Pricing of American options in discrete time using least squares estimates with complexity penalties

Michael Kohler[1] and Adam Krzyżak[2,*]

[1] *Department of Mathematics, Technische Universität Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany, email: kohler@mathematik.tu-darmstadt.de*

[2] *Department of Computer Science and Software Engineering, Concordia University, 1455 De Maisonneuve Blvd. West, Montreal, Quebec, Canada H3G 1M8, email: krzyzak@cs.concordia.ca*

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Abstract

Pricing of American options in discrete time is considered, where the option is allowed to be based on several underlying stocks. It is assumed that the price processes of the underlying stocks are given Markov processes. We use the Monte Carlo approach to generate artificial sample paths of these price processes, and then we use nonparametric regression estimates to estimate from this data so-called continuation values, which are defined as mean values of the American option for given values of the underlying stocks at time t subject to the constraint that the option is not exercised at time t. As nonparametric regression estimates we use least squares estimates with complexity penalties, which include as special cases least squares spline estimates, least squares neural networks, smoothing splines and orthogonal series estimates. General results concerning rate of convergence are presented and applied to derive results for the special cases mentioned above. Furthermore the pricing of American options is illustrated by simulated data.

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*Corresponding author. Tel: +1-514-848-2424, ext. 3007, Fax: +1-514-848-2830

Running title: *Pricing of American options using penalized least squares estimates*

1 Introduction

Monte Carlo methods for pricing of American options are very attractive in comparison to other methods in case of options which are based on several (correlated) stocks (so-called basket options). Because in this case in the standard approach, in which the whole problem is reformulated as a free boundary problem for partial differential equations (cf., e.g., Chapter 8 in Elliott and Kopp (1999)), the numerical solution of this free boundary problem gets very complicated. And alternative methods based on binomial trees (cf., e.g., Chapter 1 in Elliott and Kopp (1999)) are in practice not able to model the correlation structure of more than two stocks correctly.

In this paper we consider American options in discrete time (sometimes also called Bermudan options). The price of such an option can be represented in a risk neutral market as a solution of an optimal stopping problem

$$V_0 = \sup_{\tau \in \mathcal{T}(0,\ldots,T)} \mathbf{E}\left\{ f_\tau(X_\tau) \right\}. \tag{1}$$

Here f_t is the (discounted) payoff function, $X_0, X_1, \ldots, X_T$ is the underlying stochastic process describing e.g. the prices of the underlying assets and the financial environment (like interest rates, etc.) and $\mathcal{T}(0,\ldots,T)$ is the class of all $\{0,\ldots,T\}$-valued stopping times, i.e., $\tau \in \mathcal{T}(0,\ldots,T)$ is a measurable function of $X_0, \ldots, X_T$ satisfying

$$\{\tau = \alpha\} \in \mathcal{F}(X_0,\ldots,X_\alpha) \quad \text{for all } \alpha \in \{0,\ldots,T\}.$$

In the sequel we assume that $X_0, X_1, \ldots, X_T$ is a $\mathbb{R}^d$–valued Markov process recording all necessary information about financial variables including prices of the underlying assets as well as additional risk factors driving stochastic volatility or stochastic interest rates. Neither the Markov property nor the form of the payoff as a function of the state X_t is restrictive and can always be achieved by including supplementary variables.

The computation of (1) can be done by determination of an optimal stopping rule $\tau^* \in \mathcal{T}(0,\ldots,T)$ satisfying

$$V_0 = \mathbf{E}\{f_{\tau^*}(X_{\tau^*})\}. \tag{2}$$

Let

$$q_t(x) = \sup_{\tau \in \mathcal{T}(t+1,\ldots,T)} \mathbf{E}\left\{ f_\tau(X_\tau) | X_t = x \right\} \tag{3}$$

be the so–called continuation value describing the value of the option at time t given $X_t = x$ and subject to the constraint of holding the option at time t rather than exercising it. Here $\mathcal{T}(t+1,\ldots,T)$ is the class of all $\{t+1,\ldots,T\}$–valued stopping times. Set $q_T(x) = 0$. It can be shown that

$$\tau^* = \inf\{s \geq 0 \,:\, q_s(X_s) \leq f_s(X_s)\} \tag{4}$$

satisfies (2), i.e., τ^* is an optimal stopping time (cf., e.g., Chow, Robbins and Siegmund (1971) or Shiryayev (1978)). Therefore it suffices to compute the continuation values (3) in order to solve the optimal stopping problem (1).

One way to compute the continuation values is to use a regression representation like

$$q_t(x) \quad = \quad \mathbf{E}\left\{\max\{f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\} | X_t = x\right\} \quad (t = 0, 1, \ldots, T-1) \tag{5}$$

(cf. Tsitsiklis and Van Roy (1999), further regression representations can be found in Longstaff and Schwartz (2001) and Egloff (2005)). Typically, the underlying distributions are rather complicated, therefore it is not possible to compute the conditional expectation in (5) directly.

The basic idea of regression-based Monte Carlo methods for pricing American options is to apply recursively regression estimates to artificially created samples of

$$(X_t, \max\left\{f_{t+1}(X_{t+1}), \hat{q}_{t+1}(X_{t+1})\right\})$$

(so–called Monte Carlo samples) to construct estimates $\hat{q}_t$ of q_t. In connection with linear regression this was proposed in Tsitsiklis and Van Roy (1999), and, based on a different regression estimation than (5), in Longstaff and Schwartz (2001). Nonparametric least squares regression estimates have been applied and investigated in this context in Egloff (2005), Egloff, Kohler and Todorovic (2007) and Kohler, Krzyżak and Todorovic (2006), smoothing spline regression estimates have been analyzed in Kohler (2008b), recursive kernel regression estimates have been considered in Barty et al. (2006), and local polynomial kernel estimates have been studied in Belomestny (2009).

In this article we develop a general theory which covers the estimates of most of the above papers as well as additional ones (e.g., orthogonal series estimates). Our main theoretical results provide a unifying tool which enables to derive rate of convergence results for many estimates at once. Furthermore we illustrate the estimates by applying them to an option based on the average of three correlated stocks. The simulations show that the nonparametric estimates studied in this article produce better results than the existing parametric ones.

1.1 Notation. The sets of natural numbers, natural numbers including zero, integers, real numbers and non-negative real numbers are denoted by $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$ and $\mathbb{R}_+$, respectively. The least integer greater than or equal to a real number x will be denoted by $\lceil x \rceil$. $\log(x)$ denotes the natural logarithm of $x > 0$. For a function $f : \mathbb{R}^d \to \mathbb{R}$ the partial derivative with respect to the j-th component will be denoted by

$$\frac{\partial f}{\partial x_j}.$$

We say that $a_n = O_{\mathbf{P}}(b_n)$ if $\limsup_{n\to\infty} \mathbf{P}(a_n > c \cdot b_n) = 0$ for some finite constant c.

1.2 Outline of the paper. The precise definition of the estimates and the main theoretical results concerning rate of convergence of the estimate are given in Sections 2 and 3, respectively. The application of the estimates to simulated data will be described in Section 4, and the proofs will be given in Section 5.

2 Definition of the estimate

The definition of the estimates depends on a finite set of parameters $\mathcal{P}_n$, sets of functions $\mathcal{G}_p$ $(p \in \mathcal{P}_n)$ and penalties

$$pen_p^2(g) = pen_{n,p}^2(g) \geq 0$$

for $g \in \mathcal{G}_p$ and $p \in \mathcal{P}_n$. Here we supress the dependency of $pen_p^2(g)$ on the sample size for ease of notation. Set $n_l = \lceil n/2 \rceil$. Furthermore choose $\beta_n \in \mathbb{R}_+$ such that

$$\beta_n \to \infty \quad (n \to \infty).$$

Let

$$(X_{i,t}^{(l)})_{t\in\{0,\ldots,T\}} \quad (i \in \{1, \ldots, n\}, l \in \{0, \ldots, T-1\})$$

be independent copies of the price process

$$(X_t)_{t\in\{0,\ldots,T\}}.$$

We define recursively estimates $\hat{q}_{n,t}$ of q_t as follows: Set

$$\hat{q}_{n,T}(x) = 0 \quad (x \in \mathbb{R}^d).$$

In order to define $\hat{q}_{n,t}$ given $\hat{q}_{n,t+1}$ for $t \in \{0, 1, \ldots, T-1\}$, set

$$\hat{Y}_{i,t} = \max\left(f_{t+1}(X_{i,t+1}^{(t)}), \hat{q}_{n,t+1}(X_{i,t+1}^{(t)})\right) \quad (i \in \{1, \ldots, n\}).$$

Define for each $p \in \mathcal{P}_n$ estimates $\tilde{q}_{n_l,t}^{(p)}$ of q_t by

$$\tilde{q}_{n_l,t}^{(p)} = \arg\min_{g \in \mathcal{G}_p} \left(\frac{1}{n_l} \sum_{i=1}^{n_l} |\hat{Y}_{i,t} - g(X_{i,t}^{(t)})|^2 + pen_p^2(g) \right)$$

and set

$$\hat{q}_{n_l,t}^{(p)}(x) = T_{\beta_n} \tilde{q}_{n_l,t}^{(p)}(x) \quad (x \in \mathbb{R}^d),$$

where

$$T_{\beta_n} z = \max\{-\beta_n, \min\{\beta_n, z\}\}$$

for $z \in \mathbb{R}$. We use sample splitting to determine p as follows

$$\hat{p} = \arg\min_{p \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{Y}_{i,t} - \hat{q}_{n_l,t}^{(p)}(X_{i,t}^{(t)})|^2.$$

Our final estimate is defined by

$$\hat{q}_{n,t}(x) = \hat{q}_{n_l,t}^{(\hat{p})}(x).$$

Special choices of the function spaces $\mathcal{G}_p$ and the penalties $pen_p^2(g)$ lead to various estimates. If we set

$$pen_n^2(g) = 0 \quad \text{for all } g \in \mathcal{G}_p, p \in \mathcal{P}_n$$

(i.e., if we do not use a penalty term), then we get the least squares estimates for pricing of American options (e.g., the least squares spline estimates similar to the estimates in Egloff, Kohler and Todorovic (2007), or least squares neural network estimates like in Kohler, Krzyżak and Todorovic (2006)).

If we set $p = (k, \lambda)$ and let

$$\mathcal{G}_p = \mathcal{G}$$

be the class of all functions $g : \mathbb{R}^d \to \mathbb{R}$ with derivatives of all order and set

$$pen_p^2(g) = \lambda \cdot \sum_{\alpha_1, \ldots, \alpha_d \in \mathbb{N}_0,\, \alpha_1 + \ldots + \alpha_d = k} \frac{k!}{\alpha_1! \cdot \ldots \cdot \alpha_d!} \int_{\mathbb{R}^d} \left| \frac{\partial^k g}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}(x) \right|^2 dx$$

then we get the smoothing spline estimates for pricing of American options as in Kohler (2008b).

Furthermore we can also include in our framework the orthogonal series estimates with hard-thresholding. In this case we choose $p = c \in \mathbb{R}_+$, choose function spaces

$$\mathcal{G}_p = \mathcal{G}(X_{1,t}^{(t)}, \ldots, X_{n_l,t}^{(t)}) = span\,\{h_i \,:\, i \in \{1, \ldots, n_l\}\}\,,$$

where $h_1, \ldots, h_{n_l}$ are the generalized Haar-Wavelets from Kohler (2008a), which are orthogonal with respect to the empirical scalar product

$$< h_i, h_j >= \frac{1}{n_l} \sum_{k=1}^{n_l} h_i(X_{k,t}^{(t)}) \cdot h_j(X_{k,t}^{(t)}),$$

and set

$$pen_p^2\left(\sum_{k=1}^{n_l} a_k \cdot h_k\right) = c \cdot \log n_l \cdot \frac{\#\{1 \le k \le n_l \,:\, a_k \neq 0\}}{n_l}.$$

As it is shown in Kohler (2008a) we have in this case

$$\tilde{q}_{n_l,t}^{(p)} = \sum_{k=1}^{n_l} \eta_{\frac{\sqrt{c \cdot \log n_l}}{\sqrt{n_l}}} \left(\frac{1}{n_l} \sum_{k=1}^{n_l} \hat{Y}_{i,t} \cdot h_k(X_{k,t}^{(t)})\right) \cdot h_k,$$

where η_δ is the hard-thresholding defined by

$$\eta_\delta(a) = \begin{cases} a & \text{if} \quad |a| > \delta, \\ 0 & \text{if} \quad |a| \le \delta. \end{cases}$$

3 Theoretical results

3.1 Main result

In order to be able to formulate our main result we need the notion of covering numbers. Let $x_1, \ldots, x_n \in \mathbb{R}^d$ and set $x_1^n = (x_1, \ldots, x_n)$. Define the distance $d_2(f, g)$ between $f, g : \mathbb{R}^d \to \mathbb{R}$ by

$$d_2(f, g) = \left(\frac{1}{n} \sum_{i=1}^{n} |f(x_i) - g(x_i)|^2\right)^{1/2}.$$

An ϵ-cover of $\mathcal{F}$ (w.r.t. the distance d_2) is a set of functions $f_1, \ldots, f_\kappa : \mathbb{R}^d \to \mathbb{R}$ with the property

$$\min_{1 \le j \le \kappa} d_2(f, f_j) < \epsilon \quad \text{for all } f \in \mathcal{F}\,.$$

Let $\mathcal{N}_2(\epsilon, \mathcal{F}, x_1^n)$ denote the size κ of the smallest ϵ-cover of $\mathcal{F}$ w.r.t. the distance d_2, and set $\mathcal{N}_2(\epsilon, \mathcal{F}, x_1^n) = \infty$ if there does not exist any ϵ-cover of $\mathcal{F}$ of a finite size.

Our main theoretical result is the following theorem.

Theorem 1 *Let $L \geq 1$ and let $\beta_n \geq L$ for $n \in \mathbb{N}$. Assume that $X_0, X_1, \ldots, X_T$ is a $\mathbb{R}^d$-valued Markov process and that the discounted payoff function f_t is bounded in absolute value by L. Define the estimate $\hat{q}_{n,t}$ as in Section 2. Assume*

$$|\mathcal{P}_n| \to \infty \quad (n \to \infty) \quad \textit{and} \quad \frac{|\mathcal{P}_n|}{n^r} \to 0 \quad (n \to \infty)$$

for some $r > 0$ and let $p_n \in \mathcal{P}_n$ be an arbitrary sequence of parameters. Assume that $(\delta_n)_n$ is monotonically decreasing such that $\delta_n > c_1/n$,

$$\delta_n \to 0 \quad (n \to \infty) \quad \textit{and} \quad \frac{n \cdot \delta_n}{\beta_n^2 \cdot \log n} \to \infty \quad (n \to \infty),$$

and

$$\frac{\sqrt{n_l} \cdot \delta}{\beta_n^2} \geq c_2 \int_{c_3\delta/\beta_n^2}^{\sqrt{48\delta}} \Bigg(\log \mathcal{N}_2 \Bigg(u, \{T_{\beta_n} f - g : f \in \mathcal{G}_{p_n}, \frac{1}{n_l} \sum_{i=1}^{n_l} |T_{\beta_n} f(x_i) - g(x_i)|^2 \leq 4\delta, pen_{p_n}^2(f) \leq 4\delta\}, x_1^{n_l} \Bigg) \Bigg)^{1/2} du \quad (6)$$

for all $\delta \geq \delta_{n_l}/6$, all $g \in \mathcal{G}_{p_n} \cup \{q_t\}$, all $t \in \{0, 1, \ldots, T-1\}$ and all $x_1, \ldots, x_{n_l} \in \mathbb{R}$. Then

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}} \left(\delta_{n_l} + \max_{s \in \{t, t+1, \ldots, T-1\}} \inf_{f \in \mathcal{G}_{p_n}, \|f\|_\infty \leq \beta_n} \left(pen_{p_n}^2(f) + \int |f(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx) \right) \right) \quad (7)$$

for all $t \in \{0, 1, \ldots, T-1\}$.

Remark 1. Theorem 1 cannot be applied in case of the orthogonal series estimate, where the function spaces are data-dependent which is not allowed in Theorem 1. But in this case we can use the following modification of Theorem 1, which is implied by the proof of the theorem.

Assume that there exist sets

$$\mathcal{F}_k \subseteq \bar{\mathcal{F}}_k$$

of functions $f : \mathbb{R}^d \to \mathbb{R}$ for $k \in \{1, \ldots, n_l\}$ and penalties

$$pen_p^2(k) = pen_{n,p}^2(k) \geq 0 \quad (k \in \{1, \ldots, n_l\}, p \in \mathcal{P}_n)$$

such that the estimate $\tilde{q}_{n,t}^{(p)}$ satisfies for some (random) $k^* \in \{1, \ldots, n_l\}$

$$\tilde{q}_{n,t}^{(p)}(\cdot, ((X_{1,t}^{(t)}, \hat{Y}_{1,t}), \ldots, (X_{n_l,t}^{(t)}, \hat{Y}_{n_l,t}))) \in \bar{\mathcal{F}}_{k^*} \quad \text{and} \quad pen_p^2(\tilde{q}_{n,t}^{(p)}) = pen_p^2(k^*) \tag{8}$$

and

$$\frac{1}{n_l}\sum_{i=1}^{n_l}(\tilde{q}_{n,t}^{(p)}(X_{i,t}^{(t)}) - \hat{Y}_{i,t})^2 + pen_p^2(\tilde{q}_{n,t}^{(p)}) \leq \inf_{k \in \{1,\ldots,n_l\}} \left(\inf_{f \in \mathcal{F}_k} \frac{1}{n_l}\sum_{i=1}^{n_l}(f(X_{i,t}^{(t)}) - \hat{Y}_{i,t})^2 + pen_p^2(k) \right) \tag{9}$$

for all $p \in \mathcal{P}_n$. Let the conditions of Theorem 1 be satisfied with (6) replaced by

$$\frac{\sqrt{n_l}\cdot(\delta + pen_{p_n}^2(k))}{\beta_n^2} \geq c_2 \int_{c_3(\delta + pen_{p_n}^2(k))/\beta_n^2}^{\sqrt{48(\delta + pen_{p_n}^2(k))}} \left(\log \mathcal{N}_2\Bigg(u, \{T_{\beta_n} f - g : f \in \bar{\mathcal{F}}_k, \right.$$
$$\left. \frac{1}{n_l}\sum_{i=1}^{n_l} |T_{\beta_n} f(x_i) - g(x_i)|^2 \leq 4\delta, pen_{p_n}^2(k) \leq 4\delta\}, x_1^{n_l} \Bigg) \right)^{1/2} du$$

for all $k \in \{1, \ldots, n_l\}$, all $\delta \geq \delta_{n_l}/6$, all $g \in \bar{\mathcal{F}}_k \cup \{q_t\}$, all $t \in \{0, 1, \ldots, T-1\}$ and all $x_1, \ldots, x_{n_l} \in \mathbb{R}$. Then

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx)$$
$$= O_{\mathbf{P}}\left(\delta_{n_l} + \max_{s \in \{t, t+1, \ldots, T-1\}} \inf_{k \in \{1,\ldots,n_l\}} \left(pen_{p_n}^2(k) + \inf_{f \in \mathcal{F}_k, \|f\|_\infty \leq \beta_n} \int |f(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx) \right) \right)$$

for all $t \in \{0, 1, \ldots, T-1\}$.

3.2 Application to least squares estimates

From Theorem 1 various results concerning least squares estimates can be derived. For simplicity we restrict ourselves to the neural network estimates introduced in Kohler, Krzyżak and Todorovic (2006).

Let $\sigma : \mathbb{R} \to [0, 1]$ be a sigmoid function, i.e., assume that σ is monotonically increasing and satisfies

$$\sigma(x) \to 0 \quad (x \to -\infty) \quad \text{and} \quad \sigma(x) \to 1 \quad (x \to \infty).$$

An example of such a sigmoid function is the logistic squasher defined by $\sigma(x) = \frac{1}{1+e^{-x}}$ $(x \in \mathbb{R})$. Let $\mathcal{P}_n = \{1, \ldots, n\}$ and set

$$\mathcal{G}_k = \left\{ \sum_{i=1}^{k} c_i \cdot \sigma(a_i^T x + b_i) + c_0 \quad : \quad a_i \in \mathbb{R}^d,\ b_i \in \mathbb{R},\ \sum_{i=0}^{k} |c_i| \leq \beta_n \right\}$$

where σ is the sigmoid function from above. Define the least squares estimate as described in Section 2 using $p = k$ as parameter and $\mathcal{G}_k$ as function spaces. Then Theorem 1 implies

Corollary 1 *(Kohler, Krzyżak and Todorovic (2006)).*

Let $L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is a $\mathbb{R}^d$-valued Markov process and that the discounted payoff function f_t is bounded in absolute value by L. Define the estimate $\hat{q}_{n,t}$ as in Section 2 with $\mathcal{P}_n$ and $\mathcal{G}_p$ as above. Let $k_n \in \mathcal{P}_n$ and assume that k_n, β_n satisfy

$$\beta_n \to \infty \quad (n \to \infty), \quad k_n \to \infty \quad (n \to \infty), \quad \frac{\beta_n^2 \cdot k_n \cdot \log n}{n} \to 0 \quad (n \to \infty).$$

Then

$$\begin{aligned}
&\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\
&= O_{\mathbf{P}} \left(\frac{\beta_n^4 \cdot k_n \cdot \log n}{n} + \max_{s \in \{t,t+1,\ldots,T-1\}} \inf_{f \in \mathcal{G}_{k_n}} \int |f(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx) \right)
\end{aligned}$$

for all $t \in \{0, 1, \ldots, T\}$.

Proof. Set

$$\delta_n = c \cdot \frac{\beta_n^4 \cdot k_n \cdot \log n}{n}.$$

Then

$$\delta_n \to 0 \quad (n \to \infty) \quad \text{and} \quad \frac{n \cdot \delta_n}{\beta_n^2 \cdot \log n} \to \infty \quad (n \to \infty).$$

Furthermore Lemma 1 in Kohler, Krzyżak and Todorovic (2006) implies that (6) is satisfied (compare also step 5 in the proof of Theorem 1 in Kohler, Krzyżak and Todorovic (2006)). Hence the assertion follows from Theorem 1. □

From this one can conclude as in Kohler, Krzyżak and Todorovic (2006):

Corollary 2 *(Kohler, Krzyżak and Todorovic (2006)).*
Let $L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is an $\mathbb{R}^d$-valued Markov process, $X_t \in [-A, A]^d$ almost surely for some $A > 0$ and all $t \in \{0, 1, \ldots, T\}$, that the discounted payoff function f_t is bounded in absolute value by L, i.e.,

$$|f_t(x)| \leq L \textit{ for } x \in \mathbb{R}^d \textit{ and } t \in \{0, 1, \ldots, T\},$$

and that the Fourier transform $\tilde{Q}_t$ of q_t defined by

$$\tilde{Q}_t(\nu) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\nu^T x} q_t(x) dx \quad (\nu \in \mathbb{R}^d)$$

satisfies

$$q_t(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\nu^T x} \tilde{Q}_t(\nu) d\nu \quad \textit{and} \quad \int_{\mathbb{R}^d} \|\nu\| |\tilde{Q}_t(\nu)| d\nu \le C$$

for all $x \in \mathbb{R}^d$ *and all* $t \in \{0, \ldots, T-1\}$. *Let* $\beta_n = c_4 \cdot \log n$ *and define the estimate* $\hat{q}_{n,t}$ *as in Corollary 1. Then*

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}} \left(\left(\frac{\log^5 n}{n} \right)^{1/2} \right)$$

for all $t \in \{0, 1, \ldots, T\}$.

3.3 Application to smoothing spline estimates

Let

$$\mathcal{P}_n = \left\{ (k, \lambda) \quad : \quad \lambda = \frac{i}{n} \text{ for some } i \in \{0, 1, \ldots, n^2\}, k \in \left\{ \left\lceil \frac{d}{2} \right\rceil, \ldots, K \right\} \right\}$$

and define the smoothing spline estimate as described in Section 2. Then Theorem 1 implies the following result concerning the rate of convergence of the estimate:

Corollary 3 *(Kohler (2008b)).*
Let $A, L > 0$. *Assume that* $X_0, X_1, \ldots, X_T$ *is a* $[-A, A]^d$*–valued Markov process, that the discounted payoff function* f_t *is bounded in absolute value by* L *and that the continuation values satisfy*

$$q_t \in \{ f \in \mathcal{G}([-A, A]^d) : J_{k^*}^2(f) \le C \} \quad (t = 0, 1, \ldots, T-1)$$

for some $k^* \in \{\lceil d/2 \rceil, \ldots, K\}$ *and some* $C > 0$. *Let* $\beta_n = c_4 \cdot \log n$ *and define the estimates* $\hat{q}_{n,t}$ *as in Section 2. Then* $\hat{q}_{n,t}$ *satisfies for any* $t \in \{0, 1, \ldots, T-1\}$

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}} \left(C^{\frac{d}{2k^*+d}} \cdot \left(\frac{\log(n)}{n} \right)^{\frac{2k^*}{2k^*+d}} \right).$$

Proof. Set

$$\lambda_n^* = C^{-\frac{2k^*}{2k^*+d}} \cdot \left(\frac{\log(n)}{n} \right)^{\frac{2k^*}{2k^*+d}}$$

and $p_n = (k^*, \lambda_n^*)$. Then Lemma 3 in Kohler (2008b) implies that

$$\delta_n = C^{\frac{d}{2k^*+d}} \cdot \left(\frac{\log(n)}{n} \right)^{\frac{2k^*}{2k^*+d}}$$

satisfies inequality (6) (cf. proof of Theorem 2 in Kohler (2008b)). Theorem 1 implies the assertion. □

3.4 Application to orthogonal series estimates

Set $\mathcal{P}_n = \{2^k : k \in \mathbb{Z}, |k| \le \log(n)\}$ and let $\hat{q}_{n,t}$ be the orthogonal series estimate introduced in Section 2.

In order to formulate our main result for the orthogonal series estimates we need the following notation: Set $\Pi_1 = \{\{[-A,A]^d\}\}$ and let Π_{k+1} be the set of all partitions which one obtains from partitions in Π_k by subdividing one of the sets into 2^d equivolume subsets. For a partition π let $\mathcal{F}_c \circ \pi$ be the set of all piecewise constant functions with respect to that partition.

The following result is implied by Theorem 1:

Corollary 4 *Let $A, L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is a $[-A,A]^d$–valued Markov process, that the discounted payoff function f_t is bounded in absolute value by L. Let $\beta_n = c_4 \cdot \log n$. Then the estimates $\hat{q}_{n,t}$ introduced above satisfy for any $t \in \{0,1,\ldots,T-1\}$:*

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx)$$
$$= O_{\mathbf{P}}\left(\max_{s \in \{t,t+1,\ldots,T-1\}} \min_{k \in \{1,\ldots,n\}} \min_{\pi \in \Pi_k} \left\{ \frac{(\log(n))^4 \cdot |\pi|}{n} + \inf_{f \in \mathcal{F}_c \circ \pi} \int |f(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx) \right\} \right). \tag{10}$$

Proof. Choose $p_n = p \in \mathcal{P}_n$ as a constant. Set

$$\mathcal{F}_k = \cup_{\pi \in \cup_{l=1}^{\infty} \Pi_l, |\pi| = k} \mathcal{F}_c \circ \pi.$$

Let Π be the family of sets defined recursively as follows: Π contains $[-A,A]^d$, and for each set contained in Π also the two sets are contained which one obains if one chooses one component of the set and splits the interval there into two intervals of the same length. Let $\bar{\mathcal{F}}_k$ be the set of all piecewise constant functions with respect to a partition of $[-A,A]^d$ which can be constructed by choosing at most $2k+1$ sets of Π and by intersecting each of the sets with the complements of all remaining sets. Set for $p = c$

$$pen_p^2(k) = c \cdot \log n_l \cdot \frac{2k+1}{n}.$$

It follows from the proof of Theorem 1 in Kohler (2008a) that $\tilde{q}_{n,t}^{(p)}$ satisfies (8) and (9) and that we have for $u > c_5/n$

$$\mathcal{N}_2\left(u, \{T_{\beta_n} f - g : f \in \bar{\mathcal{F}}_k\}, x_1^{n_l}\right) \le (c_6 \cdot n)^{4k+1}$$

for all $x_1, \ldots, x_{n_l} \in \mathbb{R}^d$. Hence the assumptions of Remark 1 are satisfied if we set

$$\delta_n = \frac{\beta_n^2 \log^2 n}{n},$$

and Remark 1 implies the assertion. $\square$

As in Kohler (2008a) we can derive from Corollary 4 various results concerning the rate of convergence of the estimate, e.g. we get

Corollary 5 *Let $A, L > 0$. Assume that $X_0, X_1, \ldots, X_T$ is a $[-A, A]^d$–valued Markov process, that the discounted payoff function f_t is bounded in absolute value by L and that the continuation values are (p, C)-Hölder-smooth on $[-A, A]^d$ for some $0 < p \leq 1$ and some $C \geq 0$. Then the estimate $\hat{q}_{n,t}$ introduced in Corollary 4 satisfies for any $t \in \{0, 1, \ldots, T-1\}$*

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}}\left(C^{\frac{d}{2p+d}} \cdot \left(\frac{\log^4(n)}{n} \right)^{\frac{2p}{2p^*+d}} \right).$$

3.5 Remarks

Remark 2. In the corollaries above we assume that the stochastic process is bounded. Usually in modelling of financial processes one models them by unbounded processes. In this case we choose a large value $A > 0$ and replace X_t by its bounded approximation

$$X_t^A = X_{\min\{t, \tau_A\}} \quad \text{where} \quad \tau_A = \inf\{s \geq 0 \,:\, X_s \notin [-A, A]^d\}.$$

(Here we assume for simplicity that the stochastic process has continuous paths in order to be able to neglect an additional truncation of X_t^A). This boundedness assumption enables us to estimate the price of the American option from samples of polynomial size in the number of free parameters, in contrast to Monte Carlo estimation from standard (unbounded) Black-Scholes models, where Glasserman and Yu (2004) showed that samples of exponential size in the number of free parameters are needed. For many industrial models, the localization error can be estimated explicitely. For instance, section 4 in Egloff, Kohler and Todorovic (2007) contains apriori bounds for the localization and payoff truncation error in case of discretely sampled jump diffusion processes.

Remark 3. Assume $X_0 = x_0$ a.s. for some $x_0 \in \mathbb{R}$. We can estimate the price

$$V_0 = \max\{f_0(x_0), q_0(x_0)\}$$

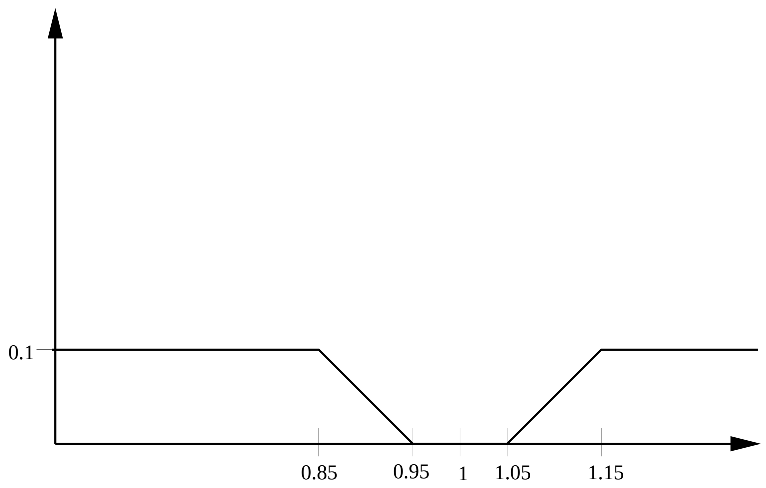

Figure 1: Strangle spread payoff function with strikes 0.85, 0.95, 1.05 and 1.15

(cf., (1) and (3)) of the American option by

$$\hat{V}_0 = \max\{f_0(x_0), \hat{q}_{n,0}(x_0)\}.$$

Since the distribution of X_0 is concentrated on x_0, we have the following error bound:

$$\begin{aligned}
|\hat{V}_0 - V_0|^2 \qquad &= |\max\{f_0(x_0), \hat{q}_{n,0}(x_0)\} - \max\{f_0(x_0), q_0(x_0)\}|^2 \\
&\leq |\hat{q}_{n,0}(x_0) - q_0(x_0)|^2 \\
&= \int |\hat{q}_{n,0}(x) - q_0(x)|^2 \mathbf{P}_{X_0}(dx).
\end{aligned}$$

Therefore the above results imply bounds on the error of $\hat{V}_0$.

4 Application to simulated data

In this section, we illustrate the finite sample behavior of our algorithms by comparing them with the Tsitsiklis–Van Roy algorithm and Longstaff–Schwartz algorithm proposed by Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001), respectively.

We consider for the pricing problem a strangle spread function with strikes 0.85, 0.95, 1.05 and 1.15 (cf. Figure 1) for the average of three correlated stock prices. The stocks are ADECCO R, BALOISE R and CIBA. We simulate the paths of the underlying stocks with the following simple Black-Scholes model:

$$X_{i,t} = x_0 \cdot e^{r \cdot t} \cdot e^{\sum_{j=1}^{3} (\sigma_{i,j} \cdot W_j(t) - \frac{1}{2} \sigma_{i,j}^2 t)} \quad (i = 1, \ldots, 3). \tag{11}$$

Here $r > 0$ is the riskless interest rate, $\sigma_i = (\sigma_{i,1}, \sigma_{i,2}, \sigma_{i,3})^T$ is the volatility of the i-th stock, x_0 is the initial stock price, and $\{W_j(t) : t \in \mathbb{R}_+\}$ ($j \in \{1,2,3\}$) are independent Wiener processes.

To fit the model to the three real stocks, we used stock prices observed from Nov. 10, 2000 until Oct. 3, 2003 on weekdays when the stock market was open for the total of 756 days. We estimate the volatility from this data observed in the past by the historical volatility

$$\sigma = (\sigma_{i,j})_{1 \le i,j \le 3} = \begin{pmatrix} 0.3024 & 0.1354 & 0.0722 \\ 0.1354 & 0.2270 & 0.0613 \\ 0.0722 & 0.0613 & 0.0717 \end{pmatrix}.$$

The time to maturity of the option is assumed to be one year. We discretize the time interval $[0,1]$ by dividing it into 48 equidistant time steps with $t_0 = 0 < t_1 < \ldots < t_{48} = 1$. The prices of the underlying stock at time points t_k ($k = 0, \ldots, 48$) are then given by (11) with t replaced by t_k and $\sigma_{i,j}$ as above. Furthermore we set $x_0 = 1$, $r = 0.05$ and use discount factors $e^{-r \cdot t_k}$ for $k = 0, \ldots, 48$.

We choose the sample size for our algorithms individually as large as possible in view of a reasonable time to compute the estimates: For least squares splines estimates, smoothing spline estimates and least squares neural networks we choose sample size $n = 4000$,$n = 2000$ and $n = 2000$, resp. For the orthogonal series estimate, which can be computed approximately in linear time in the sample size, we choose sample size $n = 30000$. For the parametric algorithms we choose sample size of 10000 since due to problems with the bias of the estimates a larger sample size did not improve the estimates.

For the least squares splines we use B-splines with degree $M \in \{0,1,2\}$ and equidistant knots with knot distance $\delta \in \{0.25, 0.5, 0.67, 1\}$. For the neural network estimate we use feedforward neural networks with one hidden layer and $k \in \{2^0, 2^1, \ldots, 2^5\}$ hidden neurons fitted to the data by backpropagation. The implementation of the orthogonal series estimate is described in Kohler (2008a). The parameters are each time selected by splitting of the sample. We set the number of learning and training samples to $n_l = 2400$ and $n_t = 1600$ for least squares splines, to $n_l = n_t = 1000$ for least squares neural networks, and to $n_l = 27000$ and $n_t = 3000$ for the orthogonal series estimate. To simplify the implementation of the smoothing spline estimate we select the smoothing parameter of this estimate by generalized cross-validation as implemented in the routine $Tps()$ from the library $fields$ in the statistics package R. For the Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms we use polynomials of degree 1.

We apply all six algorithms to 100 independently generated sets of paths. We would like to stress that all six algorithms provide lower bounds to the optimal stopping value. Since we evaluate the approximative optimal stopping rule on newly generated data, a higher MCE indicates a better performance of the algorithm. We compare the algorithms using boxplots. Observe that the higher the boxplot of the MCE the better the performance of the corresponding algorithm.

As we can see in Figure 2, all four algorithms based on nonparametric regression are superior to Longstaff–Schwartz and Tsitsiklis–Van Roy algorithms, since the higher boxplots of the MCE indicate better performance. Furthermore, the orthogonal series estimate seems to be slightly worse than the algorithms based on least squares splines, smoothing splines and neural networks. But on a modern PC this estimate needed only one hour to compute a MCE for one of the 100 set of paths, while the other algorithms needed between 2 and 6 hours. However, due to the use of tensor products it does not produce good results for $d > 3$, which is in contrast to neural networks and smoothing splines.

5 Proofs

5.1 A deterministic lemma

Let $\beta_n \geq L > 0$, $x_1, \ldots, x_n \in \mathbb{R}^d$, $y_1 \ldots, y_n \in \mathbb{R}$, $\bar{y}_1, \ldots, \bar{y}_n \in [-L, L]$. Let $\mathcal{G}$ be a set of functions $g : \mathbb{R}^d \to \mathbb{R}$ and for $g \in \mathcal{G}$ let

$$pen^2(g) \geq 0$$

be a penalty term. Define

$$\tilde{g}_n = \arg\min_{g \in \mathcal{G}} \left(\frac{1}{n} \sum_{i=1}^{n} |\bar{y}_i - g(x_i)|^2 + pen^2(g) \right) \quad \text{and} \quad \hat{g}_n = T_{\beta_n} \tilde{g}_n.$$

Let $m : \mathbb{R}^d \to \mathbb{R}$ be a fixed function and define

$$h = \arg\min_{g \in \mathcal{G}} \left(\|g - m\|_n^2 + pen^2(g) \right)$$

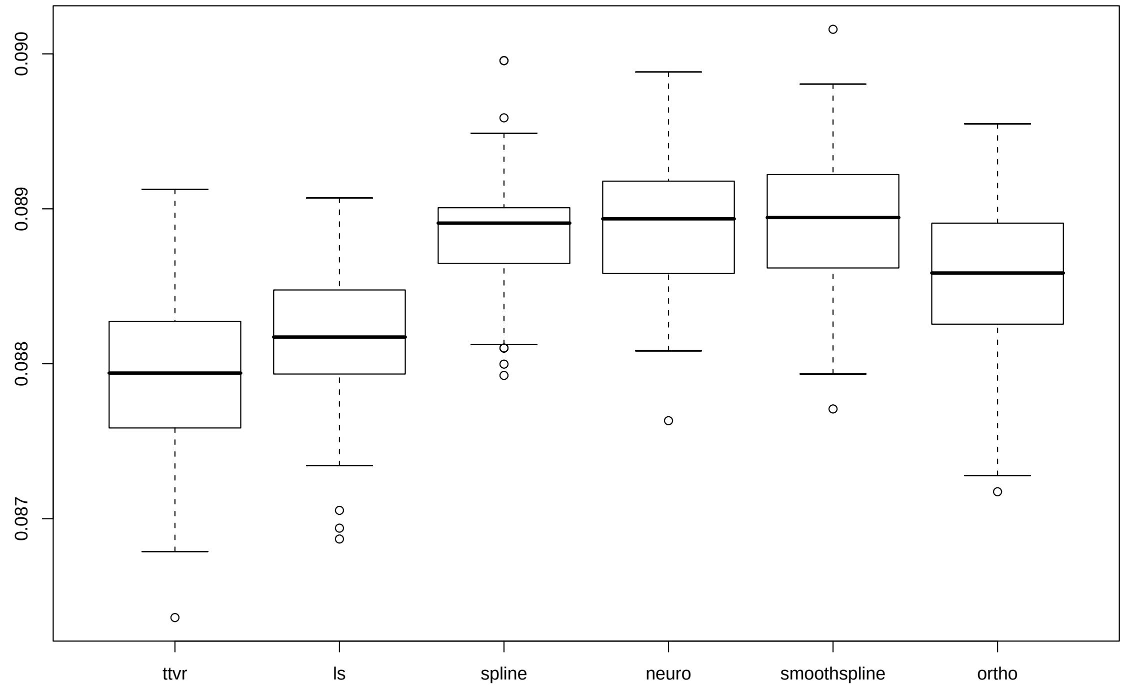

Figure 2: Realized option prices for the strangle spread-payoff of the Tsitsiklis–Van Roy (ttvr), Longstaff–Schwartz (ls), least squares spline (spline), least squares neural network (neuro), smoothing spline (smoothspline) and orthogonal series (ortho) algorithms in a 3-dimensional case. The values of the least squares spline and the least squares neural network algorithms have been computed in Todorovic (2007).

where

$$\|g\|_n^2 = \frac{1}{n}\sum_{i=1}^n |g(x_i)|^2.$$

Lemma 1 *Assume*

$$\|\hat{g}_n - m\|_n^2 + pen^2(\hat{g}_n) \geq 3(\|h - m\|_n^2 + pen^2(h)) + 128 \cdot \frac{1}{n}\sum_{i=1}^n |y_i - \bar{y}_i|^2 + \delta^2 \tag{12}$$

for some $\delta \geq 0$. *Then*

$$\frac{1}{n}\sum_{i=1}^n (y_i - m(x_i)) \cdot (\hat{g}_n(x_i) - h(x_i)) \geq \frac{1}{24} \cdot \left(\|\hat{g}_n - h\|_n^2 + pen^2(\hat{g}_n)\right) + \frac{\delta^2}{6}. \tag{13}$$

Proof. The proof is inspired by the first part of the proof of Theorem 2.1 in van de Geer (2001). Because of $|\bar{y}_i| \leq L \leq \beta_n$ we have

$$\frac{1}{n}\sum_{i=1}^n |\bar{y}_i - \hat{g}_n(x_i)|^2 \leq \frac{1}{n}\sum_{i=1}^n |\bar{y}_i - \tilde{g}_n(x_i)|^2.$$

By definition of the estimate and because of $h \in \mathcal{G}$ this implies

$$\frac{1}{n}\sum_{i=1}^n |\bar{y}_i - \hat{g}_n(x_i)|^2 + pen^2(\hat{g}_n) \leq \frac{1}{n}\sum_{i=1}^n |\bar{y}_i - h(x_i)|^2 + pen^2(h),$$

hence

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n |\bar{y}_i - m(x_i)|^2 + \frac{2}{n}\sum_{i=1}^n (\bar{y}_i - m(x_i)) \cdot (m(x_i) - \hat{g}_n(x_i)) + \|m - \hat{g}_n\|_n^2 + pen^2(\hat{g}_n) \\
&\leq \frac{1}{n}\sum_{i=1}^n |\bar{y}_i - m(x_i)|^2 + \frac{2}{n}\sum_{i=1}^n (\bar{y}_i - m(x_i)) \cdot (m(x_i) - h(x_i)) + \|m - h\|_n^2 + pen^2(h),
\end{aligned}$$

which implies

$$\begin{aligned}
&\|m - \hat{g}_n\|_n^2 + pen^2(\hat{g}_n) - \|m - h\|_n^2 - pen^2(h) \\
&\leq \frac{2}{n}\sum_{i=1}^n (\bar{y}_i - m(x_i)) \cdot (\hat{g}_n(x_i) - h(x_i)) \\
&= \frac{2}{n}\sum_{i=1}^n (\bar{y}_i - y_i) \cdot (\hat{g}_n(x_i) - h(x_i)) + \frac{2}{n}\sum_{i=1}^n (y_i - m(x_i)) \cdot (\hat{g}_n(x_i) - h(x_i)) \\
&=: T_1 + T_2.
\end{aligned}$$

We show next that $T_1 \leq T_2$. Assume to the contrary that this is not true. Then

$$\|m - \hat{g}_n\|_n^2 + pen^2(\hat{g}_n) - \|m - h\|_n^2 - pen^2(h)$$

$$
\begin{aligned}
&< \frac{4}{n}\sum_{i=1}^{n}(\bar{y}_i - y_i)\cdot(\hat{g}_n(x_i) - h(x_i)) \\
&\le 4\cdot\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\bar{y}_i - y_i)^2}\cdot\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{g}_n(x_i) - h(x_i))^2} \\
&\le 4\cdot\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\bar{y}_i - y_i)^2}\cdot\sqrt{2\|\hat{g}_n - m\|_n^2 + 2pen^2(\hat{g}_n) + 2\|h-m\|_n^2 + 2pen^2(h)}.
\end{aligned}
$$

Using (12) we see that the left-hand side of the above inequality is bounded from below by

$$
\begin{aligned}
&\frac{1}{2}\cdot\left(\|m - \hat{g}_n\|_n^2 + pen^2(\hat{g}_n)\right) + \frac{1}{2}\cdot\left(3(\|h-m\|_n^2 + pen^2(h)) + 128\cdot\frac{1}{n}\sum_{i=1}^{n}|y_i - \bar{y}_i|^2 + \delta^2\right) \\
&\quad -\|m-h\|_n^2 - pen^2(h) \\
&\ge \frac{1}{2}\cdot\left(\|\hat{g}_n - m\|_n^2 + pen^2(\hat{g}_n) + \|h-m\|_n^2 + pen^2(h)\right),
\end{aligned}
$$

which implies

$$
\frac{1}{2}\cdot\sqrt{\|\hat{g}_n - m\|_n^2 + pen^2(\hat{g}_n) + \|h-m\|_n^2 + pen^2(h)} < 4\cdot\sqrt{2}\cdot\sqrt{\frac{1}{n}\sum_{i=1}^{n}|y_i - \bar{y}_i|^2}
$$

i.e.,

$$
\|\hat{g}_n - m\|_n^2 + pen^2(\hat{g}_n) + \|h-m\|_n^2 + pen^2(h) < 128\cdot\frac{1}{n}\sum_{i=1}^{n}|y_i - \bar{y}_i|^2.
$$

But this is a contradiction to (12), so we have indeed proved $T_1 \le T_2$. As a consequence we can conclude

$$
\frac{4}{n}\sum_{i=1}^{n}(y_i - m(x_i))\cdot(\hat{g}_n(x_i) - h(x_i)) \ge \|\hat{g}_n - m\|_n^2 + pen^2(\hat{g}_n) - \|h-m\|_n^2 - pen^2(h). \tag{14}
$$

By (12) the right-hand side of (14) is bounded from below by

$$
\begin{aligned}
&\frac{1}{3}(\|\hat{g}_n - m\|_n^2 + pen^2(\hat{g}_n)) + \frac{2}{3}\left(2\|h-m\|_n^2 + 2pen^2(h) + \delta^2\right) - \|h-m\|_n^2 - pen^2(h) \\
&= \frac{1}{3}\|\hat{g}_n - m\|_n^2 + \frac{1}{3}pen^2(\hat{g}_n) + \frac{1}{3}\|h-m\|_n^2 + \frac{1}{3}pen^2(h) + \frac{2}{3}\delta^2. \qquad (15)
\end{aligned}
$$

Because of $a^2/2 - b^2 \le (a-b)^2$ $(a, b \in \mathbb{R})$ we have

$$
\frac{1}{2}\|\hat{g}_n - h\|_n^2 - \|h-m\|_n^2 \le (\|\hat{g}_n - h\|_n - \|h-m\|_n)^2 \le \|\hat{g}_n - m\|_n^2.
$$

Using this we can bound (15) from below by

$$
\begin{aligned}
&\frac{1}{6}\|\hat{g}_n - h\|_n^2 - \frac{1}{3}\|h - m\|_n^2 + \frac{1}{3}pen^2(\hat{g}_n) + \frac{1}{3}\|h - m\|_n^2 + \frac{1}{3}pen^2(h) + \frac{2}{3}\delta^2 \\
&= \frac{1}{6}\|\hat{g}_n - h\|_n^2 + \frac{1}{3}pen^2(\hat{g}_n) + \frac{1}{3}pen^2(h) + \frac{2}{3}\delta^2 \\
&\geq \frac{1}{6}\left(\|\hat{g}_n - h\|_n^2 + pen^2(\hat{g}_n)\right) + \frac{2}{3}\delta^2.
\end{aligned}
$$

Summing up the above results we get the desired inequality

$$
4 \cdot \frac{1}{n}\sum_{i=1}^{n}(y_i - m(x_i)) \cdot (\hat{g}_n(x_i) - h(x_i)) \geq \frac{1}{6} \cdot \left(\|\hat{g}_n - h\|_n^2 + pen^2(\hat{g}_n)\right) + \frac{2\delta^2}{3}.
$$

□

5.2 Results for fixed design regression

Let

$$
Y_i = m(x_i) + W_i \quad (i = 1, \ldots, n)
$$

for some $x_1, \ldots, x_n \in \mathbb{R}^d$, $m : \mathbb{R}^d \to \mathbb{R}$ and some random variables $W_1, \ldots, W_n$ which are independent and have expectation zero. We assume that the W_i's are sub-Gaussian in the sense that

$$
\max_{i=1,\ldots,n} K^2 \mathbf{E}\{e^{W_i^2/K^2} - 1\} \leq \sigma_0^2 \tag{16}
$$

for some $K, \sigma_0 > 0$. Our goal is to estimate m from $(x_1, \bar{Y}_{1,n}), \ldots, (x_n, \bar{Y}_{n,n})$, where $\bar{Y}_{1,n}, \ldots, \bar{Y}_{n,n} \in [-L, L]$ are arbitrary (bounded) random variables with the property that the average squared measurement error

$$
\frac{1}{n}\sum_{i=1}^{n} |Y_i - \bar{Y}_{i,n}|^2
$$

is "small". Let $\mathcal{F}_n$ be a set of functions $f : \mathbb{R}^d \to \mathbb{R}$ and consider the least squares estimate with complexity penalty

$$
\tilde{m}_n(\cdot) = \arg\min_{f \in \mathcal{F}_n} \left(\frac{1}{n}\sum_{i=1}^{n} |f(x_i) - \bar{Y}_{i,n}|^2 + pen_n^2(f)\right) \quad \text{and} \quad m_n = T_{\beta_n}\tilde{m}_n, \tag{17}
$$

where

$$
pen_n^2(f) \geq 0
$$

for all $f \in \mathcal{F}_n$ and $\beta_n \geq L$.

Lemma 2 *Assume that the sub-Gaussian condition (16) holds and let the estimate be defined by (17). Then there exist constants* $c_6, c_7 > 0$ *which depend only on* σ_0 *and* K *such that for any* $\delta_n > c_1/n$ *with*

$$\sqrt{n} \cdot \delta \geq c_6 \int_{\delta/(12\sigma_0)}^{\sqrt{48\delta}} \Bigg(\log \mathcal{N}_2 \Bigg(u, \{T_{\beta_n} f - g : f \in \mathcal{F}_n, \\ \frac{1}{n} \sum_{i=1}^{n} |T_{\beta_n} f(x_i) - g(x_i)|^2 + pen_n^2(f) \leq 4\delta\}, x_1^n \Bigg) \Bigg)^{1/2} du \qquad (18)$$

for all $\delta \geq \delta_n/6$ *and all* $g \in \mathcal{F}_n$ *we have*

$$\mathbf{P}\Bigg\{ \|m_n - m\|_n^2 + pen_n^2(\tilde{m}_n) > c_7 \left(\frac{1}{n} \sum_{i=1}^{n} |Y_i - \bar{Y}_{i,n}|^2 + \delta_n + \min_{f \in \mathcal{F}_n} \left(\|f - m\|_n^2 + pen_n^2(f) \right) \right) \Bigg\} \\ \leq c_7 \cdot \exp\left(- \frac{n \cdot \min\{\delta_n, \sigma_0^2\}}{c_7} \right).$$

Lemma 2 follows from Lemma 1 and the techniques introduced in the proof of Theorem 2.1 in van de Geer (2000). For the sake of completeness we give nevertheless a detailed proof, which is in fact a modification of the proof of Lemma 2 in Kohler (2006).

Proof. Set

$$m_n^*(\cdot) = \arg\min_{f \in \mathcal{F}_n} \left(\|f - m\|_n^2 + pen_n^2(f) \right).$$

By Lemma 1,

$$\mathbf{P}\Bigg\{ \|m_n - m\|_n^2 + pen_n^2(\tilde{m}_n) \geq 128 \frac{1}{n} \sum_{i=1}^{n} |Y_i - \bar{Y}_{i,n}|^2 + \delta_n + 3 \left(\|m_n^* - m\|_n^2 + pen_n^2(m_n^*) \right) \Bigg\} \\ \leq \mathbf{P} \left\{ \|m_n - m_n^*\|_n^2 + pen_n^2(\tilde{m}_n) + 4\delta_n \leq \frac{24}{n} \sum_{i=1}^{n} (m_n(x_i) - m_n^*(x_i)) \cdot W_i \right\} \\ \leq P_1 + P_2$$

where

$$P_1 = \mathbf{P} \left\{ \frac{1}{n} \sum_{i=1}^{n} W_i^2 > 2\sigma_0^2 \right\}$$

and

$$P_2 = \\ \mathbf{P} \left\{ \frac{1}{n} \sum_{i=1}^{n} W_i^2 \leq 2\sigma_0^2, \|m_n - m_n^*\|_n^2 + pen_n^2(\tilde{m}_n) + 4\delta_n \leq \frac{24}{n} \sum_{i=1}^{n} (m_n(x_i) - m_n^*(x_i)) \cdot W_i \right\}.$$

Application of Chernoff's exponential bounding method (cf. Chernoff (1952)) together with (16) yields

$$\begin{aligned} P_1 &= \mathbf{P}\left\{\sum_{i=1}^n W_i^2/K^2 > 2n\sigma_0^2/K^2\right\} \\ &\leq \mathbf{P}\left\{\exp\left(\sum_{i=1}^n W_i^2/K^2\right) > \exp\left(2n\sigma_0^2/K^2\right)\right\} \\ &\leq \exp\left(-2n\sigma_0^2/K^2\right)\cdot \mathbf{E}\left\{\exp(\sum_{i=1}^n W_i^2/K^2)\right\} \\ &\leq \exp\left(-2n\sigma_0^2/K^2\right)\cdot\left(1+\sigma_0^2/K^2\right)^n \\ &\leq \exp\left(-2n\sigma_0^2/K^2\right)\cdot\exp\left(n\cdot\sigma_0^2/K^2\right) = \exp\left(-n\sigma_0^2/K^2\right). \end{aligned}$$

To bound P_2, we observe first that $1/n\sum_{i=1}^n W_i^2 \leq 2\sigma_0^2$ together with the Cauchy-Schwarz inequality implies

$$\begin{aligned} \frac{24}{n}\sum_{i=1}^n (m_n(x_i)-m_n^*(x_i))\cdot W_i &\leq 24\cdot\sqrt{\frac{1}{n}\sum_{i=1}^n (m_n(x_i)-m_n^*(x_i))^2}\cdot\sqrt{2\sigma_0^2} \\ &\leq 24\cdot\sqrt{\frac{1}{n}\sum_{i=1}^n (m_n(x_i)-m_n^*(x_i))^2 + pen_n^2(\tilde{m}_n)}\cdot\sqrt{2\sigma_0^2} \end{aligned}$$

hence inside of P_2 we have

$$\frac{1}{n}\sum_{i=1}^n (m_n(x_i)-m_n^*(x_i))^2 + pen_n^2(\tilde{m}_n) \leq 1152\sigma_0^2.$$

Set

$$S = \min\{s\in\mathbb{N}_0 \,:\, 4\cdot 2^s\delta_n > 1152\sigma_0^2\}.$$

Application of the peeling device (cf. Section 5.3 in van de Geer (2000)) yields

$$\begin{aligned} P_2 &= \sum_{s=1}^S \mathbf{P}\Bigg\{\frac{1}{n}\sum_{i=1}^n W_i^2 \leq 2\sigma_0^2, 4\cdot 2^{s-1}\delta_n\cdot I_{\{s\neq 1\}} \leq \|m_n-m_n^*\|_n^2 + pen_n^2(\tilde{m}_n) < 4\cdot 2^s\delta_n, \\ &\qquad \|m_n-m_n^*\|_n^2 + pen_n^2(\tilde{m}_n) + 4\delta_n \leq \frac{24}{n}\sum_{i=1}^n (m_n(x_i)-m_n^*(x_i))\cdot W_i\Bigg\} \\ &\leq \sum_{s=1}^S \mathbf{P}\Bigg\{\frac{1}{n}\sum_{i=1}^n W_i^2 \leq 2\sigma_0^2, \|m_n-m_n^*\|_n^2 + pen_n^2(\tilde{m}_n) < 4\cdot 2^s\delta_n, \\ &\qquad 4\cdot 2^{s-1}\delta_n \leq \frac{24}{n}\sum_{i=1}^n (m_n(x_i)-m_n^*(x_i))\cdot W_i\Bigg\} \end{aligned}$$

The probabilities in the above sum can be bounded by Corollary 8.3 in van de Geer (2000) (use there $R = \sqrt{4 \cdot 2^s \delta_n}$, $\delta = \frac{1}{12} \cdot 2^s \delta_n$ and $\sigma = \sqrt{2}\sigma_0$). This yields

$$
\begin{aligned}
P_2 &\leq \sum_{s=1}^{\infty} c_9 \exp\left(- \frac{n \cdot (\frac{1}{12} \cdot 2^s \delta_n)^2}{4 c_9 \cdot 4 \cdot 2^s \delta_n} \right) = \sum_{s=1}^{\infty} c_9 \exp\left(- \frac{n \cdot 2^s \cdot \delta_n}{c_{10}} \right) \\
&\leq \sum_{s=1}^{\infty} c_9 \exp\left(- \frac{n \cdot (s+1) \cdot \delta_n}{c_{10}} \right) \leq c_{11} \exp\left(- \frac{n \delta_n}{c_{11}} \right).
\end{aligned}
$$

□

5.3 Results for random design regression

Let $(X, Y), (X_1, Y_1), \ldots$ be independent and identically distributed $\mathbb{R}^d \times \mathbb{R}$ valued random variables with $\mathbf{E}Y^2 < \infty$. Let $m(x) = \mathbf{E}\{Y|X = x\}$ be the corresponding regression function. Assume that we want to estimate m from observed data, but instead of a sample

$$\mathcal{D}_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$$

of (X, Y) we have only available a set of data

$$\bar{\mathcal{D}}_n = \{(X_1, \bar{Y}_{1,n}), \ldots, (X_n, \bar{Y}_{n,n})\}$$

where the only assumption on $\bar{Y}_{1,n}, \ldots, \bar{Y}_{n,n}$ is that the measurement error

$$\frac{1}{n} \sum_{i=1}^{n} |Y_i - \bar{Y}_{i,n}|^2 \tag{19}$$

is small.

Define the estimate m_n by

$$\tilde{m}_n(\cdot) = \arg \min_{f \in \mathcal{F}_n} \left(\frac{1}{n} \sum_{i=1}^{n} |f(X_i) - \bar{Y}_{i,n}|^2 + pen_n^2(f) \right),$$

where $\mathcal{F}_n$ is a set of functions $f : \mathbb{R}^d \to \mathbb{R}$ and for $f \in \mathcal{F}_n$

$$pen_n^2(f) \geq 0$$

is the penalty term penalizing the complexity of f. Set

$$m_n = T_{\beta_n} \tilde{m}_n.$$

Then the following result holds.

Lemma 3 *Assume that $Y - m(X)$ is sub-Gaussian in the sense that*

$$C^2 \mathbf{E}\left\{ e^{(Y-m(X))^2/C^2} - 1 | X \right\} \leq \sigma_0^2 \quad \textit{almost surely} \tag{20}$$

for some $C, \sigma_0 > 0$. Let $\beta_n \geq L \geq 1$ and assume that the regression function is bounded in absolute value by L. Let $\mathcal{F}_n$ be a set of functions $f : \mathbb{R}^d \to \mathbb{R}$ and define the estimate m_n as above. Then there exist constants $c_{12}, c_{13}, c_{14} > 0$ depending only on σ_0 and C such that for any $\delta_n > c_{12}/n$ which satisfies

$$\delta_n \to 0 \quad (n \to \infty) \quad \textit{and} \quad \frac{n \cdot \delta_n}{\beta_n^2} \to \infty \quad (n \to \infty),$$

$$c_{12} \frac{\sqrt{n}\delta}{\beta_n^2} \geq \int_{c_{13}\delta/\beta_n^2}^{\sqrt{\delta}} \Bigg(\log \mathcal{N}_2 \Bigg(u, \{ T_{\beta_n} f - m \, : \, f \in \mathcal{F}_n, \\ \frac{1}{n} \sum_{i=1}^n |T_{\beta_n} f(x_i) - m(x_i)|^2 \leq \frac{\delta}{\beta_n^2}, pen_n^2(f) \leq 2\delta \}, x_1^n \Bigg) \Bigg)^{1/2} du$$

for all $\delta \geq \delta_n$ and all $x_1, \ldots, x_n \in \mathbb{R}^d$ and

$$\sqrt{n} \cdot \delta \geq c_{12} \int_{\delta/(12\sigma_0)}^{\sqrt{48\delta}} \Bigg(\log \mathcal{N}_2 \Bigg(u, \{ T_{\beta_n} f - g : f \in \mathcal{F}_n, \\ \frac{1}{n} \sum_{i=1}^n |T_{\beta_n} f(x_i) - g(x_i)|^2 + pen_n^2(f) \leq 4\delta \}, x_1^n \Bigg) \Bigg)^{1/2} du$$

for all $\delta \geq \delta_n/6$, all $x_1, \ldots, x_n \in \mathbb{R}^d$ and all $g \in \mathcal{F}_n$ we have

$$\mathbf{P}\Bigg\{ \int |m_n(x) - m(x)|^2 \mathbf{P}_X(dx) > \\ c_{14} \left(\frac{1}{n} \sum_{i=1}^n |Y_i - \bar{Y}_{i,n}|^2 + \delta_{n_l} + \inf_{f \in \mathcal{F}_n, \|f\|_\infty \leq \beta_n} \left(pen_n^2(f) + \int |f(x) - m(x)|^2 \mathbf{P}_X(dx) \right) \right) \Bigg\} \to 0$$

for $n \to \infty$.

Proof. We have

$$\mathbf{P}\Bigg\{ \int |m_n(x) - m(x)|^2 \mathbf{P}_X(dx) > \\ 3c_7 \frac{1}{n} \sum_{i=1}^n |Y_i - \bar{Y}_{i,n}|^2 + (6c_7 + 1)\delta_n + 9c_7 \inf_{f \in \mathcal{F}_n, \|f\|_\infty \leq \beta_n} \left(pen_n^2(f) + \int |f(x) - m(x)|^2 \mathbf{P}_X(dx) \right) \Bigg\} \\ \leq P_{1,n} + P_{2,n} + P_{3,n},$$

where

$$P_{1,n} = \mathbf{P}\left\{ \int |m_n(x) - m(x)|^2 \mathbf{P}_X(dx) > \delta_n + 3 \cdot pen_n^2(\tilde{m}_n) + 3\frac{1}{n}\sum_{i=1}^n |m_n(X_i) - m(X_i)|^2 \right\},$$

$$\begin{aligned} P_{2,n} &= \mathbf{P}\Bigg\{ \inf_{f\in\mathcal{F}_n, \|f\|_\infty \le \beta_n} \left(pen_n^2(f) + \frac{1}{n}\sum_{i=1}^n |f(X_i) - m(X_i)|^2 \right) > \\ & \qquad \delta_n + 3 \inf_{f\in\mathcal{F}_n, \|f\|_\infty \le \beta_n} \left(pen_n^2(f) + \int |f(x) - m(x)|^2 \mathbf{P}_X(dx) \right) \Bigg\} \end{aligned}$$

and

$$\begin{aligned} & P_{3,n} = \\ & \mathbf{P}\Bigg\{ \int |m_n(x) - m(x)|^2 \mathbf{P}_X(dx) > \\ & \quad 3c_7 \frac{1}{n}\sum_{i=1}^n |Y_i - \bar{Y}_{i,n}|^2 + (6c_7+1)\delta_n + 9c_7 \inf_{f\in\mathcal{F}_n, \|f\|_\infty \le \beta_n} \left(pen_n^2(f) + \int |f(x) - m(x)|^2 \mathbf{P}_X(dx) \right), \\ & \quad \int |m_n(x) - m(x)|^2 \mathbf{P}_X(dx) \le \delta_n + 3 \cdot pen_n^2(\tilde{m}_n) + 3\frac{1}{n}\sum_{i=1}^n |m_n(X_i) - m(X_i)|^2, \\ & \quad \inf_{f\in\mathcal{F}_n, \|f\|_\infty \le \beta_n} \left(pen_n^2(f) + \frac{1}{n}\sum_{i=1}^n |f(X_i) - m(X_i)|^2 \right) \le \\ & \qquad \delta_n + 3 \inf_{f\in\mathcal{F}_n, \|f\|_\infty \le \beta_n} \left(pen_n^2(f) + \int |f(x) - m(x)|^2 \mathbf{P}_X(dx) \right) \Bigg\}. \end{aligned}$$

We have

$$\begin{aligned} & P_{3,n} \\ & \le \mathbf{P}\Bigg\{ \delta_n + 3 \cdot pen_n(\tilde{m}_n) + 3\frac{1}{n}\sum_{i=1}^n |m_n(X_i) - m(X_i)|^2 > \\ & \quad 3c_7 \frac{1}{n}\sum_{i=1}^n |Y_i - \bar{Y}_{i,n}|^2 + (3c_7+1)\delta_n + 3c_7 \inf_{f\in\mathcal{F}_n, \|f\|_\infty \le \beta_n} \left(pen_n^2(f) + \frac{1}{n}\sum_{i=1}^n |f(X_i) - m(X_i)|^2 \right) \Bigg\} \\ & = \mathbf{P}\Bigg\{ \frac{1}{n}\sum_{i=1}^n |m_n(X_i) - m(X_i)|^2 + pen_n^2(\tilde{m}_n) > \\ & \quad c_7 \left(\frac{1}{n}\sum_{i=1}^n |Y_i - \bar{Y}_{i,n}|^2 + \delta_n + \inf_{f\in\mathcal{F}_n, \|f\|_\infty \le \beta_n} \left(pen_n^2(f) + \frac{1}{n}\sum_{i=1}^n |f(X_i) - m(X_i)|^2 \right) \right) \Bigg\} \\ & \to 0 \quad (n \to \infty) \end{aligned}$$

by Lemma 2, where we have used that

$$\min_{f\in\mathcal{F}_n} \left(pen_n^2(f) + \|f - m\|_n^2 \right) \le \inf_{f\in\mathcal{F}_n, \|f\|_\infty \le \beta_n} \left(pen_n^2(f) + \|f - m\|_n^2 \right).$$

To bound $P_{1,n}$ we use

$$\begin{aligned}
&P_{1,n}\\
&= \mathbf{P}\Bigg\{ 2\int |m_n(x)-m(x)|^2 \mathbf{P}_X(dx) - 2\|m_n - m\|_n^2 \\
&\qquad\qquad > \delta_n + 3\cdot pen_n^2(\tilde{m}_n) + \int |m_n(x)-m(x)|^2 \mathbf{P}_X(dx) + \|m_n - m\|_n^2 \Bigg\}\\
&\le \mathbf{P}\Bigg\{ \exists f \in \mathcal{F}_n : \frac{\left|\int |T_{\beta_n} f(x) - m(x)|^2 \mathbf{P}_X(dx) - \|T_{\beta_n} f - m\|_n^2\right|}{\delta_n + 3\cdot pen_n^2(f) + \int |T_{\beta_n} f(x)-m(x)|^2 \mathbf{P}_X(dx) + \|T_{\beta_n} f - m\|_n^2} > \frac{1}{2} \Bigg\}\\
&\le \sum_{s=1}^{\infty} \mathbf{P}\Bigg\{ \exists f \in \mathcal{F}_n : pen_n^2(f) \le 2^s \delta_n,\\
&\qquad\qquad \frac{\left|\int |T_{\beta_n} f(x) - m(x)|^2 \mathbf{P}_X(dx) - \|T_{\beta_n} f - m\|_n^2\right|}{2^{s-1}\delta_n + \int |T_{\beta_n} f(x)-m(x)|^2 \mathbf{P}_X(dx) + \|T_{\beta_n} f - m\|_n^2} > \frac{1}{2} \Bigg\}.
\end{aligned}$$

The probabilities in the above sum can be bounded by Theorem 19.2 in Györfi et al. (2002) (which we apply with $K = 4\beta_n^2$, $\epsilon = 1/2$, and $\alpha = 2^{s-1}\delta_n$). This yields

$$P_{1,n} \le \sum_{s=1}^{\infty} 15 \cdot \exp\left(- \frac{n \cdot 2^s \cdot \delta_n}{c_{15}\beta_n^2} \right) \le c_{16} \cdot \exp\left(- \frac{n\cdot\delta_n}{c_{16}\beta_n^2}\right) \to 0 \quad (n\to\infty).$$

To bound $P_{2,n}$ we use

$$\{ f \in \mathcal{F}_n \quad : \quad \|f\|_\infty \le \beta_n \} \subseteq T_L \mathcal{F}_n$$

and conclude

$$\begin{aligned}
P_{2,n} \;&\le\; \mathbf{P}\Bigg\{ \exists f \in \mathcal{F}_n : \|T_{\beta_n} f - m\|_n^2 > \delta_n + 2\cdot pen_n^2(f) + 3\int |T_{\beta_n} f(x) - m(x)|^2 \mathbf{P}_X(dx) \Bigg\}\\
&\le\; \sum_{s=1}^{\infty} \mathbf{P}\Bigg\{ \exists f \in \mathcal{F}_n : pen_n^2(f) \le 2^s \delta_n,\\
&\qquad\qquad \frac{\left|\int |T_{\beta_n} f(x) - m(x)|^2 \mathbf{P}_X(dx) - \|T_{\beta_n} f - m\|_n^2\right|}{2^{s-1}\delta_n + \int |T_{\beta_n} f(x)-m(x)|^2 \mathbf{P}_X(dx) + \|T_{\beta_n} f - m\|_n^2} > \frac{1}{2} \Bigg\},
\end{aligned}$$

which can be bounded as above. $\square$

The above lemma enables us to analyze the rate of convergence of the estimate for fixed function space. Next we explain how we can use the data to choose an appropriate parameter consisting of a function space $\mathcal{F}_{n,k}$ belonging to a finite collection

$$\{ \mathcal{F}_{n,k} \;:\; k \in \mathcal{P}_n \}$$

of function spaces and corresponding penalty terms

$$pen_{n,k}^2(f) \geq 0 \quad (f \in \mathcal{F}_{n,k}).$$

To do this we split the sample into a learning sample

$$\hat{\mathcal{D}}_{n_l} = \left\{(X_1, \bar{Y}_{1,n}), \ldots, (X_{n_l}, \bar{Y}_{n_l,n})\right\}$$

of size $n_l = \lceil n/2 \rceil$ and a testing sample

$$\left\{(X_{n_l+1}, \bar{Y}_{n_l+1,n}), \ldots, (X_n, \bar{Y}_{n,n})\right\}$$

of size $n_t = n - n_l$. For fixed $k \in \mathcal{P}_n$ we use the learning sample to define a estimate $m_{n_l}^k$ by

$$\tilde{m}_{n_l}^k(\cdot) = \arg\min_{f \in \mathcal{F}_{n,k}} \left(\frac{1}{n_l} \sum_{i=1}^{n_l} |f(X_i) - \bar{Y}_{i,n}|^2 + pen_{n,k}^2(f) \right)$$

and

$$m_{n_l}^k(x) = T_{\beta_n} \tilde{m}_{n_l}^k(x) \quad (x \in \mathbb{R}^d).$$

Next we choose $\hat{k} \in \mathcal{P}_n$ by minimizing the empirical L_2 risk on the testing sample, i.e., we set

$$m_n(x) = m_{n_l}^{\hat{k}}(x) \quad (x \in \mathbb{R}^d),$$

where

$$\hat{k} = \arg\min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^{n} |m_{n_l}^k(X_i) - \bar{Y}_{i,n}|^2.$$

Then the following result holds.

Lemma 4 *Assume that $Y - m(X)$ is sub-Gaussian in the sense that (20) holds for some $C, \sigma > 0$ and assume $|\mathcal{P}_n| \to \infty$ $(n \to \infty)$. Assume furthermore that conditioned on $X_1, \ldots, X_n$ the data sets*

$$\hat{\mathcal{D}}_{n_l} \quad \textit{and} \quad \{Y_{n_l+1}, \ldots, Y_n\}$$

are independent. Let for each $k \in \mathcal{P}_n$ a set $\mathcal{F}_{n,k}$ of functions $f : \mathbb{R}^d \to \mathbb{R}$ be given and let the estimate m_n be defined as above. Then

$$\frac{1}{n_t} \sum_{i=n_l+1}^{n} |m_n(X_i) - m(X_i)|^2$$
$$= O_{\mathbf{P}} \left(\frac{\beta_n^2 \cdot \log |\mathcal{P}_n|}{n_t} + \frac{1}{n_t} \sum_{i=n_l+1}^{n} |Y_i - \bar{Y}_{i,n}|^2 + \min_{k \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^{n} |m_{n_l}^k(X_i) - m(X_i)|^2 \right).$$

Proof. The results follows by applying Lemma 3 conditioned on $\hat{\mathcal{D}}_{n_l}$ and $X_1, \dots, X_n$ and with

$$\mathcal{F}_n = \{m_{n_l}^k \ : \ k \in \mathcal{P}_n\} \quad \text{and} \quad pen_n^2(f) = 0.$$

Here we bound the covering number by the finite cardinality $|\mathcal{P}_n|$ of the set of estimates.

□

5.4 Proof of Theorem 1

Before we start with the proof, observe that the boundedness of the discounted payoff function f_t by L implies $|q_t(x)| \leq L$ for $x \in \mathbb{R}^d$.

In the sequel we will show

$$\int |\hat{q}_{n,s}(x) - q_s(x)|^2 \mathbf{P}_{X_s}(dx)$$
$$= O_{\mathbf{P}} \left(\delta_{n_l} + \max_{t \in \{s,s+1,\dots,T-1\}} \inf_{f \in \mathcal{G}_{p_n}, \|f\|_\infty \leq \beta_n} \left(pen_{p_n}^2(f) + \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \right) \right) \quad (21)$$

for all $s \in \{0, 1, \dots, T\}$.

For $s = T$ we have $\hat{q}_{n,T}(x) = 0 = q_T(x)$, so the assertion is trivial. So let $t < T$ and assume that the assertion holds for $s \in \{t+1, \dots, T\}$. By induction it suffices to show (21) for $s = t$, which we will show in the sequel in seven steps.

In the first step of the proof we show

$$\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) = O_{\mathbf{P}} \left(\frac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n,t}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 + \frac{\beta_n^2 \cdot \log |\mathcal{P}_n|}{n_t} \right).$$

Let $\mathcal{D}_{n,t}$ be the set of all $X_{j,s}^{(r)}$ with either $r \geq t+1, s \in \{0, \dots, T\}$ and $j \in \{1, \dots, n\}$ or $r = t, s \in \{0, \dots, T\}$ and $j \in \{1, \dots, n_l\}$. Conditioned on $\mathcal{D}_{n,t}$,

$$\{\hat{q}_{n_l,t}^{(p)} : p \in \mathcal{P}_n\}$$

consists of $|\mathcal{P}_n|$ different functions. Furthermore, because of boundedness of $\hat{q}_{n_l,t}^{(p)}$ and q_t by β_n we have

$$\begin{aligned}
\sigma_p^2 &:= \mathbf{Var}\{|\hat{q}_{n_l,t}^{(p)}(X_{n_l+1,t}^{(t)}) - q_t(X_{n_l+1,t}^{(t)})|^2 | \mathcal{D}_{n,t}\} \\
&\leq \mathbf{E}\{|\hat{q}_{n_l,t}^{(p)}(X_{n_l+1,t}^{(t)}) - q_t(X_{n_l+1,t}^{(t)})|^4 | \mathcal{D}_{n,t}\} \\
&\leq 4\beta_n^2 \ \int |\hat{q}_{n_l,t}^{(p)}(x) - q_t(x)|^2 \ \mathbf{P}_{X_t}(dx).
\end{aligned}$$

Using this and the Bernstein inequality (cf., e.g., Lemma A.2 in Györfi et al. (2002)) we get using the notation $\epsilon_n = c_{17} \cdot \beta_n^2 \log |\mathcal{P}_n|/n_t$:

$$
\begin{aligned}
&\mathbf{P}\{\int |\hat{q}_{n,t}(x) - q_t(x)|^2 \, \mathbf{P}_{X_t}(dx) > 2 \cdot \tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n,t}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 + \epsilon_n \, |\mathcal{D}_{n,t}\} \\
&\leq |\mathcal{P}_n| \cdot \max_{p \in \mathcal{P}_n} \mathbf{P}\{\int |\hat{q}_{n_l,t}^{(p)}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\
&\qquad\qquad > 2 \cdot \tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n_l,t}^{(p)}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 + \epsilon_n \mid \mathcal{D}_{n,t}\} \\
&\leq |\mathcal{P}_n| \cdot \max_{p \in \mathcal{P}_n} \mathbf{P}\{2 \int |\hat{q}_{n_l,t}^{(p)}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\
&\qquad\qquad > 2 \cdot \tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n_l,t}^{(p)}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 + \epsilon_n \; + \tfrac{\sigma_p^2}{4\beta_n^2} | \, \mathcal{D}_{n,t}\} \\
&\leq |\mathcal{P}_n| \cdot \max_{p \in \mathcal{P}_n} \mathbf{P}\{\int |\hat{q}_{n_l,t}^{(p)}(x) - q_t(x)|^2 \, \mathbf{P}_{X_t}(dx) - \tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n_l,t}^{(p)}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 \\
&\qquad\qquad > \tfrac{1}{2} \cdot \left(\tfrac{\sigma_p^2}{4\beta_n^2} + \epsilon_n \right) |\mathcal{D}_{n,t}\} \\
&\leq |\mathcal{P}_n| \cdot \max_{p \in \mathcal{P}_n} \exp \left(\frac{-n_t \cdot \left(\frac{\sigma_p^2}{8\beta_n^2} + \frac{\epsilon_n}{2} \right)^2}{2\sigma_p^2 + 2\frac{\sigma_p^2}{8\beta_n^2} + \frac{\epsilon_n}{2} \cdot \frac{4\beta_n^2}{3}} \right) \\
&\leq |\mathcal{P}_n| \cdot \max_{p \in \mathcal{P}_n} \exp \left(- \frac{n_t(\sigma_p^2 + \epsilon_n)}{16\beta_n^2 + \frac{8\beta_n^2}{3}} \right) \\
&\leq |\mathcal{P}_n| \cdot \exp \left(- \tfrac{1}{(32+8/3) \cdot \beta_n^2} \cdot \tfrac{n_t \epsilon_n}{\log |\mathcal{P}_n|} \cdot \log |\mathcal{P}_n| \right) \\
&\to 0 \qquad (n \to \infty).
\end{aligned}
$$

In the second step of the proof we show

$$
\begin{aligned}
&\frac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n,t}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 \\
&= O_{\mathbf{P}} \Bigg(\frac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 + \frac{\beta_n^2 \log |\mathcal{P}_n|}{n_t} \\
&\qquad\qquad + \min_{p \in \mathcal{P}_n} \frac{1}{n_t} \sum_{i=n_l+1}^{n} |q_{n_l,t}^{(p)}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 \Bigg).
\end{aligned}
$$

To do this we apply Lemma 4. In the context of Lemma 4 we have $X_i = X_{i,t}^{(t)}$,

$$
Y_i = \max\{f_{t+1}(X_{i,t+1}^{(t)}), q_{t+1}(X_{i,t+1}^{(t)})\} \text{ and } \bar{Y}_{i,n} = \max\{f_{t+1}(X_{i,t+1}^{(t)}), \hat{q}_{n,t+1}(X_{i,t+1}^{(t)})\}. \quad (22)
$$

Observing

$$\frac{1}{n_t} \sum_{i=n_l+1}^{n} |Y_i - \bar{Y}_{i,n}|^2 \le \frac{1}{n_t} \sum_{i=n_l+1}^{n} |q_{t+1}(X_{i,t+1}^{(t)}) - \hat{q}_{n,t+1}(X_{i,t+1}^{(t)})|^2$$

the assertion follows from Lemma 4 if we apply it conditioned on $\mathcal{D}_{n,t}$.

In the third step of the proof we show

$$\tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 = O_{\mathbf{P}} \left(\int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \tfrac{\beta_n^2 \log |\mathcal{P}_n|}{n_t} \right).$$

Using

$$\begin{aligned}
&\mathbf{P}\{\tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 \\
&\qquad\qquad > 2 \int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \epsilon_n | \mathcal{D}_{n,t}\} \\
&= \mathbf{P}\{\tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n,t+1}(X_{i,t+1}^{(t)}) - q_{t+1}(X_{i,t+1}^{(t)})|^2 - \int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)^2 \mathbf{P}_{X_{t+1}}(dx) \\
&\qquad\qquad > \int |\hat{q}_{n,t+1}(x) - q_{t+1}(x)|^2 \mathbf{P}_{X_{t+1}}(dx) + \epsilon_n | \mathcal{D}_{n,t}\}
\end{aligned}$$

this follows as in the first step by an application of the Bernstein inequality.

In the fourth step of the proof we show

$$\min_{p \in \mathcal{P}_n} \tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n_l,t}^{(p)}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 = O_{\mathbf{P}} \left(\int |\hat{q}_{n_l,t}^{(p_n)}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) + \tfrac{\beta_n^2 \log |\mathcal{P}_n|}{n_t} \right).$$

To see this, we observe that we have as in the third step of the proof

$$\tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n_l,t}^{(p_n)}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 = O_{\mathbf{P}} \left(\int |\hat{q}_{n_l,t}^{(p_n)}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) + \tfrac{\beta_n^2 \log |\mathcal{P}_n|}{n_t} \right),$$

hence the assertion follows from

$$\min_{p \in \mathcal{P}_n} \tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n_l,t}^{(p)}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2 \le \tfrac{1}{n_t} \sum_{i=n_l+1}^{n} |\hat{q}_{n_l,t}^{(p_n)}(X_{i,t}^{(t)}) - q_t(X_{i,t}^{(t)})|^2.$$

In the fifth step of the proof we show

$$\begin{aligned}
&\int |\hat{q}_{n_l,t}^{(p_n)}(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \\
&= O_{\mathbf{P}} \left(\tfrac{1}{n_l} \sum_{i=1}^{n_l} |Y_i - \hat{Y}_{i,n}|^2 + \delta_{n_l} + \inf_{f \in \mathcal{G}_{p_n}, \|f\|_\infty \le \beta_n} \left(pen_{p_n}^2(f) + \int |f(x) - q_t(x)|^2 \mathbf{P}_{X_t}(dx) \right) \right),
\end{aligned}$$

where Y_i and $\hat{Y}_{i,n}$ are defined in (22). This follows immediately from Lemma 3 (applied with $\bar{Y}_{i,n} = \hat{Y}_{i,n}$). Here we observe that the sub-Gaussian condition is satisfied because the random variables are bounded and that (6) implies

$$\sqrt{n_l}\cdot\delta \;\;\geq\;\; c_{12}\int_{c_{13}\delta}^{\sqrt{48\delta}}\Bigg(\log\mathcal{N}_2\Bigg(u,\{T_{\beta_n}f-g : f\in\mathcal{G}_{p_n},$$

$$\frac{1}{n_l}\sum_{i=1}^{n_l}|T_{\beta_n}f(x_i)-g(x_i)|^2+pen_{p_n}^2(f)\leq 4\delta\},x_1^{n_l}\Bigg)\Bigg)^{1/2}du \quad (23)$$

for all $\delta\geq\delta_{n_l}/6$, all $g\in\mathcal{G}_{p_n}$ and all $x_1,\ldots,x_{n_l}\in\mathbb{R}$, and

$$\frac{\sqrt{n_l}\cdot\delta}{\beta_n^2} \;\;\geq\;\; c_{12}\int_{c_{13}\delta/\beta_n^2}^{\sqrt{\delta}}\Bigg(\log\mathcal{N}_2\Bigg(u,\{T_{\beta_n}f-q_t : f\in\mathcal{G}_{p_n},$$

$$\frac{1}{n_l}\sum_{i=1}^{n_l}|T_{\beta_n}f(x_i)-q_t(x_i)|^2\leq\frac{\delta}{\beta_n^2},pen_{p_n}^2(f)\leq 2\delta\},x_1^{n_l}\Bigg)\Bigg)^{1/2}du \quad (24)$$

for all $\delta\geq\delta_{n_l}$, all $t\in\{0,1,\ldots,T-1\}$ and all $x_1,\ldots,x_{n_l}\in\mathbb{R}$.

In the sixth step of the proof we show

$$\tfrac{1}{n_l}\sum_{i=1}^{n_l}|Y_i-\hat{Y}_{i,n}|^2=O_{\mathbf{P}}\Bigg(\int|\hat{q}_{n,t+1}(x)-q_{t+1}(x)|^2\mathbf{P}_{X_{t+1}}(dx)+\tfrac{\beta_n^2\log|\mathcal{P}_n|}{n_l}\Bigg).$$

First we observe

$$\frac{1}{n_l}\sum_{i=1}^{n_l}|Y_i-\hat{Y}_{i,n}|^2\leq\frac{1}{n_l}\sum_{i=1}^{n_l}|\hat{q}_{n,t+1}(X_{i,t+1}^{(t)})-q_{t+1}(X_{i,t+1}^{(t)})|^2.$$

To show

$$\tfrac{1}{n_l}\sum_{i=1}^{n_l}|\hat{q}_{n,t+1}(X_{i,t+1}^{(t)})-q_{t+1}(X_{i,t+1}^{(t)})|^2=O_{\mathbf{P}}(\textstyle\int|\hat{q}_{n,t+1}(x)-q_{t+1}(x)|^2\mathbf{P}_{X_{t+1}}(dx)+\tfrac{\beta_n^2\log|\mathcal{P}_n|}{n_l})$$

we condition on all data points $X_{j,s}^{(r)}$ with $r\geq t+1, s\in\{0,\ldots,T\}$ and $j\in\{1,\ldots,n\}$. Then the assertion follows by an application of Bernstein inequality as in steps 1 and 3.

In the seventh (and last) step of the proof we observe that we get by induction

$$\textstyle\int|\hat{q}_{n,t+1}(x)-q_{t+1}(x)|^2\mathbf{P}_{X_{t+1}}(dx)$$

$$=O_{\mathbf{P}}\Bigg(\delta_{n_l}+\max_{s\in\{t+1,\ldots,T-1\}}\inf_{f\in\mathcal{G}_{(p_n)},\|f\|_\infty\leq L}\left(pen_{p_n}^2(f)+\textstyle\int|f(x)-q_t(x)|^2\mathbf{P}_{X_s}(dx)\right)\Bigg).$$

We complete the proof by using

$$\frac{\beta_n^2\log|\mathcal{P}_n|}{n}=O(\delta_n),$$

which follows from

$$\frac{n\cdot\delta_n}{\beta_n^2\cdot\log n}\to\infty\quad(n\to\infty)\quad\text{and}\quad\frac{|\mathcal{P}_n|}{n^r}\to 0\quad(n\to\infty),$$

and by gathering the above results. □

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