Proof of Lemma 1. We prove equation (3) in Lemma 1 by induction. The assertion is trivial for $t = L$ because in this case we have

$$\tau^*_{L-1} = L \quad \text{and} \quad \tau = L$$

for any $\tau \in \mathcal{T}(L)$.

Let $t \in \{0, \ldots, L-1\}$ and assume that

$$V_s(z^s_{-\infty}) = \mathbf{E}\left\{ g_{\tau^*_{s-1}}(Z_1^{\tau^*_{s-1}}) | Z^s_{-\infty} = z^s_{-\infty} \right\}$$

holds for $s = t + 1$. In the sequel we prove equation (3) in Lemma 1. To do this, let $\tau \in \mathcal{T}(t, \ldots, L)$ be arbitrary. On $\{\tau > t\}$ we have $\tau = \max\{\tau, t+1\}$, hence

$$\begin{aligned} g_\tau(Z_1^\tau) &= g_\tau(Z_1^\tau) \cdot 1_{\{\tau=t\}} + g_\tau(Z_1^\tau) \cdot 1_{\{\tau>t\}} \\ &= g_t(Z_1^t) \cdot 1_{\{\tau=t\}} + g_{\max\{\tau,t+1\}}(Z_1^{\max\{\tau,t+1\}}) \cdot 1_{\{\tau>t\}}. \end{aligned}$$

Since $1_{\{\tau=t\}}$ and $1_{\{\tau>t\}} = 1 - 1_{\{\tau\le t\}}$ are measurable with respect to $Z^t_{-\infty}$ we have

$$\begin{aligned} &\mathbf{E}\{g_\tau(Z_1^\tau)|Z^t_{-\infty}\} \\ &= g_t(Z_1^t) \cdot 1_{\{\tau=t\}} + 1_{\{\tau>t\}} \cdot \mathbf{E}\{g_{\max\{\tau,t+1\}}(Z_1^{\max\{\tau,t+1\}})|Z^t_{-\infty}\}. \end{aligned}$$

Using the definition of V_{t+1} together with $\max\{\tau, t+1\} \in \mathcal{T}(t+1, \ldots, L)$ we get

$$\begin{aligned} &\mathbf{E}\{g_{\max\{\tau,t+1\}}(Z_1^{\max\{\tau,t+1\}})|Z^t_{-\infty}\} \\ &= \mathbf{E}\{\mathbf{E}\{g_{\max\{\tau,t+1\}}(Z_1^{\max\{\tau,t+1\}})|Z^{t+1}_{-\infty}\}|Z^t_{-\infty}\} \\ &\le \mathbf{E}\{V_{t+1}(Z^{t+1}_{-\infty})|Z^t_{-\infty}\}, \end{aligned}$$

from which we can conclude

$$\begin{aligned} &\mathbf{E}\{g_\tau(Z_1^\tau)|Z^t_{-\infty}\} \\ &\le g_t(Z_1^t) \cdot 1_{\{\tau=t\}} + 1_{\{\tau>t\}} \cdot \mathbf{E}\{V_{t+1}(Z^{t+1}_{-\infty})|Z^t_{-\infty}\} \\ &\le \max\{g_t(Z_1^t), \mathbf{E}\{V_{t+1}(Z^{t+1}_{-\infty})|Z^t_{-\infty}\}\}. \end{aligned} \tag{1}$$

Now we make the same calculations using $\tau = \tau^*_{t-1}$. We get

$$\begin{aligned} &\mathbf{E}\{g_{\tau^*_{t-1}}(Z_1^{\tau^*_{t-1}})|Z^t_{-\infty}\} \\ &= g_t(Z_1^t) \cdot 1_{\{\tau^*_{t-1}=t\}} + 1_{\{\tau^*_{t-1}>t\}} \cdot \mathbf{E}\{g_{\max\{\tau^*_{t-1},t+1\}}(Z_1^{\max\{\tau^*_{t-1},t+1\}})|Z^t_{-\infty}\}. \end{aligned}$$

By definition of τ_t^* we have on $\{\tau_{t-1}^* > t\}$

$$\max\{\tau_{t-1}^*, t+1\} = \tau_t^*.$$

Using this and the induction hypothesis we can conclude

$$\begin{aligned}
&\mathbf{E}\{g_{\tau_{t-1}^*}(Z_1^{\tau_{t-1}^*})|Z_{-\infty}^t\} \\
&= g_t(Z_1^t) \cdot 1_{\{\tau_{t-1}^*=t\}} \\
&\qquad +1_{\{\tau_{t-1}^*>t\}} \cdot \mathbf{E}\{\mathbf{E}\{g_{\tau_t^*}(Z_1^{\tau_t^*})|Z_{-\infty}^{t+1}\}|Z_{-\infty}^t\} \\
&= g_t(Z_1^t) \cdot 1_{\{\tau_{t-1}^*=t\}} \\
&\qquad +1_{\{\tau_{t-1}^*>t\}} \cdot \mathbf{E}\{V_{t+1}(Z_{-\infty}^{t+1})|Z_{-\infty}^t\}. \qquad (2)
\end{aligned}$$

Next we show

$$\mathbf{E}\{V_{t+1}(Z_{-\infty}^{t+1})|Z_{-\infty}^t\} = q_t(Z_{-\infty}^t). \qquad (3)$$

To see this, we observe that by the induction hypothesis and because of $\tau_t^* \in \mathcal{T}(t+1,\ldots,L)$ we have

$$\begin{aligned}
&\mathbf{E}\{V_{t+1}(Z_{-\infty}^{t+1})|Z_{-\infty}^t\} \\
&= \mathbf{E}\{\mathbf{E}\{g_{\tau_t^*}(Z_1^{\tau_t^*})|Z_{-\infty}^{t+1}\}|Z_{-\infty}^t\} \\
&= \mathbf{E}\{g_{\tau_t^*}(Z_1^{\tau_t^*})|Z_{-\infty}^t\} \\
&\leq q_t(Z_{-\infty}^t).
\end{aligned}$$

Furthermore the definition of V_{t+1} implies

$$\begin{aligned}
&\mathbf{E}\{V_{t+1}(Z_{-\infty}^{t+1})|Z_{-\infty}^t\} \\
&= \mathbf{E}\left\{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}(t+1,\ldots,L)} \mathbf{E}\left\{g_\tau(Z_1^\tau)|Z_{-\infty}^{t+1}\right\}|Z_{-\infty}^t\right\} \\
&\geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(t+1,\ldots,L)} \mathbf{E}\left\{\mathbf{E}\left\{g_\tau(Z_1^\tau)|Z_{-\infty}^{t+1}\right\}|Z_{-\infty}^t\right\} \\
&= q_t(Z_{-\infty}^t),
\end{aligned}$$

which concludes the proof of (3). Using (3) and the definition of τ_{t-1}^* we get

$$\begin{aligned}
&g_t(Z_1^t) \cdot 1_{\{\tau_{t-1}^*=t\}} + 1_{\{\tau_{t-1}^*>t\}} \cdot \mathbf{E}\{V_{t+1}(Z_{-\infty}^{t+1})|Z_{-\infty}^t\} \\
&= g_t(Z_1^t) \cdot 1_{\{\tau_{t-1}^*=t\}} + 1_{\{\tau_{t-1}^*>t\}} \cdot q_t(Z_{-\infty}^t) \\
&= \max\{g_t(Z_1^t), q_t(Z_{-\infty}^t)\}. \qquad (4)
\end{aligned}$$

Summarizing the above results we have

$$
\begin{aligned}
&V_t(z_{-\infty}^t) \\
&:= \operatorname{ess} \sup_{\tau \in \mathcal{T}(t,t+1,\ldots,L)} \mathbf{E}\left\{ g_\tau(Z_1^\tau) | Z_{-\infty}^t = z_{-\infty}^t \right\} \\
&\overset{(1)}{\leq} \max\{g_t(z_1^t), \mathbf{E}\{V_{t+1}(Z_{-\infty}^{t+1}) | Z_{-\infty}^t = z_{-\infty}^t\}\} \\
&\overset{(3)}{=} \max\{g_t(z_1^t), q_t(z_{-\infty}^t)\} \\
&\overset{(2),(4)}{=} \mathbf{E}\{g_{\tau_{t-1}^*}(Z_1^{\tau_{t-1}^*}) | Z_{-\infty}^t = z_{-\infty}^t\},
\end{aligned}
$$

from which we conclude

$$
\begin{aligned}
&V_t(z_{-\infty}^t) \\
&= \max\{g_t(z_1^t), q_t(z_{-\infty}^t)\} \\
&= \mathbf{E}\{g_{\tau_{t-1}^*}(Z_1^{\tau_{t-1}^*}) | Z_{-\infty}^t = z_{-\infty}^t\}, \qquad (5)
\end{aligned}
$$

which completes the proof of equation (3) in Lemma 1. In order to prove equation (4) in Lemma 1 we observe that

$$
\begin{aligned}
V_0^* &:= \sup_{\tau \in \mathcal{T}(0,\ldots,L)} \mathbf{E}\left\{ g_\tau(Z_1^\tau) \right\} \\
&= \sup_{\tau \in \mathcal{T}(0,\ldots,L)} \mathbf{E}\left\{ g_0 \cdot 1_{\{\tau=0\}} + g_{\max\{\tau,1\}}(Z_1^{\max\{\tau,1\}}) \cdot 1_{\{\tau>0\}} \right\} \\
&\geq \mathbf{E}\left\{ g_0 \cdot 1_{\{g_0 \geq q_0(Z_{-\infty}^0)\}} + g_{\tau_0^*}(Z_1^{\tau_0^*}) \cdot 1_{\{g_0 < q_0(Z_{-\infty}^0)\}} \right\} \\
&\overset{(5)}{=} \mathbf{E}\left\{ g_0 \cdot 1_{\{g_0 \geq q_0(Z_{-\infty}^0)\}} \right. \\
&\qquad\qquad \left. + \mathbf{E}\{V_1(Z_{-\infty}^1) | Z_{-\infty}^0\} \cdot 1_{\{g_0 < q_0(Z_{-\infty}^0)\}} \right\} \\
&\overset{(3)}{=} \mathbf{E}\left\{ g_0 \cdot 1_{\{g_0 \geq q_0(Z_{-\infty}^0)\}} + q_0(Z_{-\infty}^0) \cdot 1_{\{g_0 < q_0(Z_{-\infty}^0)\}} \right\} \\
&= \mathbf{E}\left\{ \max\{g_0, q_0(Z_{-\infty}^0)\} \right\} \\
&\overset{(5)}{=} \mathbf{E}\left\{ \operatorname{ess} \sup_{\tau \in \mathcal{T}(0,\ldots,L)} \mathbf{E}\{g_\tau(Z_1^\tau) | Z_{-\infty}^0\} \right\} \\
&\geq \sup_{\tau \in \mathcal{T}(0,\ldots,L)} \mathbf{E}\{g_\tau(Z_1^\tau)\} \\
&= V_0^*,
\end{aligned}
$$

which yields equation (4) in Lemma 1. □

Proof of Lemma 2. Equation (5) in Lemma 2 is implied by (3) and (5). In order to prove equation (6) in Lemma 2 we observe that we have by (3) and Lemma 1

$$
\begin{aligned}
& q_t(Z_{-\infty}^t) \\
& = \mathbf{E}\{V_{t+1}(Z_{-\infty}^{t+1})|Z_{-\infty}^t\} \\
& = \mathbf{E}\{\mathbf{E}\{g_{\tau_t^*}(Z_1^{\tau_t^*})|Z_{-\infty}^{t+1}\}|Z_{-\infty}^t\} \\
& = \mathbf{E}\{g_{\tau_t^*}(Z_1^{\tau_t^*})|Z_{-\infty}^t\}.
\end{aligned}
$$

□