Universally consistent upper bounds for Bermudan options based on Monte Carlo and nonparametric regression

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Abstract
Monte Carlo evaluation of American options in discrete time is considered. Upper bounds on the price of such options can be constructed by the dual approach, where the maximal difference between the payoff and a martingale is minimized. In this article techniques from nonparametric regression are used to estimate so-called continuation values, and nested Monte Carlo is used to compute the optimal martingale approximately. It is shown that the resulting upper bounds on the option tend to the true price regardedless of the structure of the continuation values. Furthermore it is illustrated by simulated data that in this context nonparametric regression leads to better bounds than linear regression.

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1 Introduction

Monte Carlo methods for pricing American options are very attractive compared to other methods when the number of underlying assets or state variables is large. One way to apply such methods is to represent the price of an American option in discrete time (also

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Running title: *Upper bounds for Bermudan options* called Bermudan option) in the risk-neutral market as a solution of an optimal stopping problem

$$V_0 = \mathbf{E}\left\{f_{\tau^*}(X_{\tau^*})\right\} = \sup_{\tau \in \mathcal{T}(0,\dots,T)} \mathbf{E}\left\{f_\tau(X_\tau)\right\}. \tag{1}$$

Here f_t is the (discounted) payoff function, $X_0, X_1, \dots, X_T$ is the underlying stochastic process describing e.g. the prices of the underlyings and the financial environment (like interest rates, etc.) and $\mathcal{T}(0,\dots,T)$ is the class of all $\{0,\dots,T\}$-valued stopping times, i.e., $\tau \in \mathcal{T}(0,\dots,T)$ is a measurable function of $X_0, \dots, X_T$ satisfying

$$\{\tau = \alpha\} \in \mathcal{F}(X_0,\dots,X_\alpha) \quad \text{for all } \alpha \in \{0,\dots,T\}.$$

In the sequel we assume that $X_0, X_1, \dots, X_T$ is a $\mathbb{R}^d$–valued Markov process recording all necessary information about financial variables including prices of the underlying assets as well as additional risk factors driving stochastic volatility or stochastic interest rates. Neither the Markov property nor the form of the payoff as a function of the state X_t is restrictive and can always be achieved by including supplementary variables.

The general theory of optimal stopping (cf., e.g., Chow, Robbins and Siegmund (1971) or Shiryayev (1978)) implies that the optimal stopping time τ^* is given by

$$\tau^* = \inf\{s \geq 0 \,:\, q_s(X_s) \leq f_s(X_s)\}, \tag{2}$$

where

$$q_t(x) = \sup_{\tau \in \mathcal{T}(t+1,\dots,T)} \mathbf{E}\left\{f_\tau(X_\tau) | X_t = x\right\} \tag{3}$$

($t \in \{0,\dots,T-1\}$) are the so–called continuation values describing the value of the option at time t given $X_t = x$ and subject to the constraint of holding the option at time t rather than exercising it. Here $\mathcal{T}(t+1,\dots,T)$ is the class of all $\{t+1,\dots,T\}$–valued stopping times. Furthermore we set $q_T = 0$. Unfortunately, since

$$V_0 = \mathbf{E}\left\{\max\{f_0(X_0), q_0(X_0)\}\right\},$$

computation of the continuation values is not easier than computation of the price of the option. But by using a regression representation like

$$q_t(x) = \mathbf{E}\left\{\max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\} | X_t = x\right\} \quad (t = 0, 1, \dots, T-1) \tag{4}$$

(cf. Tsitsiklis and Van Roy (1999)), estimates $\hat{q}_0, \ldots, \hat{q}_{T-1}$ can be computed recursively by applying regression estimates to Monte Carlo samples of

$$(X_t, \max\{f_{t+1}(X_{t+1}), \hat{q}_{t+1}(X_{t+1})\})$$

(where $\hat{q}_T = q_T = 0$). In connection with linear regression this was proposed by Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001), where the latter article used a different regression representation than (4). Egloff (2005) used nonparametric regression estimates (cf., e.g., Györfi et al. (2002)) in this context and analyzed them theoretically. Unfortunately the definition of the estimates was so complicated, that it seems to be hard to implement them. Nonparametric regression estimates of continuation values which are easy to compute in practice have been introduced in Egloff, Kohler and Todorovic (2007), Kohler, Krzyżak and Todorovic (2006), and Kohler (2008) and it was shown that these estimates achieve better results for simulated data than linear regression. This implies that techniques from nonparametric regression are really useful in this context.

The above estimates yield estimates

$$\hat{\tau} = \inf\{s \geq 0 \quad : \quad \hat{q}_s(X_s) \leq f_s(X_s)\}$$

of the optimal stopping time τ^*. By Monte Carlo these estimates yields estimates of V_0, such that expectation

$$\mathbf{E}\{f_{\hat{\tau}}(X_{\hat{\tau}})\}$$

of the estimate is less than or equal to the true price V_0. It was proposed independently by Rogers (2001) and Haugh and Kogan (2004) that by using a dual method Monte Carlo estimates can be constructed such that the expectation of the estimate is greater than or equal to V_0. The key idea is to show that

$$V_0 = \inf_{M \in \mathcal{M}} \mathbf{E}\left\{\max_{t=0,\ldots,T} (f_t(X_t) - M_t)\right\}, \tag{5}$$

where $\mathcal{M}$ is the set of all martingales $M_0, \ldots, M_T$ with $M_0 = 0$. Here the optimal martingale achieving the infimum in (5) can be expressed with the aid of the continuation values by

$$M_t^* = \sum_{s=1}^{t} (\max\{f_s(X_s), q_s(X_s)\} - \mathbf{E}\{\max\{f_s(X_s), q_s(X_s)\}|X_{s-1}\}) \tag{6}$$

(cf., e.g., Section 8.7 in Glasserman (2004)). Given estimate $\hat{q}_s$ ($s \in \{0, 1, \ldots, T\}$) of the continuation values, we can estimate this martingale by

$$\hat{M}_t = \sum_{s=1}^{t} \left(\max\{f_s(X_s), \hat{q}_s(X_s)\} - \mathbf{E}^* \left\{ \max\{f_s(X_s), \hat{q}_s(X_s)\} | X_{s-1} \right\} \right). \tag{7}$$

Provided we use unbiased and $\mathcal{F}(X_0, \ldots, X_t)$-measurable estimates $\mathbf{E}^*$ of the inner expectation in (6) (which can be constructed, e.g., by nested Monte Carlo) this leads to a martingale, too. This in turn can be used to construct Monte Carlo estimates of V_0, for which the expectation

$$\mathbf{E} \left\{ \max_{t=0,\ldots,T} \left(f_t(X_t) - \hat{M}_t \right) \right\}$$

is greater than or equal to V_0. As a consequence we get two kind of estimates with expectation lower and higher than V_0, resp., so we have available an interval in which our true price should be contained. In connection with linear regression these kind of estimates have been studied in Rogers (2001) and Haugh and Kogan (2004). Jamshidian (2007) studies multiplicative versions of this method. A comparative study of multiplicative and additive duals is contained in Chen and Glasserman (2007). Andersen and Broadie (2004) derive upper and lower bounds for American options based on duality. Belomestny, Bender and Schoenmakers (2007) propose in a Brownian motion setting estimates with expectation greater than or equal to the true price, which can be computed without nested Monte Carlo (and hence are quite easy to compute).

In this article we study the above dual method in connection with nonparametric regression. This leads to estimates with expectation greater than or equal to V_0. The estimates will be based on nested Monte Carlo and are applicable for general Markov processes. Our main theoretical result is that these estimates will be universally consistent provided we choose the estimates of the continuation values properly, i.e., provided the estimated price will tend to the true price for all (bounded) Markov processes. This is in contrast to estimates based on linear regression, which are based on the assumption that the continuation values can be approximated well by a linear combination of the used basis functions, which are chosen independently of the sample size. By using simulated data we show furthermore that the new estimates proposed in this article also have for finite sample size a better performance than estimates based on linear regression.

The precise definition of the estimates and the main theoretical result concerning consistency of the estimates are given in Sections 2 and 3, respectively. The application of the estimates to simulated data will be described in Section 4, and the proofs will be given in Section 5. Section 6 contains a conclusion summarizing the main results.

2 Definition of the estimate

Let $X_0, X_1, \ldots, X_T$ be a $\mathbb{R}^d$–valued Markov process and let f_t be the discounted payoff function which we assume to be bounded in absolute value by L. We assume that the data generating process is completely known, i.e., that all parameters of this process are already estimated from historical data. In this section we describe a dual Monte Carlo method for estimation of V_0.

To do this we generate artificial independent Markov processes $\{X_{i,t}\}_{t=0,\ldots,T}$ $(i = 1, 2, \ldots, n)$ which are identically distributed as $\{X_t\}_{t=0,\ldots,T}$. Set $n = n_q + n_M$. In a first step we use the first n_q replication of the Markov process to define regression based Monte Carlo estimates $\hat{q}_{n,t}$ of q_t. Here any of the estimates described in Egloff, Kohler and Todorovic (2007), Kohler, Krzyżak and Todorovic (2006) or Kohler (2008) can be used. For simplicity we describe in the sequel only the estimate of Kohler (2008) in detail.

Here we start with

$$\hat{q}_{n,T}(x) = 0 \quad (x \in \mathbb{R}^d).$$

Given an estimate $\hat{q}_{n,t+1}$ of q_{t+1}, we set

$$\hat{Y}_{i,t}^{(t)} = \max\{f_{t+1}(X_{i,t+1}^{(t)}), \hat{q}_{n,t+1}(X_{i,t+1}^{(t)})\}$$

(where we have suppressed the dependency of $\hat{Y}_{i,t}^{(t)}$ on n) and use the data

$$\left\{ \left(X_{i,t}^{(t)}, \hat{Y}_{i,t}^{(t)} \right) \quad : \quad i = 1, \ldots, n_q \right\}.$$

to define $\hat{q}_{n,t}$ as follows: we subdivide the data in a learning sample of size $n_l = \lceil n_q/2 \rceil$ and a testing sample of size $n_t = n_q - n_l$ and define for given $\lambda \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$ a regression estimate of q_t by

$$\tilde{q}_{n_l,t}^{(k,\lambda)}(\cdot) = \arg \min_{f \in W^k([-A,A]^d)} \left(\frac{1}{n_l} \sum_{i=1}^{n_l} |f(X_{i,t}^{(t)}) - \hat{Y}_{i,t}^{(t)}|^2 + \lambda \cdot J_k^2(f) \right)$$

and

$$q_{n_l,t}^{(k,\lambda)}(x) = T_L \tilde{q}_{n_l,t}^{(k,\lambda)}(x) \quad (x \in \mathbb{R}^d).$$

Here $W^k([-A,A]^d)$ denotes the Sobolev space

$$\left\{ f \,:\, \frac{\partial^k f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} \in L_2([-A,A]^d) \text{ for all } \alpha_1, \dots, \alpha_d \in \mathbb{N} \text{ with } \alpha_1 + \dots + \alpha_d = k \right\},$$

we set

$$J_k^2(f) = \sum_{\alpha_1,\dots,\alpha_d \in \mathbb{N},\, \alpha_1+\dots+\alpha_d=k} \frac{k!}{\alpha_1! \cdot \dots \cdot \alpha_d!} \int_{\mathbb{R}^d} \left| \frac{\partial^k f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}(x) \right|^2 dx,$$

and $T_L z = \max\{-L, \min\{L, z\}\}$ for $z \in \mathbb{R}$. Then we minimize the empirical L_2 error on the discrete parameter set

$$\mathcal{P}_n = \left\{ (k,\lambda) \quad : \quad \lambda = \frac{i}{n} \text{ for some } i \in \{0,1,\dots,n^2\}, k \in \left\{ \lceil \frac{d}{2} \rceil, \dots, K \right\} \right\}$$

(where $K \geq \lceil d/2 \rceil$ is a given natural number) in order to choose the value of the parameter. I.e., we choose

$$(\lambda_t^*, k_t^*) = \arg \min_{(k,\lambda) \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^{n} |q_{n_l,t}^{(k,\lambda)}(X_{i,t}^{(t)}) - \hat{Y}_{i,t}^{(t)}|^2$$

and define our final estimate of q_t by

$$\hat{q}_{n,t}(x) = q_{n_l,t}^{(\lambda_t^*, k_t^*)}(x) \quad (x \in \mathbb{R}^d).$$

In a second step we estimate the martingale (6). Here we approximate

$$\max\{f_s(X_{s,i}), q_s(X_{s,i})\}$$

by replacing q_s by its estimate $\hat{q}_{n,s}$. In order to estimate

$$\mathbf{E}\left\{ \max\{f_s(X_{s,i}), q_s(X_{s,i})\} | X_{s-1,i} \right\} \tag{8}$$

we use nested Monte Carlo. We generate independent copies $X_{s,i}^{(1)}, \dots, X_{s,i}^{(l_n)}$ of $X_{s,i}$ such that conditional on $X_{s-1,i}$ the newly generated data is independent from all previously generated data and independent and identically distributed as $X_{s,i}$. Using this data we estimate (8) by

$$\frac{1}{l_n} \sum_{j=1}^{l_n} \max\{f_s(X_{s,i}^{(j)}), \hat{q}_{n,s}(X_{s,i}^{(j)})\} \tag{9}$$

and the resulting estimate of $M_t^*(X_{t,i})$ is

$$\hat{M}_{t,i} = \sum_{s=1}^{t} \left(\max\{f_s(X_{s,i}), \hat{q}_{n,s}(X_{s,i})\} - \frac{1}{l_n} \sum_{j=1}^{l_n} \max\{f_s(X_{s,i}^{(j)}), \hat{q}_{n,s}(X_{s,i}^{(j)})\} \right),$$

where $\hat{M}_{0,i} = 0$. Finally we use

$$\hat{V}_0 = \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \max_{t=0,\dots,T} \left(f_t(X_{t,i}) - \hat{M}_{t,i} \right)$$

as estimate of

$$V_0 = \mathbf{E} \left\{ \max_{t=0,\dots,T} \left(f_t(X_t) - M_t^* \right) \right\}.$$

Since (9) is a conditionally unbiased estimate of

$$\mathbf{E} \left\{ \max\{f_s(X_{s,i}), \hat{q}_{n,s}(X_{s,i})\} | X_{s-1,i}, X_{t,j} \, (t \in \{0,\dots,T\}, j \in \{1,\dots,n_q\}) \right\},$$

it is easy to see that

$$\left(\hat{M}_{t,i} \right)_{t=0,\dots,T}$$

is indeed a martingale with respect to the filtration

$$\mathcal{F}_{t,i} = \mathcal{F}\left(X_{0,i}, \dots, X_{t,i}, X_{0,1}, \dots, X_{T,1}, \dots, X_{0,n_q}, \dots, X_{T,n_q} \right)$$

$(i = n_q + 1, \dots, n)$. Consequently the expectation of $\hat{V}_0$ is greater than or equal to V_0 (cf. (5))

3 Main theorem

In the sequel we will use the notation $\mathbf{P}_{X_t}$ for the distribution of X_t. Our main theoretical result is the following theorem.

Theorem 1 *Let $L > 0$, let $X_0, X_1, \dots, X_T$ be a $\mathbb{R}^d$–valued Markov process and assume that the discounted payoff function f_t is bounded in absolute value by L. Let the estimate $\hat{V}_0$ be defined as in Section 2. Assume that the estimates $\hat{q}_{n,t}$ of q_t are bounded in absolute value by L and satisfy*

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \to 0 \quad \textit{in probability,} \tag{10}$$

and that

$$n_M \to \infty \quad (n \to \infty) \quad \text{and} \quad \frac{l_n}{\log n_M} \to \infty \quad (n \to \infty). \tag{11}$$

Then

$$\hat{V}_0 \to V_0 \quad \text{in probability.}$$

The estimates defined in Egloff, Kohler and Todorovic (2007), Kohler, Krzyżak and Todorovic (2006) and Kohler (2008) satisfy (10) for all bounded Markov processes. Hence if we use any of these estimates in the definition of our new estimate, we get universally consistent upper bounds on the price of V_0.

Corollary 1 *Let $A, L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is a $[-A, A]^d$–valued Markov process and that the discounted payoff function f_t is bounded in absolute value by L. Let the estimate $\hat{V}_0$ be defined as in Section 2 where q_t is estimated by least squares splines as in Egloff, Kohler and Todorovic (2007), by least squares neural networks as in Kohler, Krzyżak and Todorovic (2006) or by smoothing splines as in Kohler (2008) (i.e., as in Section 2). Choose n_q, n_M and l_n such that*

$$n_q \to \infty \quad (n \to \infty), \quad n_M \to \infty \quad (n \to \infty) \quad \text{and} \quad \frac{l_n}{\log n_M} \to \infty \quad (n \to \infty).$$

Then

$$\hat{V}_0 \to V_0 \quad \text{in probability.}$$

Proof. The assertion follows from Theorem 1 above and Theorem 4.1 in Egloff, Kohler and Todorovic (2007), Corollary 1 in Kohler, Krzyżak and Todorovic (2006) and Theorem 1 in Kohler (2008). □

4 Application to simulated data

In this section, we illustrate the finite sample behavior of our algorithm by comparing it with algorithms for computing dual upper bounds with linear regression using the regression representations proposed by Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001), respectively.

We consider an American option based on the average of three correlated stock prices. The stocks are ADECCO R, BALOISE R and CIBA. The stock prices were observed

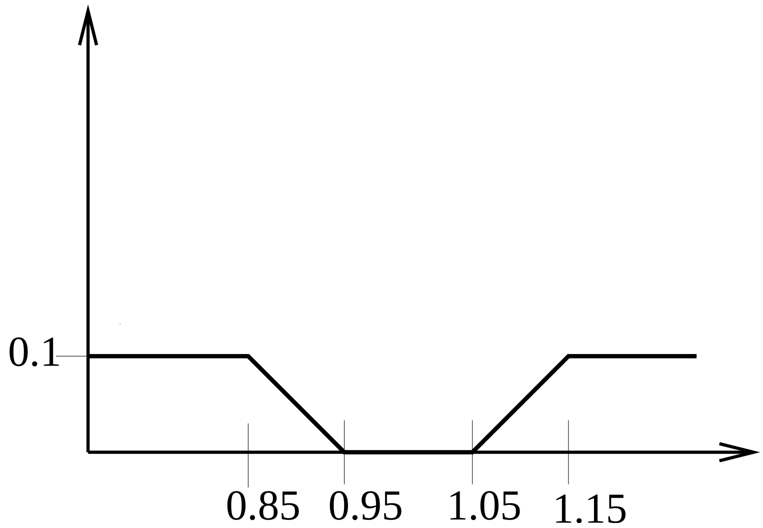

Figure 1: Strangle spread payoff with strike prices 0.85, 0.95, 1.05 and 1.15.

from Nov. 10, 2000 until Oct. 3, 2003 on weekdays when the stock market was open for the total of 756 days. We estimate the volatility from data observed in the past by the historical volatility

$$\sigma = (\sigma_{i,j})_{1 \le i,j \le 3} = \begin{pmatrix} 0.3024 & 0.1354 & 0.0722 \\ 0.1354 & 0.2270 & 0.0613 \\ 0.0722 & 0.0613 & 0.0717 \end{pmatrix}.$$

We simulate the paths of the underlying stocks with a Black-Scholes model by

$$X_{i,t} = x_0 \cdot e^{r \cdot t} \cdot e^{\sum_{j=1}^{3} (\sigma_{i,j} \cdot W_j(t) - \frac{1}{2} \cdot \sigma_{i,j}^2 t)} \quad (i = 1, \ldots, 3),$$

where $\{W_j(t) : t \in \mathbb{R}_+\}$ $(j = 1, \ldots, 3)$ are three independent Wiener processes and where the parameters are chosen as follows: $x_0 = 1$, $r = 0.05$ and components $\sigma_{i,j}$ of the volatility matrix as above. The time to maturity is assumed to be one year. To compute the payoff of the option we use a strangle spread function (cf. Figure 1) with strikes 0.85, 0.95, 1.05 and 1.15 applied to the average of the three correlated stock prices.

We discretize the time interval $[0, 1]$ by dividing it into $m = 48$ equidistant time steps with $t_0 = 0 < t_1 < \ldots < t_m = 1$ and consider a Bermudan option with payoff function as above and exercise dates restricted to $\{t_0, t_1, \ldots, t_m\}$. We choose discount factors e^{-rt_j} for $j = 0, \ldots, m$. For all three algorithms we use parameters $n_q = 2000$, $n_M = 1000$ and

$l_n = 100$.

For our newly proposed algorithm we use smoothing splines as implemented in the routine *Tps*() from the library "fields" in the statistics package *R*, where the smoothing parameter is chosen by generalized cross-validation. For the Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms we use linear regression as implemented in *R*.

We apply all three algorithms to 100 independently generated sets of paths and we compare the algorithms using boxplots for the 100 upper bounds computed for each algorithm. We would like to stress that for all three algorithms the expectation of the values are upper bounds to the true option price, hence lower values indicates a better performance of the algorithms.

As we can see in Figure 2, our algorithm is superior to Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms, since the lower boxplot of the upper bounds for our algorithm indicates better performance. Of course the simulations with linear regression can be improved by choosing the basis functions in a clever way. One way to do this is to use the payoff function as one of the basis functions. In this case it turns out that the algorithms based on linear regression produce similarly good values as the algorithm using nonparametric regression in Figure 2, but if we increase the sample size of the regression estimate from $n_q = 2000$ to $n_q = 40000$, the values of the algorithm using nonparametric regression are again better than the values of the algorithms using linear regression.

5 Proof of Theorem 1

Set

$$M^*_{t,i} = \sum_{s=1}^{t} \left(\max\{f_s(X_{s,i}), q_s(X_{s,i})\} - \mathbf{E}\left\{\max\{f_s(X_{s,i}), q_s(X_{s,i})\} | X_{s-1,i}\right\}\right)$$

$(t \in \{1, \dots, T\})$ and $M^*_{0,i} = 0$ $(i \in \{1, \dots, n\})$.

In the first step of the proof we observe that by the law of large numbers, (5) and (6) we have

$$\begin{aligned} \bar{V}_0 &= \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \max_{t=0,\dots,T} \left(f_t(X_{t,i}) - M^*_{t,i}\right) \\ &\to \mathbf{E}\left\{\max_{t=0,\dots,T} \left(f_t(X_t) - M^*_t\right)\right\} = V_0 \quad \text{in probability.} \end{aligned}$$

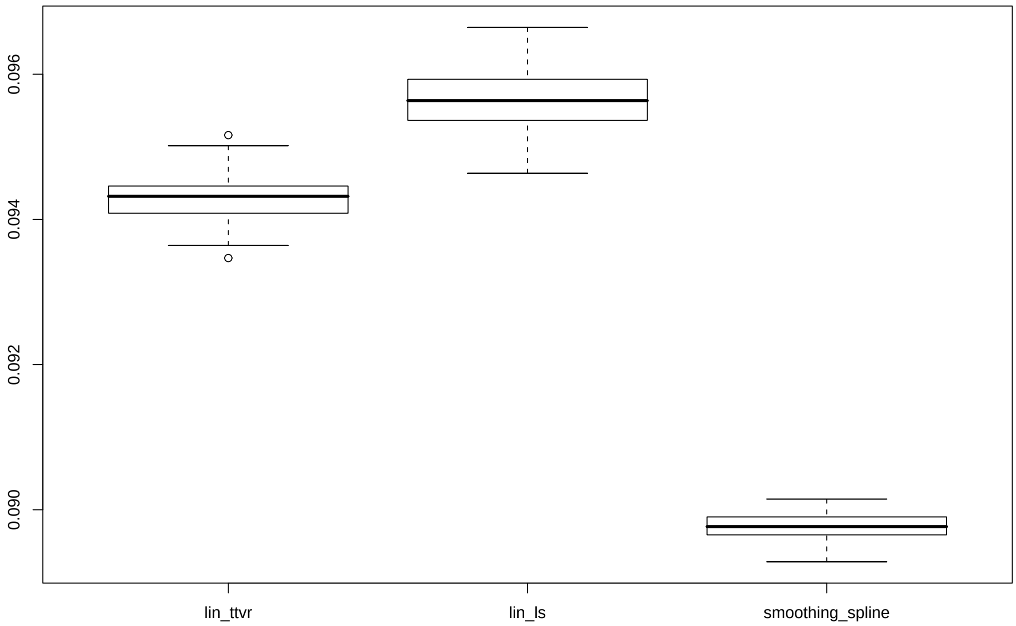

Figure 2: Upper bounds computed with a dual Monte-Carlo method based on linear regression and the Tsitsiklis–Van Roy algorithm (lin–ttvr), linear regression and the Longstaff–Schwartz algorithm (lin–ls), and the newly proposed smoothing spline estimate (smoothing–spline) in a 3-dimensional case.

Hence it suffices to show

$$\hat{V}_0 - \bar{V}_0 \to 0 \quad \text{in probability.} \tag{12}$$

In the second step of the proof we show that (12) is implied by

$$\frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} |\hat{q}_{n,s}(X_{s,i}) - q_s(X_{s,i})| \to 0 \quad \text{in probability} \tag{13}$$

for all $s \in \{0, \ldots, T\}$,

$$\frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \frac{1}{l_n} \sum_{j=1}^{l_n} |\hat{q}_{n,s}(X_{s,i}^{(j)}) - q_s(X_{s,i}^{(j)})| \to 0 \quad \text{in probability} \tag{14}$$

for all $s \in \{0, \ldots, T\}$ and

$$\frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \left| \frac{1}{l_n} \sum_{j=1}^{l_n} \max\left\{ f_s(X_{s,i}^{(j)}), q_s(X_{s,i}^{(j)}) \right\} - \mathbf{E}\left\{ \max\left\{ f_s(X_{s,i}), q_s(X_{s,i}) \right\} | X_{s-1,i} \right\} \right| \to 0 \tag{15}$$

in probability for all $s \in \{0, \ldots, T\}$. This follows from

$$\begin{aligned}
&|\hat{V}_0 - \bar{V}_0| \\
&= \left| \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \max_{t=0,\ldots,T} \left(f_t(X_{t,i}) - \hat{M}_{t,i} \right) - \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \max_{t=0,\ldots,T} \left(f_t(X_{t,i}) - M_{t,i}^* \right) \right| \\
&\leq \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \max_{t=1,\ldots,T} \left| \hat{M}_{t,i} - M_{t,i}^* \right| \\
&\leq \sum_{t=1}^{T} \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \left| \hat{M}_{t,i} - M_{t,i}^* \right| \\
&\leq \sum_{t=1}^{T} \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \left| \sum_{s=1}^{t} (\max\{f_s(X_{s,i}), \hat{q}_{n,s}(X_{s,i})\} - \max\{f_s(X_{s,i}), q_s(X_{s,i})\}) \right| \\
&\quad + \sum_{t=1}^{T} \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \left| \sum_{s=1}^{t} \frac{1}{l_n} \sum_{j=1}^{l_n} \left(\max\{f_s(X_{s,i}^{(j)}), \hat{q}_{n,s}(X_{s,i}^{(j)})\} - \max\{f_s(X_{s,i}^{(j)}), q_s(X_{s,i}^{(j)})\} \right) \right| \\
&\quad + \sum_{t=1}^{T} \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \left| \sum_{s=1}^{t} \left(\frac{1}{l_n} \sum_{j=1}^{l_n} \max\{f_s(X_{s,i}^{(j)}), q_s(X_{s,i}^{(j)})\} \right. \right. \\
&\qquad \left. \left. - \mathbf{E}\left\{ \max\left\{ f_s(X_{s,i}), q_s(X_{s,i}) \right\} | (X_{s-1,i} \right\} \right) \right| \\
&\leq T \cdot \sum_{s=1}^{T} \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} |\hat{q}_{n,s}(X_{s,i}) - q_s(X_{s,i})|
\end{aligned}$$

$$+T \cdot \sum_{s=1}^{T} \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \frac{1}{l_n} \sum_{j=1}^{l_n} |\hat{q}_{n,s}(X_{s,i}^{(j)}) - q_s(X_{s,i}^{(j)})|$$

$$+T \cdot \sum_{s=1}^{T} \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \Bigg| \Big(\frac{1}{l_n} \sum_{j=1}^{l_n} \max\{f_s(X_{s,i}^{(j)}), q_s(X_{s,i}^{(j)})\}$$

$$-\mathbf{E}\left\{\max\left\{f_s(X_{s,i}), q_s(X_{s,i})\right\}\right\} |X_{s-1,i}\Big) \Bigg|.$$

In the third step of the proof we show (13). Set

$$\mathcal{D}_{n_q} = \{X_{s,i} \quad : \quad s = 0, \dots, T, i = 1, \dots, n_q\}.$$

By Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\mathbf{E}\left\{ \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} |\hat{q}_{n,s}(X_{s,i}) - q_s(X_{s,i})| \Big| \mathcal{D}_{n_q} \right\} \\
&= \mathbf{E}\left\{ |\hat{q}_{n,s}(X_s) - q_s(X_s)| \Big| \mathcal{D}_{n_q} \right\} \\
&\leq \sqrt{\mathbf{E}\left\{ |\hat{q}_{n,s}(X_s) - q_s(X_s)|^2 \Big| \mathcal{D}_{n_q} \right\}} \\
&= \sqrt{\int |\hat{q}_{n,s}(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx)}.
\end{aligned}$$

Since $\hat{q}_{n,s}$ and q_s are bounded, assumption (10) together with the dominated convergence theorem implies

$$\mathbf{E}\left\{ \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} |\hat{q}_{n,s}(X_{s,i}) - q_s(X_{s,i})| \right\} \leq \mathbf{E}\sqrt{\int |\hat{q}_{n,s}(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx)} \to 0 \quad (n \to \infty),$$

which in turn implies (13).

In the fourth step of the proof we show

$$\frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \left| \frac{1}{l_n} \sum_{j=1}^{l_n} |\hat{q}_{n,s}(X_{s,i}^{(j)}) - q_s(X_{s,i}^{(j)})| - \mathbf{E}\left\{ |\hat{q}_{n,s}(X_{s,i}) - q_s(X_{s,i})| \Big| X_{s-1,i}, \mathcal{D}_{n_q} \right\} \right| \to 0 \tag{16}$$

in probability. Let $\epsilon > 0$ be arbitrary. Then

$$\mathbf{P}\Bigg\{ \frac{1}{n_M} \sum_{i=n_q+1}^{n_q+n_M} \Bigg| \frac{1}{l_n} \sum_{j=1}^{l_n} |\hat{q}_{n,s}(X_{s,i}^{(j)}) - q_s(X_{s,i}^{(j)})|$$

$$-\mathbf{E}\left\{|\hat{q}_{n,s}(X_{s,i})-q_s(X_{s,i})||X_{s-1,i},\mathcal{D}_{n_q}\right\}\Bigg|>\epsilon\Bigg|X_{s-1,i},\mathcal{D}_{n_q}\Bigg\}$$

$$\leq \mathbf{P}\Bigg\{\max_{i=n_q+1,\dots,n_q+n_M}\Bigg|\frac{1}{l_n}\sum_{j=1}^{l_n}|\hat{q}_{n,s}(X_{s,i}^{(j)})-q_s(X_{s,i}^{(j)})|$$

$$-\mathbf{E}\left\{|\hat{q}_{n,s}(X_{s,i})-q_s(X_{s,i})||X_{s-1,i},\mathcal{D}_{n_q}\right\}\Bigg|>\epsilon\Bigg|X_{s-1,i},\mathcal{D}_{n_q}\Bigg\}$$

$$\leq n_M\cdot\max_{i=n_q+1,\dots,n_q+n_M}\mathbf{P}\Bigg\{\Bigg|\frac{1}{l_n}\sum_{j=1}^{l_n}|\hat{q}_{n,s}(X_{s,i}^{(j)})-q_s(X_{s,i}^{(j)})|$$

$$-\mathbf{E}\left\{|\hat{q}_{n,s}(X_{s,i})-q_s(X_{s,i})||X_{s-1,i},\mathcal{D}_{n_q}\right\}\Bigg|>\epsilon\Bigg|X_{s-1,i},\mathcal{D}_{n_q}\Bigg\}.$$

Conditioned on $X_{s-1,i}$ and $\mathcal{D}_{n_q}$ the random variables

$$|\hat{q}_{n,s}(X_{s,i}^{(1)})-q_s(X_{s,i}^{(1)})|,\dots,|\hat{q}_{n,s}(X_{s,i}^{(l_n)})-q_s(X_{s,i}^{(l_n)})|$$

are independent and identically distributed with expectation

$$\mathbf{E}\left\{|\hat{q}_{n,s}(X_{s,i})-q_s(X_{s,i})||X_{s-1,i},\mathcal{D}_{n_q}\right\}.$$

Since $\hat{q}_{n,s}$ and q_s are both bounded by L, these random variables are bounded by $2L$. By an application of Hoeffding‘s inequality (cf., e.g., Lemma A.3 in Györfi et al. (2002)) we get

$$n_M\cdot\max_{i=n_q+1,\dots,n_q+n_M}\mathbf{P}\Bigg\{\Bigg|\frac{1}{l_n}\sum_{j=1}^{l_n}|\hat{q}_{n,s}(X_{s,i}^{(j)})-q_s(X_{s,i}^{(j)})|$$

$$-\mathbf{E}\left\{|\hat{q}_{n,s}(X_{s,i})-q_s(X_{s,i})||X_{s-1,i},\mathcal{D}_{n_q}\right\}\Bigg|>\epsilon\Bigg|X_{s-1,i},\mathcal{D}_{n_q}\Bigg\}$$

$$\leq n_M\cdot 2\cdot\exp\left(-\frac{2\cdot l_n\cdot\epsilon^2}{4L^2}\right)=2\cdot\exp\left(-l_n\cdot\left(\frac{2\cdot\epsilon^2}{4L^2}-\frac{\log n_M}{l_n}\right)\right).$$

By assumption (11) the last term converges to zero, which proves (16).

In the fifth step of the proof we show (14). Because of (16) it suffices to show

$$\frac{1}{n_M}\sum_{i=n_q+1}^{n_q+n_M}\mathbf{E}\left\{|\hat{q}_{n,s}(X_{s,i})-q_s(X_{s,i})|\Big|X_{s-1,i},\mathcal{D}_{n_q}\right\}\to 0\quad\text{in probability.}$$

This in turn follows from

$$\frac{1}{n_M}\sum_{i=n_q+1}^{n_q+n_M}\mathbf{E}\left\{|\hat{q}_{n,s}(X_{s,i})-q_s(X_{s,i})|\Big|X_{s-1,i},\mathcal{D}_{n_q}\right\}-\mathbf{E}\left\{|\hat{q}_{n,s}(X_s)-q_s(X_s)|\Big|\mathcal{D}_{n_q}\right\}\to 0 \tag{17}$$

in probability and

$$\mathbf{E}\left\{|\hat{q}_{n,s}(X_s) - q_s(X_s)|\middle|\mathcal{D}_{n_q}\right\} \to 0 \quad \text{in probability,} \tag{18}$$

which we show next.

Conditioned on $\mathcal{D}_{n_q}$, the random variables

$$\mathbf{E}\left\{|\hat{q}_{n,s}(X_{s,i}) - q_s(X_{s,i})|\middle|X_{s-1,i}, \mathcal{D}_{n_q}\right\} \quad (i = n_q + 1, \ldots, n_q + n_M)$$

are independent and identically distributed with expectation

$$\mathbf{E}\left\{|\hat{q}_{n,s}(X_s) - q_s(X_s)|\middle|\mathcal{D}_{n_q}\right\}.$$

Since they are furthermore bounded by $2L$, (17) follows from another application of Hoeffding‘s inequality.

To show (18) we observe

$$\mathbf{E}\left\{|\hat{q}_{n,s}(X_s) - q_s(X_s)|\middle|\mathcal{D}_{n_q}\right\} = \int |\hat{q}_{n,s}(x) - q_s(x)|\mathbf{P}_{X_s}(dx).$$

By an application of the Cauchy-Schwarz inequality (18) follows from (10).

In the sixth step of the proof we show (15). We proceed similarily to the fourth step of the proof. For $\epsilon > 0$ we have

$$\begin{aligned}
&\mathbf{P}\Bigg\{\frac{1}{n_M}\sum_{i=n_q+1}^{n_q+n_M}\Bigg|\frac{1}{l_n}\sum_{j=1}^{l_n}|\max\left\{f_s(X_{s,i}^{(j)}), q_s(X_{s,i}^{(j)})\right\} \\
&\qquad\qquad -\mathbf{E}\left\{\max\left\{f_s(X_{s,i}), q_s(X_{s,i})\right\}|X_{s-1,i}\right\}\Bigg| > \epsilon\Bigg\} \\
&\leq n_M \cdot \max_{i=n_q+1,\ldots,n_q+n_M}\mathbf{P}\Bigg\{\Bigg|\frac{1}{l_n}\sum_{j=1}^{l_n}|\max\left\{f_s(X_{s,i}^{(j)}), q_s(X_{s,i}^{(j)})\right\} \\
&\qquad\qquad\qquad -\mathbf{E}\left\{\max\left\{f_s(X_{s,i}), q_s(X_{s,i})\right\}|X_{s-1,i}\right\}\Bigg| > \epsilon\Bigg\}.
\end{aligned}$$

Application of Hoeffding‘s inequality conditioned on $X_{s-1,i}$ yields that the right-hand side above is bounded by

$$n_M \cdot 2 \cdot \exp\left(-\frac{2 \cdot l_n \cdot \epsilon^2}{4L^2}\right) = 2 \cdot \exp\left(-l_n \cdot \left(\frac{2 \cdot \epsilon^2}{4L^2} - \frac{\log n_M}{l_n}\right)\right) \to 0 \quad (n \to \infty).$$

Gathering the above results, the proof is complete. □

6 Conclusion

In this paper methods from nonparametric regression have been used to construct dual Monte Carlo estimates for pricing American options in discrete time. It was shown that the estimates are consistent for all bounded Markov processes. I.e., whenever the underlying price process is a bounded Markov process, the estimated value will tend to the true value for sample size tending to infinity regardedless of the distribution of the price process. Furthermore it was illustrated by using simulated data that in the context of dual Monte Carlo estimates the use of nonparametric regression yields better results than corresponding estimates based on linear regression.

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