. \square

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