Upper bounds for Bermudan options on Markovian data using nonparametric regression and a reduced number of nested Monte Carlo steps

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Summary: This paper is concerned with evaluation of American options also called Bermudan options in discrete time. We use the dual approach to derive upper bounds on the price of such options using only a reduced number of nested Monte Carlo steps. The key idea is to apply nonparametric regression to estimate continuation values and all other required conditional expectations and to combine the resulting estimate with another estimate computed by using only a reduced number of nested Monte Carlo steps. The expectation of the resulting estimate is an upper bound on the option price. It is shown that the estimates of the option prices are universally consistent, i.e., they converge to the true price regardless of the structure of the continuation values. The finite sample behavior is validated by experiments on simulated data.

1 Introduction

The main advantage of Monte Carlo methods for pricing American options in discrete time, also called Bermudan options, is that they can be computed quickly compared to other methods when the number of underlying assets or state variables is large. One way to apply them is to use the dual representation of the price V_0 of an American option in discrete time given by

$$V_0 = \inf_{M \in \mathcal{M}} \mathbf{E} \left\{ \max_{t=0,\ldots,T} \left(f_t(X_t) - M_t \right) \right\}. \tag{1.1}$$

Here X_0, X_1, $\ldots$, X_T denote the underlying Markovian process describing, e.g., the prices of the underlyings and the financial environment (like interest rates, etc.), f_t is the discounted payoff function and $\mathcal{M}$ is the set of all martingales M_0, $\ldots$, M_T with $M_0 = 0$ (cf. Rogers (2001), Haugh and Kogan (2004), or

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Section 8.7 in Glasserman (2004)). Neither the Markov property nor the form of the payoff as a function of the state X_t is restrictive and can always be achieved by including supplementary variables.

We next describe the optimal martingale M_t^* at which the infimum in (1.1) is reached. Let $\mathcal{T}(t+1,\dots,T)$ be the class of all $\{t+1,\dots,T\}$–valued stopping times, i.e., of all functions $\tau = \tau(X_0,\dots,X_T)$ with values in $\{t+1,\dots,T\}$ satisfying

$$\{\tau = \alpha\} \in \mathcal{F}(X_0,\dots,X_\alpha) \quad \text{for all } \alpha \in \{t+1,\dots,T\}.$$

Let

$$q_t(x) = \sup_{\tau \in \mathcal{T}(t+1,\dots,T)} \mathbf{E}\left\{f_\tau(X_\tau)|X_t = x\right\}$$

$(t \in \{0,\dots,T-1\})$ be the so–called continuation values describing the value of the option at time t given $X_t = x$ and subject to the constraint of holding the option at time t rather than exercising it. It can be shown that the optimal martingale M_t^* is given by

$$M_t^* = \sum_{s=1}^{t} \left(\max\{f_s(X_s), q_s(X_s)\} - q_{s-1}(X_{s-1})\right) \quad (t \in \{1,\dots,T\}) \tag{1.2}$$

(cf., e.g., Section 8.7 in Glasserman (2004)). Because of

$$q_{s-1}(X_{s-1}) = \mathbf{E}\left\{\max\{f_s(X_s), q_s(X_s)\}|X_{s-1}\right\} \tag{1.3}$$

(cf. Tsitsiklis and van Roy (1999)) M_t^* is indeed a martingale.

Using (1.3) (or other regression representations like the ones in Longstaff and Schwartz (2001) or Egloff (2005)) the continuation values can be estimated recursively by the Monte Carlo method. This approach has been proposed for linear regression in Tsitsiklis and van Roy (1999) and Longstaff and Schwartz (2001). Egloff (2005), Egloff, Kohler and Todorovic (2007), Kohler, Krzyżak and Todorovic (2006), and Kohler (2008a) introduced various estimates based on nonparametric regression.

With such estimates $\hat{q}_s$ of q_s the optimal martingale M_t^* can be estimated by

$$\hat{M}_t = \sum_{s=1}^{t} \Big(\max\{f_s(X_s), \hat{q}_s(X_s)\} - \hat{\mathbf{E}}\left\{\max\{f_s(X_s), \hat{q}_s(X_s)\}|X_{s-1}\right\}\Big) \quad (t \in \{1,\dots,T\}) \tag{1.4}$$

and $\hat{M}_0 = 0$. As long as $\hat{\mathbf{E}}$ is an unbiased estimate of the corresponding conditional expectation, $\hat{M}_t$ will be a martingale and according to (1.1)

$$\mathbf{E}\left\{\max_{t=0,\dots,T} \left(f_t(X_t) - \hat{M}_t\right)\right\}$$

will be an upper bound on the price of the option. Similar estimates have been introduced in Rogers (2001) and Haugh and Kogan (2004), where linear regression was used to estimate the continuation values recursively, and where nested Monte Carlo was used to get unbiased estimates $\hat{\mathbf{E}}$ of the conditional expectation occuring in (1.2) (cf. (1.3)). Jamshidian (2007) studied multiplicative versions of this method. A comparative study of multiplicative and additive duals is contained in Chen and Glasserman (2007). Andersen and Broadie (2004) derive upper and lower bounds for American options based on duality.

Kohler (2008b) applied nonparametric regression in this context. It was shown that the resulting estimates of the option price converge to the true values regardless of the structure of the continuation values, and that their performance on simulated data was superior to the estimates based on linear regression. However, the use of nested Monte Carlo substantially increased the computational burden. In a Brownian motion setting Belomestny, Bender and Schoenmakers (2007) proposed dual estimates of option prices which do not require nested Monte Carlo and hence can be computed significantly faster.

In this article we introduce for general Markovian processes dual Monte Carlo estimates based on nonparametric regression which do not require many nested Monte Carlo steps. The key idea is to define dual estimates where all conditional expectations are estimated by nonparametric regression. In general there is no guarantee that the expectation of this kind of estimate is an upper bound on the option price. However, by combining it with a dual estimate of the option price based on nonparametric regression and nested Monte Carlo we construct another estimate, which has this property, and which requires only a reduced number of nested Monte Carlo steps. We show that our new estimates are universally consistent, i.e., they converge to the true price regardless of the structure of the continuation values. We illustrate the finite sample behavior of our estimates by experiments on simulated data.

The definition of the estimates is given in Section 2. Our main theoretical result concerning consistency of the estimates is presented in Section 3 and proven in Section 5. Section 4 contains an application of our method to simulated data.

2 Definition of the estimate

Let $X_0, X_1, \ldots, X_T$ be a $\mathbb{R}^d$–valued Markov process and let f_t be the discounted payoff function which we assume to be bounded in absolute value by L. We assume that the data generating process is completely known, i.e., that all parameters of this process are already estimated from historical data. In this section we describe dual Monte Carlo methods for estimation of V_0.

We start with the algorithm from Kohler (2008b) using nested Monte Carlo and nonparametric regression. The algorithm uses artificially generated independent Markov processes $\{X_{i,t}\}_{t=0,\ldots,T}$ $(i = 1, 2, \ldots, n + N_n)$ which are identically distributed as $\{X_t\}_{t=0,\ldots,T}$. In addition we use random variables $\{X_{i,t}^{(k)}\}_{t=0,\ldots,T}$ $(i = n+1, \ldots, n+N_n, k = 1, \ldots, K_n)$ which are constructed in such a way that given $X_{i,t-1}$ the random variables

$$X_{i,t}, X_{i,t}^{(1)}, \ldots, X_{i,t}^{(K_n)}$$

are i.i.d. and such that given $X_{i,t-1}$ the random variables $X_{i,t}^{(1)}, \ldots, X_{i,t}^{(K_n)}$ are independent of all other random variables introduced above. In a first step the first n replications $\{X_{i,t}\}_{t=0,\dots,T}$ $(i=1,2,\dots,n)$ of the Markov process are used to define regression based Monte Carlo estimates $\hat{q}_{n,t}$ of q_t. Here any of the estimates described in Egloff, Kohler and Todorovic (2007), Kohler, Krzyżak and Todorovic (2006) or Kohler (2008a) can be applied. In a second step the martingale (1.2) is estimated by

$$\bar{M}_{t,j} = \sum_{s=1}^{t} \left(\max\{f_s(X_{s,j}), \hat{q}_{n,s}(X_{s,j})\} - \frac{1}{K_n} \sum_{k=1}^{K_n} \max\{f_s(X_{s,j}^{(k)}), \hat{q}_{n,s}(X_{s,j}^{(k)})\} \right) \tag{2.1}$$

$(t \in \{1,\dots,T\})$ and $\bar{M}_0 = 0$. Since

$$\frac{1}{K_n} \sum_{k=1}^{K_n} \max\{f_s(X_{s,j}^{(k)}), \hat{q}_{n,s}(X_{s,j}^{(k)})\} \tag{2.2}$$

is an unbiased estimate of the corresponding expectation (conditioned on all data $\mathcal{D}_n$ used in the definition of $\hat{q}_{n,s}$ and conditioned on $X_{s-1,j}$), this is indeed a martingale. Consequently the expectation of the estimate

$$\hat{V}_{0,n} = \frac{1}{N_n} \sum_{j=n+1}^{n+N_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \bar{M}_{t,j} \right) \tag{2.3}$$

is an upper bound on V_0.

What makes the computation of the estimate time consuming are the nested Monte Carlo steps needed in (2.2). Here we need K_n successors of the random variable $X_{t,j}$ for each $j \in \{n+1,\dots,n+N_n\}$, so we need to simulate $N_n \cdot K_n$ random variables for each time step. The problem with this is that we need a large number N_n in order to ensure that the estimate (2.3) is close to its expectation.

In the sequel we want to modify the definition of the estimate in such a way that the estimate can be computed faster. The main idea is to use a regression estimate instead of (2.2). A simple way to define such an estimate is to set

$$\tilde{M}_t = \sum_{s=1}^{t} \left(\max\{f_s(X_s), \hat{q}_{n,s}(X_s)\} - \hat{q}_{n,s-1}(X_{s-1}) \right) \quad (t \in \{1,\dots,T\})$$

and to estimate the price of the option by

$$\mathbf{E}^* \left\{ \max_{t=0,\dots,T} \left(f_t(X_t) - \tilde{M}_t \right) \right\}, \tag{2.4}$$

where $\mathbf{E}^*$ denotes the expectation conditioned on $\mathcal{D}_n$. However, since for $t>0$

$$f_t(X_t) - \tilde{M}_t \le f_t(X_t) - \sum_{s=1}^{t-1} \left(\hat{q}_{n,s}(X_s) - \hat{q}_{n,s-1}(X_{s-1}) \right) - \left(f_t(X_t) - \hat{q}_{n,t-1}(X_{t-1}) \right) = \hat{q}_{n,0}(X_0),$$

where we have equality in case that t is the first index with $f_t(X_t) \ge \hat{q}_{n,t}(X_t)$, (2.4) is in fact equal to

$$\mathbf{E}^* \left\{ \max\{f_0(X_0), \hat{q}_{n,0}(X_0)\} \right\},$$

and this will in general be no longer an upper bound on V_0, which satisfies

$$V_0 = \mathbf{E} \left\{ \max\{f_0(X_0), q_0(X_0)\} \right\}$$

(cf., e.g., Section 8.7 in Glasserman (2004)).

Instead, we use a second regression estimate $\hat{q}_{\tilde{n},s-1}$ in order to estimate the conditional expectation corresponding to (2.2). Here we will use a sample size $\tilde{n}$ larger than n. To be able to compute this estimate quickly, we set $\tilde{n} = K_n \cdot n$ and compute the estimate by applying it to a sample of n i. i. d. random variables with the same distribution as

$$\left(X_{s-1}, \frac{1}{K_n} \sum_{k=1}^{K_n} \max\{f_s(X_s^{(k)}), \hat{q}_{n,s}(X_s^{(k)})\} \right).$$

The regression function of this sample is

$$q^*_{n,s-1}(x) = \mathbf{E}^* \left\{ \max\{f_s(X_s), \hat{q}_{n,s}(X_s)\} \middle| X_{s-1} = x \right\},$$

which is indeed the function we want to approximate.

Our corresponding estimate of the option price is

$$\hat{V}_{1,n} = \frac{1}{N_n} \sum_{j=n+1}^{n+N_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right) \tag{2.5}$$

where

$$\hat{M}_{t,j} = \sum_{s=1}^{t} \left(\max\{f_s(X_{s,j}), \hat{q}_{n,s}(X_{s,j})\} - \hat{q}_{\tilde{n},s-1}(X_{s-1,j}) \right). \tag{2.6}$$

Unfortunately, there is no guarantee that the expectation of this estimate is indeed an upper bound on the option price. To construct an estimate with that property, we use an idea similar to control variates (cf., e.g., Section 4.1 in Glasserman (2004)) and combine (2.5) with (2.3). To do this, we define the estimate

$$\begin{aligned} \hat{V}_{2,n} \;=\; & \frac{1}{N_n} \sum_{j=n+1}^{n+N_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right) \\ & + \frac{1}{\bar{N}_n} \sum_{j=n+1}^{n+\bar{N}_n} \left(\max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \bar{M}_{t,j} \right) - \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right) \right), \end{aligned} \tag{2.7}$$

where $\bar{N}_n \leq N_n$ is an additional parameter of the estimate (apparently in the case $\bar{N}_n = N_n$ the estimates (2.3) and (2.7) coincide). Thus in the present paper estimate (2.3), which has been proposed in Kohler (2008b), is replaced by estimates (2.5) and (2.7). Clearly, the expectation of estimate (2.7) is equal to the expectation of $\hat{V}_{0,n}$ and hence it provides an upper bound on the option price. We conjecture that

$$\max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \bar{M}_{t,j} \right) \quad \text{and} \quad \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right)$$

are close and therefore the standard deviation of

$$\max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \bar{M}_{t,j} \right) - \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right) \tag{2.8}$$

is smaller than the standard deviation of

$$\max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right). \tag{2.9}$$

As we will see in Section 4, this is indeed true in our simulation. There the standard deviation of (2.8) will be approximately half of the standard deviation of (2.9). Since the error of a Monte Carlo estimate of an expectation is of order

$$\frac{s}{\sqrt{N_n}},$$

where s is the standard deviation and N_n is the sample size, this allows us to choose $\bar{N}_n \approx N_n/4$, which for (2.7) drastically reduces the number of nested Monte Carlo steps compared to (2.3).

3 Main theorem

Our main theoretical result is the following theorem:

Theorem 1 *Let $L > 0$, let $X_0, X_1, \ldots, X_T$ be a $\mathbb{R}^d$–valued Markov process and assume that the discounted payoff function f_t is bounded in absolute value by L. Let the estimates $\hat{V}_{1,n}$ and $\hat{V}_{2,n}$ be defined as in Section 2. Assume that the estimates $\hat{q}_{n,t}$ of q_t are bounded in absolute value by L and satisfy*

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \to 0 \quad \textit{in probability}, \tag{3.1}$$

and that

$$N_n \to \infty,\, \bar{N}_n \to \infty \quad and \quad \frac{K_n}{\log \bar{N}_n} \to \infty \quad (n \to \infty). \tag{3.2}$$

Then

$$\hat{V}_{1,n} \to V_0 \quad \textit{in probability} \tag{3.3}$$

and

$$\hat{V}_{2,n} \to V_0 \quad \textit{in probability}. \tag{3.4}$$

The estimates defined in Egloff, Kohler and Todorovic (2007), Kohler, Krzyżak and Todorovic (2006) and Kohler (2008) satisfy (3.1) for all bounded Markov processes. Hence if we use any of these estimates in the definition of our new estimate, we get universally consistent upper bounds on the price of V_0.

Corollary 2 *Let $A, L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is a $[-A, A]^d$–valued Markov process and that the discounted payoff function f_t is bounded in absolute value by L. Let the estimates $\hat{V}_{1,n}$ and $\hat{V}_{2,n}$ be defined as in Section 2 where q_t is estimated by the least squares splines as in Egloff, Kohler and Todorovic (2007), by the least squares neural networks as in Kohler, Krzyżak and Todorovic (2006) or by the smoothing splines as in Kohler (2008a). Choose N_n, $\bar{N}_n$ and K_n such that*

$$N_n \to \infty,\, \bar{N}_n \to \infty \quad and \quad \frac{K_n}{\log \bar{N}_n} \to \infty, \quad (n \to \infty).$$

Then

$$\hat{V}_{1,n} \to V_0 \quad \textit{in probability}$$

and

$$\hat{V}_{2,n} \to V_0 \quad \textit{in probability}.$$

Proof. The assertion follows from Theorem 1 above and Theorem 4.1 in Egloff, Kohler and Todorovic (2007), Corollary 1 in Kohler, Krzyżak and Todorovic (2006) and Theorem 1 in Kohler (2008). □

Remark. As stated in the last paragraph of Section 2 the expectation of $\hat{V}_{2,n}$ is an upper bound on V_0, i.e.,

$$\mathbf{E}\hat{V}_{2,n} \geq V_0.$$

4 Application to simulated data

In this section, we illustrate the finite sample behavior of our algorithm by comparing it with algorithms for computing dual upper bounds with linear regression using the regression representations proposed by Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001), respectively, and by comparing it with the algorithm in Kohler (2008b).

We consider an American option based on the average of five correlated stock prices. The stocks are ADECCO R, BALOISE R, CIBA, CLARIANT and CREDIT SUISSE R. The stock prices were observed from Nov. 10, 2000 until Oct. 3, 2003 on weekdays when the stock market was open for the total of 756 days. We estimate the volatility from data observed in the past by the historical volatility

$$\sigma = \begin{pmatrix} 0.3024 & 0.1354 & 0.0722 & 0.1367 & 0.1641 \\ 0.1354 & 0.2270 & 0.0613 & 0.1264 & 0.1610 \\ 0.0722 & 0.0613 & 0.0717 & 0.0884 & 0.0699 \\ 0.1367 & 0.1264 & 0.0884 & 0.2937 & 0.1394 \\ 0.1641 & 0.1610 & 0.0699 & 0.1394 & 0.2535 \end{pmatrix}.$$

We simulate the paths of the underlying stocks with a Black-Scholes model by

$$X_{i,t} = x_0 \cdot e^{r \cdot t} \cdot e^{\sum_{j=1}^{5} (\sigma_{i,j} \cdot W_j(t) - \frac{1}{2} \cdot \sigma_{i,j}^2 t)} \quad (i = 1, \ldots, 5),$$

where $\{W_j(t) : t \in \mathbb{R}_+\}$ $(j = 1, \ldots, 5)$ are five independent Wiener processes and where the parameters are chosen as follows: $x_0 = 100$, $r = 0.05$ and components $\sigma_{i,j}$ of the volatility matrix as above. The time to maturity is assumed to be one year. To compute the payoff of the option we use a strangle spread function (cf. Figure 4.1) with strikes 75, 90, 110 and 125 applied to the average of the five correlated stock prices.

We discretize the time interval $[0, 1]$ by dividing it into $m = 48$ equidistant time steps with $t_0 = 0 < t_1 < \ldots < t_m = 1$ and consider a Bermudan option with payoff function as above and exercise dates restricted to $\{t_0, t_1, \ldots, t_m\}$. We choose discount factors e^{-rt_j} for $j = 0, \ldots, m$. For the algorithm in Kohler (2008b) we set $n_q = 2000$, $n_M = 1000$ and $l_n = 100$, and for our newly proposed estimates we set

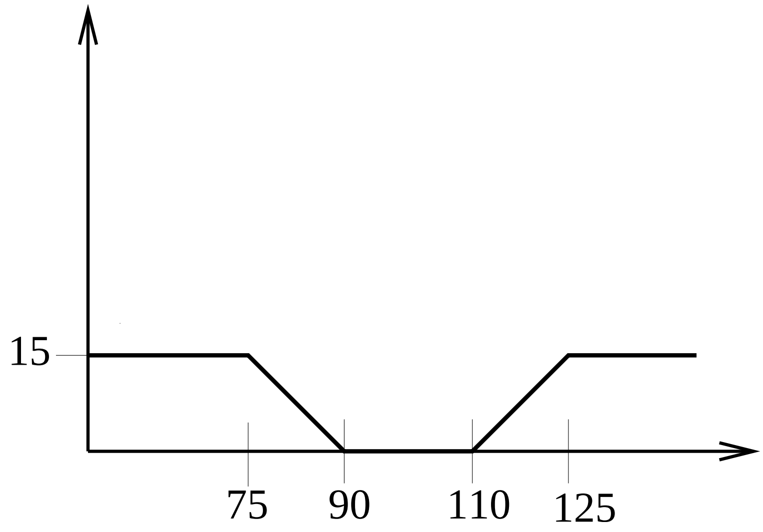

Figure 4.1 Strangle spread payoff with strike prices 75, 90, 110 and 125.

$n = 2000$, $K_n = 20$, $N_n = 1000$ and $\bar{N}_n = 250$. For the other algorithms we use parameters $n = 2000$, $N_n = 1000$ and $K_n = 100$.

For the algorithms using nonparametric regression we use smoothing splines as implemented in the routine $Tps()$ from the library "fields" in the statistics package R, where the smoothing parameter is chosen by generalized cross-validation. For the Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms we use linear regression as implemented in R.

We apply all five algorithms to 100 independently generated sets of paths. For each algorithm and each of the 100 sets of paths we compute our Monte Carlo estimate of the option price. We would like to stress that for all estimates except $\hat{V}_{1,n}$ the expectations are upper bounds to the true option price, hence lower values indicate a better performance of these algorithms.

We compare the algorithms using boxplots for the 100 upper bounds computed for each algorithm. As we can see in Figure 4.2, all algorithm using nonparametric regression are superior to Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms, since the lower boxplot of the upper bounds indicates better performance.

Furthermore we can see that our newly proposed estimate $\hat{V}_{2,n}$ achieves similar values to the estimate proposed in Kohler (2008b). However, $\hat{V}_{2,n}$ can be computed for sample size $N_n = 1000$ approximately 20% faster. This computational advantage concerning computing time will grow if we want to have estimates which are closer to their expectations and therefore increase N_n.

In Figure 4.3 we compare the empirical standard deviations of the values occuring in the arithmetic

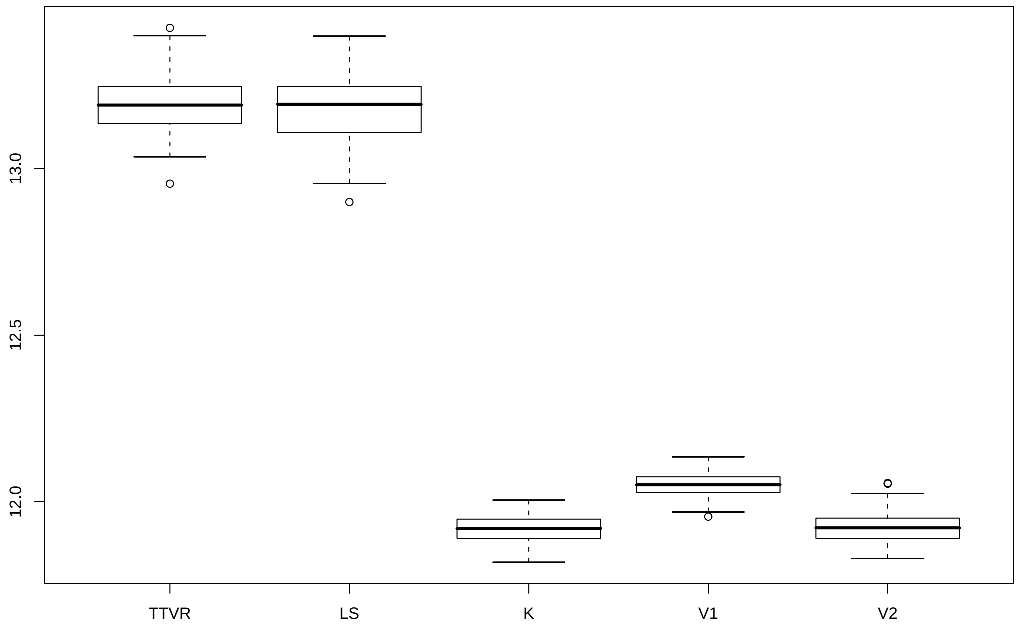

Figure 4.2 Upper bounds computed with a dual Monte-Carlo method based on linear regression and the Tsitsiklis–Van Roy algorithm (TTVR), linear regression and the Longstaff–Schwartz algorithm (LS), the algorithm proposed in Kohler (2008b) (K) and the newly proposed smoothing spline estimates $\hat{V}_{1,n}$ (V1) and $\hat{V}_{2,n}$ (V2) in a 5-dimensional case.

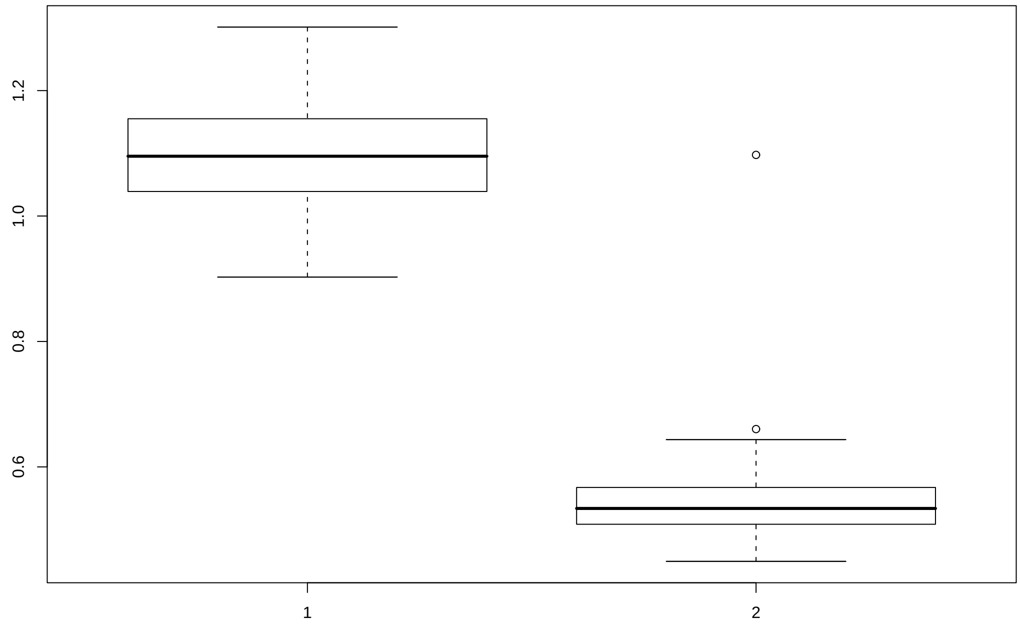

Figure 4.3 Comparison of the standard deviations occurring in our simulations during computation of $\hat{V}_{2,n}$

means in

$$\frac{1}{N_n} \sum_{j=n+1}^{n+N_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right)$$

and

$$\frac{1}{\bar{N}_n} \sum_{j=n+1}^{n+\bar{N}_n} \left(\max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \bar{M}_{t,j} \right) - \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right) \right)$$

occurring in our simulations marked by 1 and 2, resp. As one can see, the standard deviations of the terms occurring in the second sum are indeed most of the time approximately at most half as large as the standard deviations of the terms occurring in the first terms. This shows that we can use $\bar{N}_n$ of approximately one quarter size of N_n.

5 Proof of Theorem 1

In the proof we will use the notation

$$\bar{V}_{0,n} = \frac{1}{N_n} \sum_{j=n+1}^{n+N_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - M^*_{t,j} \right),$$

where

$$M^*_{t,j} = \sum_{s=1}^{t} \left(\max\{f_s(X_{s,j}), q_s(X_{s,j})\} - q_{s-1}(X_{s-1,j}) \right) \quad (t \in \{1,\dots,T\}).$$

In the first step of the proof we show

$$\bar{V}_{0,n} \to V_0 \quad \text{in probability.} \tag{5.1}$$

Let $(\tilde{X}_{t,j})_{t=0,\dots,T}$ $(j = 1,\dots,N_n)$ be independent copies of $(X_t)_{t=0,\dots,T}$, which are independent of all previously introduced data. Because of

$$V_0 = \mathbf{E}\left\{ \max_{t=0,\dots,T} \left(f_t(X_t) - M^*_t\right) \right\}$$

(cf. (1.1) and (1.2)) we have for any $\epsilon > 0$

$$\begin{aligned}
&\mathbf{P}\left\{\left|\bar{V}_{0,n} - V_0\right| > \epsilon\right\} \\
&= \mathbf{P}\Bigg\{\Bigg| \frac{1}{N_n} \sum_{j=n+1}^{n+N_n} \max_{t=0,\dots,T} \Bigg(f_t(X_{t,j}) - \sum_{s=1}^{t} \Big(\max\{f_s(X_{s,j}), q_s(X_{s,j})\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - q_{s-1}(X_{s-1,j}) \Big) \Bigg) - V_0 \Bigg| > \epsilon \Bigg\} \\
&= \mathbf{P}\Bigg\{\Bigg| \frac{1}{N_n} \sum_{j=1}^{N_n} \max_{t=0,\dots,T} \Bigg(f_t(\tilde{X}_{t,j}) - \sum_{s=1}^{t} \Big(\max\{f_s(\tilde{X}_{s,j}), q_s(\tilde{X}_{s,j})\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - q_{s-1}(\tilde{X}_{s-1,j}) \Big) \Bigg) - V_0 \Bigg| > \epsilon \Bigg\}.
\end{aligned}$$

¿From this we can conclude (5.1) by an application of the law of large numbers.

In the second step of the proof we show that for all $s \in \{1,\dots,T-1\}$

$$\frac{1}{N_n} \sum_{j=n+1}^{n+N_n} |\hat{q}_{n,s}(X_{s,j}) - q_s(X_{s,j})| \to 0 \quad (n \to \infty) \quad \text{in probability} \tag{5.2}$$

and

$$\frac{1}{N_n} \sum_{j=n+1}^{n+N_n} |\hat{q}_{\tilde{n},s}(X_{s,j}) - q_s(X_{s,j})| \to 0 \quad (n \to \infty) \quad \text{in probability.} \tag{5.3}$$

Set

$$\mathcal{D}_n = \{X_{i,s} \quad : \quad i = 1,\dots,n, s = 1,\dots,T\}.$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\mathbf{E}\left\{\frac{1}{N_n}\sum_{j=n+1}^{n+N_n}|\hat{q}_{n,s}(X_{s,j})-q_s(X_{s,j})|\Big|\mathcal{D}_n\right\}\\
&=\mathbf{E}\left\{|\hat{q}_{n,s}(X_s)-q_s(X_s)|\Big|\mathcal{D}_n\right\}\\
&\leq\sqrt{\mathbf{E}\left\{|\hat{q}_{n,s}(X_s)-q_s(X_s)|^2\Big|\mathcal{D}_n\right\}}\\
&=\sqrt{\int|\hat{q}_{n,s}(x)-q_s(x)|^2\mathbf{P}_{X_s}(dx)}.
\end{aligned}$$

Since $\hat{q}_{n,s}$ and q_s are bounded, assumption (3.1) together with the dominated convergence theorem yields

$$\mathbf{E}\left\{\frac{1}{N_n}\sum_{j=n+1}^{n+N_n}|\hat{q}_{n,s}(X_{s,j})-q_s(X_{s,j})|\right\}\leq\mathbf{E}\sqrt{\int|\hat{q}_{n,s}(x)-q_s(x)|^2\mathbf{P}_{X_s}(dx)}\to 0\quad(n\to\infty),$$

which in turn implies (5.2). By replacing $\hat{q}_{n,s}$ by $\hat{q}_{\tilde{n},s}$ in the above proof we get (5.3) as well.

In the third step of the proof we show (3.3). Observe

$$\begin{aligned}
&\left|\hat{V}_{1,n}-\bar{V}_{0,n}\right|\\
&\leq\frac{1}{N_n}\sum_{j=n+1}^{n+N_n}\max_{t=1,\ldots,T}\left|\hat{M}_{t,j}-M^*_{t,j}\right|\\
&\leq\frac{1}{N_n}\sum_{j=n+1}^{n+N_n}\max_{t=1,\ldots,T}\left|\sum_{s=1}^{t}(\max\{f_s(X_{s,j}),\hat{q}_{n,s}(X_{s,j})\}-\max\{f_s(X_{s,j}),q_s(X_{s,j})\})\right|\\
&\quad+\frac{1}{N_n}\sum_{j=n+1}^{n+N_n}\max_{t=1,\ldots,T}\left|\sum_{s=1}^{t}(\hat{q}_{\tilde{n},s-1}(X_{s,j})-q_{s-1}(X_{s-1,j}))\right|\\
&\leq\sum_{s=1}^{T-1}\frac{1}{N_n}\sum_{j=n+1}^{n+N_n}|\hat{q}_{n,s}(X_{s,j})-q_s(X_{s,j})|\\
&\quad+\sum_{s=1}^{T-1}\frac{1}{N_n}\sum_{j=n+1}^{n+N_n}|\hat{q}_{\tilde{n},s-1}(X_{s,j})-q_{s-1}(X_{s-1,j})|,
\end{aligned}$$

where we have used

$$\hat{q}_{n,T}(x)=\hat{q}_{\tilde{n},T}(x)=q_T(x)=0\quad(x\in\mathbb{R}^d).$$

By applying (5.2) and (5.3) we get

$$\hat{V}_{1,n}-\bar{V}_{0,n}\to 0\quad\text{in probability.}\tag{5.4}$$

Finally (5.4) together with (5.1) yields (3.3).

In the fourth step of the proof we show (3.4). Here we observe that the first three steps of the proof yield

$$\hat{V}_{1,n}\to V_0\quad\text{in probability.}$$

In the same way one can prove

$$\frac{1}{\bar{N}_n} \sum_{j=n+1}^{n+\bar{N}_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right) \to V_0 \quad \text{in probability.}$$

Furthermore, by Theorem 1 in Kohler (2008b) we know

$$\frac{1}{\bar{N}_n} \sum_{j=n+1}^{n+\bar{N}_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \bar{M}_{t,j} \right) \to V_0 \quad \text{in probability.}$$

¿From this we conclude

$$\begin{aligned} \hat{V}_{2,n} &= \frac{1}{N_n} \sum_{j=n+1}^{n+N_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right) \\ &+ \frac{1}{\bar{N}_n} \sum_{j=n+1}^{n+\bar{N}_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \bar{M}_{t,j} \right) - \frac{1}{\bar{N}_n} \sum_{j=n+1}^{n+\bar{N}_n} \max_{t=0,\dots,T} \left(f_t(X_{t,j}) - \hat{M}_{t,j} \right) \\ &\to V_0 + V_0 - V_0 = V_0 \quad \text{in probability.} \end{aligned}$$

The proof is complete. □

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