Over-parametrized deep neural networks do not generalize well

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Abstract
Recently it was shown in several papers that backpropagation is able to find the global minimum of the empirical risk on the training data using over-parametrized deep neural networks. In this paper a similar result is shown for deep neural networks with the sigmoidal squasher activation function in a regression setting, and a lower bound is presented which proves that these networks do not generalize well on a new data in the sense that they do not achieve the optimal minimax rate of convergence for estimation of smooth regression functions.

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1 Introduction

Deep neural networks belong to the most successful approaches in multivariate statistical applications, see, e.g., Schmidhuber (2015) and the literature cited therein. Motivated by the practical success of these networks there has been in recent years an increasing interest in studying the corresponding estimators both practically and theoretically. This is often done in the context of nonparametric regression with random design. Here, \((X,Y)\) is an \(\mathbb{R}^d \times \mathbb{R}\)-valued random vector satisfying \(E\{Y^2\} < \infty\), and given a sample of \((X,Y)\) of size \(n\), i.e., given a data set

\[
D_n = \{(X_1,Y_1), \ldots, (X_n,Y_n)\},
\]

where \((X,Y), (X_1,Y_1), \ldots, (X_n,Y_n)\) are i.i.d. random variables, the aim is to construct an estimate

\[
m_n(\cdot) = m_n(\cdot, D_n) : \mathbb{R}^d \to \mathbb{R}
\]
of the regression function \( m : \mathbb{R}^d \to \mathbb{R} \), \( m(x) = \mathbb{E}\{Y|X = x\} \) such that the \( L_2 \) error

\[
\int |m_n(x) - m(x)|^2 \text{P}_X(dx)
\]

is “small” (see, e.g., Győrfi et al. (2002) for a systematic introduction to nonparametric regression and a motivation for the \( L_2 \) error). In the sequel we want to use (deep) neural networks in order to estimate a regression function. The starting point in defining a neural network is the choice of an activation function \( \sigma : \mathbb{R} \to \mathbb{R} \). Here, we use in the sequel so-called squashing functions, which are nondecreasing and satisfy \( \lim_{x \to -\infty} \sigma(x) = 0 \) and \( \lim_{x \to \infty} \sigma(x) = 1 \). An example of a squashing function is the so-called sigmoidal or logistic squasher

\[
\sigma(x) = \frac{1}{1 + \exp(-x)} \quad (x \in \mathbb{R}).
\]

In applications, also unbounded activation functions are often used, e.g., the famous ReLU activation function

\[
\sigma(x) = \max\{x, 0\}.
\]

The network architecture \((L, k)\) depends on a positive integer \( L \) called the number of hidden layers and a width vector \( k = (k_1, \ldots, k_L) \in \mathbb{N}^L \) that describes the number of neurons in the first, second, \ldots, \( L \)-th hidden layer. A multilayer feedforward neural network with architecture \((L, k)\) and activation function \( \sigma \) is a real-valued function \( f : \mathbb{R}^d \to \mathbb{R} \) defined by

\[
f_c(x) = \sum_{i=1}^{k_L} c_{i,1}^{(L)} \cdot f_i^{(L)}(x) + c_{0}^{(L)}
\]

for some \( c_{1,0}^{(L)}, \ldots, c_{1,k_L}^{(L)} \in \mathbb{R} \) and for \( f_i^{(L)} \)'s recursively defined by

\[
f_i^{(r)}(x) = \sigma \left( \sum_{j=1}^{k_{r-1}} c_i^{(r-1)} \cdot f_j^{(r-1)}(x) + c_{i,0}^{(r-1)} \right)
\]

for some \( c_{i,0}^{(r-1)}, \ldots, c_{i,k_{r-1}}^{(r-1)} \in \mathbb{R} \) \((r = 2, \ldots, L)\) and

\[
f_i^{(1)}(x) = \sigma \left( \sum_{j=1}^{d} c_{i,j}^{(0)} \cdot x^{(j)} + c_{i,0}^{(0)} \right)
\]

for some \( c_{i,0}^{(0)}, \ldots, c_{i,d}^{(0)} \in \mathbb{R} \).

In the sequel we want to use the data (1) in order to choose the weights \( c = (c_{i,j}^{(s)})_{i,j,s} \) of the neural network such that the resulting function \( f_c \) defined by (3)–(5) is a good estimate of the regression function. This can be done for instance by applying the principle of the least squares. Here one defines a suitable class \( \mathcal{F}_n \) of neural networks
and chooses that function from this class which minimizes the error on the training data, i.e., one defines the so-called least squares neural network estimate by

\[ m_n(\cdot) = \arg \min_{f \in F_n} \frac{1}{n} \sum_{i=1}^{n} |f(X_i) - Y_i|^2. \]

Recently it was shown in several articles, that such least squares estimates based on deep neural networks achieve nice rates of convergence if suitable structural constraints on the regression function are imposed, cf., e.g., Kohler and Krzyżak (2017), Bauer and Kohler (2019), Kohler and Langer (2019) and Schmidt-Hieber (2019). Eckle and Schmidt-Hieber (2019) and Kohler, Krzyżak and Langer (2019) showed that the least squares neural network regression estimates based on deep neural networks can achieve the rate of convergence results similar to piecewise polynomial partition estimates where the partition is chosen in an optimal way. Results concerning estimation by neural networks of piecewise polynomial regression functions with partitions having rather general smooth boundaries have been obtained by Imaizumi and Fukamizu (2019).

Unfortunately it is not possible to compute the least squares neural networks regression estimate exactly, because such computation requires minimization of the non-convex and nonlinear function

\[ F_n(c) = \frac{1}{n} \sum_{i=1}^{n} |f_c(X_i) - Y_i|^2 \]

with respect to the weight vector \( c \). In practice, one uses gradient descent in order to compute the minimum of the above function approximately. Here one chooses a random starting value \( c^{(0)} \) for the weight vector, and then defines \( c^{(t+1)} = c^{(t)} - \lambda_n \cdot (\nabla_c F_n)(c^{(t)}) \) \( (t = 0, \ldots, t_n - 1) \) for some suitably chosen stepsize \( \lambda_n > 0 \) and the number of gradient descent steps \( t_n \in \mathbb{N} \). Then the regression estimate is defined by \( m_n(x) = f_{c^{(t_n)}}(x) \).

There are quite a few papers which try to prove that backpropagation works theoretically for deep neural networks. The most popular approach in this context is the so-called landscape approach. Choromanska et al. (2015) used random matrix theory to derive a heuristic argument showing that the risk of most of the local minima of the empirical \( L_2 \) risk \( F_n(c) \) is not much larger than the risk of the global minimum. For neural networks with special activation function it was possible to validate this claim, see, e.g., Arora et al. (2018), Kawaguchi (2016), and Du and Lee (2018), which have analyzed gradient descent for neural networks with linear or quadratic activation function. But for such neural networks there do not exist good approximation results, consequently, one cannot derive from these results good rates of convergence for neural network regression estimates. Du et al. (2018) analyzed gradient descent applied to neural networks with one hidden layer in case of an input with a Gaussian distribution. They used the expected gradient instead of the gradient in their gradient descent routine, and therefore, their result cannot be used to derive the rate of convergence results for a neural network regression estimate learned by the gradient descent. Liang et al. (2018) applied gradient
descent to a modified loss function in classification, where it is assumed that the data can be interpolated by a neural network. Here, as we will show in this paper (cf., Theorem 2 below), the last assumption does not lead to good rates of convergence in nonparametric regression, and it is unclear whether the main idea (of simplifying the estimation by a modification of the loss function) can also be used in a regression setting.

Recently it was shown in several papers, see, e.g., Allen-Zhu, Li and Song (2019), Kawaguchi and Huang (2019) and the literature cited therein, that gradient descent leads to a small empirical $L_2$ risk in over-parametrized neural networks. Here the results in Allen-Zhu, Li and Song (2019) are proven for the ReLU activation function and neural networks with a polynomial size in the sample size. The neural networks in Kawaguchi and Huang (2019) use squashing activation functions and are much smaller (in fact, they require only a linear size in the sample size). In contrast to Allen-Zhu, Li and Song (2019) there the learning rate is set to zero for all neurons except for neurons in the output layer, so actually they compute a linear least squares estimate with gradient descent, which is not used in practice.

In this paper we show a related result for a deep neural network regression estimate with the logistic squasher activation function, where the learning rate is nonzero for all neurons of the network. By analyzing the minimax rate of convergence of this estimate in case of a general design distribution we are able to show that this estimate does not generalize well to new (independent) data in a sense that it does not achieve the optimal minimax rate of convergence in case of a smooth regression function. Here the main trick is that we also allow discrete design distributions and prove a general result which shows that any estimate which achieves with high probability a very small error on the training data in case of such distributions does not achieve the optimal minimax error. This is in contrast to a recent trend in machine learning, where one tries to argue that such estimates can achieve good rates of convergence (see, e.g., Bartlett et al. (2019), Belkin et al. (2019), Hastie et al. (2019) and the literature cited therein). We would like to point out that our result above is not a contradiction to Belkin, Rakhlin and Tsybakov (2018), who show that learning method which interpolates the training data can achieve the optimal rates for nonparametric regression problems, because it is assumed there that the design variable has a density with respect to the Lebesgue-Borel measure, which is bounded away from zero and infinity.

Throughout the paper, the following notation is used: The sets of natural numbers, natural numbers including 0, and real numbers are denoted by $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{R}$, respectively. The Euclidean norm of $x \in \mathbb{R}^d$ is denoted by $\|x\|$ and $\|x\|_\infty$ denotes its supremum norm. For $f : \mathbb{R}^d \to \mathbb{R}$

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$$

is its supremum norm. Let $p = q + s$ for some $q \in \mathbb{N}_0$ and $0 < s \leq 1$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called $(p,C)$-smooth, if for every $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}_0^d$ with $\sum_{j=1}^d \alpha_j = q$ the partial derivative $\frac{\partial^q f}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_d}}$ exists and satisfies

$$\left| \frac{\partial^q f}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_d}}(x) - \frac{\partial^q f}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_d}}(z) \right| \leq C \cdot \|x - z\|^s$$

4
for all $x, z \in \mathbb{R}^d$.

The outline of this paper is as follows: In Section 2 the over-parametrized neural network regression estimates are defined. The main results are presented in Section 3 and proven in Section 4.

## 2 An over-parametrized neural network regression estimator

In the sequel we use the logistic squasher $\sigma(x) = 1/(1 + e^{-x})$ as the activation function, and we use a network topology where we compute the linear combination of $k_n$ fully connected neural networks with $L$ layers and $k_0$ neurons per layer. Thus we define our neural networks by

$$f_c(x) = \sum_{i=1}^{k_n} c_{1,1,i}^{(L)} \cdot f_{i,i}^{(L)}(x) + c_{1,1,0}^{(L)}$$

for some $c_{1,1,0}^{(L)}, \ldots, c_{1,1,k_n}^{(L)} \in \mathbb{R}$, where $f_{i,i}^{(L)}$ are recursively defined by

$$f_{k,i}^{(r)}(x) = \sigma \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f_{k,j}^{(r-1)}(x) + c_{k,i,0}^{(r-1)} \right)$$

for some $c_{k,i,0}^{(r-1)}, \ldots, c_{k,i,k_0}^{(r-1)} \in \mathbb{R}$ ($r = 2, \ldots, L$) and

$$f_{k,i}^{(1)}(x) = \sigma \left( \sum_{j=1}^{d} c_{k,i,j}^{(0)} \cdot x^{(j)} + c_{k,i,0}^{(0)} \right)$$

for some $c_{k,i,0}^{(0)}, \ldots, c_{k,i,d}^{(0)} \in \mathbb{R}$.

We learn the weight vector $c = (c_{k,i,j}^{(s)})_{k,i,j,s}$ of our neural work by the gradient descent. We initialize $c^{(0)}$ by setting

$$c_{1,1,k}^{(L)} = 0 \quad \text{for } k = 0, \ldots, k_n,$$

and by choosing all others weights randomly such that all weights $c_{k,i,j}^{(s)}$ with $s < L$ are independent uniformly distributed on $[-n^4, n^4]$, and we set

$$c^{(t+1)} = c^{(t)} - \lambda_n \cdot (\nabla c F_n)(c^{(t)}) \quad (t = 0, \ldots, t_n - 1)$$

where

$$F_n(c) = \frac{1}{n} \sum_{i=1}^{n} |f_c(X_i) - Y_i|^2$$

is the empirical $L_2$ risk of the network $f_c$ on the training data. The the step size $\lambda_n > 0$ and the number $t_n$ of gradient descent steps will be chosen below.

Because of (9) we have

$$F_n(c^{(0)}) = \frac{1}{n} \sum_{i=1}^{n} |Y_i|^2.$$
3 Main results

Our first result shows that our estimate is able to achieve with high probability a very small error on the training data in case that $k_n$, $\lambda_n$ and $t_n$ are suitably chosen.

**Theorem 1** Let $k_0 \in \mathbb{N}$ with $k_0 \geq 2 \cdot d$, let $L \in \mathbb{N}$ with $L \geq 2$, set

$$k_n = n^{5(L-2)-(d^2+k_0)+5k_0(d+2)+7},$$

$$\lambda_n = \frac{1}{n^{8(L-2)-(d^2+k_0)+8k_0(d+2)+16L+15}},$$

and

$$t_n = 2 \cdot n^{8(L-2)-(d^2+k_0)+8k_0(d+2)+16L+17},$$

and define the estimate as in Section 2. Then for sufficiently large $n$ we have on the event

$$\inf \{ \| X_i - X_j \|_\infty : 1 \leq i,j \leq n, X_i \neq X_j \} \geq \frac{1}{(n+1)^3},$$

$$\max \{ \| X_i \|_\infty : 1 \leq i \leq n \} \leq 1 \quad \text{and} \quad \max \{ |Y_i| : 1 \leq i \leq n \} \leq n^2$$

that with probability at least $1 - 1/n$ the random choice of $c^{(0)}$ leads to

$$\frac{1}{n} \sum_{i=1}^{n} |f_{c^{(\ell)}}(X_i) - Y_i|^2 \leq \min_{g: \mathbb{R}^d \rightarrow \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 + \frac{1}{n \cdot \log n}. \quad (10)$$

**Remark 1.** A corresponding result was shown in Kawaguchi and Huang (2019) for a fully connected network of much smaller size (linear instead of polynomial in the sample size as in Theorem 1 above), however there the learning rate of the gradient descent was set to zero for all weights $c^{(r)}_{k,i,j}$ with $r < L$. In contrast in our result the learning rate is positive for all weights.

As our next result shows, any estimate which (as our estimate from Theorem 1) achieves with high probability a very small error on the training data does in general not generalize well on a new independent data (provided we allow the distributions of $X$ which are concentrated on finite sets).

**Theorem 2** Let $(X,Y)$, $(X_1,Y_1)$, ... be independent and identically distributed $\mathbb{R}^d \times \mathbb{R}$-valued random variables with $\mathbb{E}Y^2 < \infty$, and let $U$ be an $\mathbb{R}^K$-valued random variable independent of the above random variables. Let $C_n$ be a subset of $\mathbb{R}^K$, and let

$$m_n(\cdot) = m_n(\cdot, (X_1,Y_1), \ldots, (X_n,Y_n), U) : \mathbb{R}^d \rightarrow \mathbb{R}$$

be an estimate of $m$. Let $\kappa_n > 0$ and let $\delta_n \leq 1/(n+1)^3$ and assume that $m_n$ satisfies

$$\frac{1}{n} \sum_{i=1}^{n} |m_n(X_i) - Y_i|^2 \leq \min_{g: \mathbb{R}^d \rightarrow \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 + \kappa_n$$
whenever
\[
\inf \{ \|X_i - X_j\|_\infty : 1 \leq i, j \leq n, X_i \neq X_j \} \geq \delta_n \quad \text{and} \quad U \in C_n.
\]

Then there exists a distribution of \((X, Y)\) such that \(X \in [0, 1]^d\) a.s., \(Y \in \{-1, 1\}\) a.s., \(m(x) = 0\) for all \(x \in [0, 1]^d\) and such that we have for \(n \geq 10\)
\[
\mathbb{E} \int |m_n(x) - m(x)|^2 \mathbb{P}_X(dx) \geq \frac{1}{5} - \frac{n \cdot \kappa_n}{n} - \frac{1}{2} \cdot \mathbb{P}_U(C_n^c).
\]

**Corollary 1** Let \(p, C, c_1 > 0\) and let \(\mathcal{D}^{(p, C)}\) be the class of all distributions of \((X, Y)\) which satisfy
1. \(X \in [0, 1]^d\) a.s.
2. \(\sup_{x \in [0,1]^d} \mathbb{E}\{|Y^2|X = x\} \leq c_1\)
3. \(m(\cdot) = \mathbb{E}\{Y|X = \cdot\}\) is \((p, C)\)-smooth.

Let \(m_n\) be the neural network regression estimate from Theorem 1. Then we have for \(n\) sufficiently large
\[
\sup_{(X,Y) \in \mathcal{D}^{(p, C)}} \mathbb{E} \int |m_n(x) - m(x)|^2 \mathbb{P}_X(dx) \geq \frac{1}{6}.
\]

**Proof.** Let \(U\) be the values for the random initialization of the weights of the estimate in Theorem 1. By Theorem 1 we know that there exists a set \(C_n\) of weights such that (10) holds for \(n\) sufficiently large whenever \(U \in C_n\), where \(\mathbb{P}_U(C_n^c) \leq 1/n\). Hence the assumptions of Theorem 2 are satisfied with \(\kappa_n = 1/(n \cdot \log n)\). Let \((X, Y)\) be the distribution from Theorem 2. Then for \(n\) sufficiently large
\[
\sup_{(X,Y) \in \mathcal{D}^{(p, C)}} \mathbb{E} \int |m_n(x) - m(x)|^2 \mathbb{P}_X(dx) \geq \mathbb{E} \int |m_n(x) - m(x)|^2 \mathbb{P}_X(dx)
\]
\[
\geq \frac{1}{5} - \frac{n \cdot \frac{1}{n \cdot \log n}}{2} - \frac{1}{2} \cdot \frac{1}{n}
\]
\[
\geq \frac{1}{6}.
\]

\(\square\)

**Remark 2.** Let \(\mathcal{D}^{(p, C)}\) be the class of distributions of \((X, Y)\) introduced in Corollary 1. It is well-known that there exist estimates \(m_n\) which satisfy
\[
\sup_{(X,Y) \in \mathcal{D}^{(p, C)}} \mathbb{E} \int |m_n(x) - m(x)|^2 \mathbb{P}_X(dx) \leq c_2 \cdot n^{-\frac{2n}{2p+d}}
\]
and that no estimate can achieve a better rate of convergence (cf., Stone (1982) and Chapters 3 and 11 in Győrfi et al. (2002)). Hence Corollary 1 implies that the estimate of Theorem 1 does not achieve the optimal minimax rate of convergence for the class \(\mathcal{D}^{(p, C)}\), in fact the minimax \(L_2\) error for this class does not even converge to zero, let alone in contrast to the optimal value.
4 Proofs

4.1 Proof of Theorem 1

Lemma 1 Let $F : \mathbb{R}^K \to \mathbb{R}$ be differentiable, let $L_n > 0$, set
\[
\lambda_n = \frac{1}{L_n},
\]
let $a_1 \in \mathbb{R}^K$ and set
\[
a_2 = a_1 - \lambda_n \cdot (\nabla aF)(a_1).
\]
Then
\[
\| (\nabla aF)(a) - (\nabla aF)(a_1) \| \leq L_n \cdot \| a - a_1 \| \tag{11}
\]
for all $a = a_1 + s \cdot (a_2 - a_1)$, $s \in [0, 1]$ implies
\[
F(a_2) \leq F(a_1) - \frac{1}{2} \cdot L_n \cdot \| (\nabla aF)(a_1) \|^2.
\]

Proof. See proof of Lemma 1 in Braun, Kohler and Walk (2019). □

Set
\[
F_n(c) = \frac{1}{n} \sum_{i=1}^{n} |f_c(X_i) - Y_i|^2
\]
where $f_c$ is defined by (6)–(8).

Lemma 2 Let $f_c$ be defined by (6)–(8) and assume that for any $i \in \{1, \ldots, n\}$ there exists $j_i \in \{1, \ldots, k_n\}$ such that
\[
f^{(L)}_{j_i,j_i}(X_i) \geq 1 - \frac{2}{n^2} \quad \text{and} \quad \sup_{t \in \{1, \ldots, n\}, X_i \neq X_t} f^{(L)}_{j_i,j_i}(X_i) \leq \frac{2}{n^2} \tag{12}
\]
hold. Then we have for any $n \geq 5$
\[
\| (\nabla cF_n(c)) \|^2 \geq \frac{1}{n} \cdot \left( \frac{1}{n} \sum_{i=1}^{n} |f_c(X_i) - Y_i|^2 - \min_{g : \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right).
\]

Proof. Set
\[
\bar{m}_n(x) = \frac{\sum_{i=1}^{n} Y_i \cdot I_{\{X_i = x\}}}{\sum_{i=1}^{n} I_{\{X_i = x\}}} \quad (x \in \mathbb{R}^d),
\]
where we use the convention $0/0 = 0$. We have
\[
\frac{1}{n} \sum_{i=1}^{n} |f(X_i) - Y_i|^2 = \frac{1}{n} \sum_{i=1}^{n} |f(X_i) - \bar{m}_n(X_i)|^2 + \frac{1}{n} \sum_{i=1}^{n} |\bar{m}_n(X_i) - Y_i|^2,
\]
since
\[
\frac{1}{n} \sum_{i=1}^{n} (f(X_i) - \bar{m}_n(X_i)) \cdot (\bar{m}_n(X_i) - Y_i)
\]
\[
= \frac{1}{n} \sum_{x \in \{X_1, \ldots, X_n\}} (f(x) - \bar{m}_n(x)) \cdot \sum_{1 \leq i \leq n : X_i = x} (\bar{m}_n(X_i) - Y_i) = 0.
\]

This implies
\[
\min_{g : \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 = \frac{1}{n} \sum_{i=1}^{n} |\bar{m}_n(X_i) - Y_i|^2
\]
and
\[
\frac{1}{n} \sum_{i=1}^{n} |f(X_i) - \bar{m}_n(X_i)|^2 = \frac{1}{n} \sum_{i=1}^{n} |f(X_i) - Y_i|^2 - \min_{g : \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2
\]
for any \(f : \mathbb{R}^d \to \mathbb{R}\).

Next we observe
\[
\|D_c F_n(c)\|^2 = \sum_{k,i,j,s} \left| \frac{\partial}{\partial c_{k,j,i}} \frac{\partial c_{s,j} F_n(c)}{F_n(c)} \right|^2 \geq \sum_{i \in \{1, \ldots, n\}, j_i \neq j} \text{for all } t < i \left( \frac{2}{n} \sum_{t=1}^{n} (f_{c}(X_t) - Y_t) \cdot \frac{\partial}{\partial c_{1,j_i}} f_{c}(X_t) \right)^2
\]
\[
= \sum_{i \in \{1, \ldots, n\}, j_i \neq j} \left( \frac{2}{n} \sum_{t=1}^{n} (f_{c}(X_t) - Y_t) \cdot f_{j_i,j_i}^{(L)}(X_t) \right)^2
\]
\[
\geq \sum_{i \in \{1, \ldots, n\}, j_i \neq j} \left( \frac{1}{2} \cdot \frac{2}{n} \cdot \sum_{t \in \{1, \ldots, n\}, X_t = X_i} (f_{c}(X_t) - Y_t) \cdot f_{j_i,j_i}^{(L)}(X_t) \right)^2 - \frac{2}{n} \cdot \sum_{t \in \{1, \ldots, n\}, X_t \neq X_i} (f_{c}(X_t) - Y_t) \cdot f_{j_i,j_i}^{(L)}(X_t)^2
\]
where the last inequality followed from \(b^2 \leq 2(b - a)^2 + 2a^2\) which implies
\[
a^2 \geq \frac{1}{2} b^2 - (b - a)^2 \quad (a, b \in \mathbb{R}).
\]

Using
\[
\sum_{t \in \{1, \ldots, n\}, X_t = X_i} (f_{c}(X_t) - Y_t) = |\{1 \leq k \leq n : X_k = X_i\}| \cdot (f_{c}(X_i) - \bar{m}_n(X_i)),
\]
\[
\sum_{t \in \{1, \ldots, n\}, X_t \neq X_i} (f_{c}(X_t) - Y_t) \cdot f_{j_i,j_i}^{(L)}(X_t)
\]
for some $x_i \in \{x_i, x_{i-1}, x_{i-2}, \ldots, x_1\}$

\[
\sum_{t \in \{1, \ldots, n\}, i \neq j} \frac{(f_e(x_i) - \bar{m}_e(x_i))^2}{n} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} n \cdot \left| f_e(x_i) - \bar{m}_e(x_i) \right|^2
\]

\[
= \left( \frac{2}{n} \cdot \left( 1 - \frac{2}{n^2} \right)^2 - \frac{16}{n^3} \right) \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} n \cdot \left| f_e(x_i) - \bar{m}_e(x_i) \right|^2
\]

\[
= \left( \frac{2}{n} - \frac{8}{n^3} + \frac{8}{n^3} - \frac{16}{n^3} \right) \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} n \cdot \left| f_e(x_i) - \bar{m}_e(x_i) \right|^2
\]

\[
\geq \frac{1}{n} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} n \cdot \left( f_e(x_i) - \bar{m}_e(x_i) \right)^2
\]

\[
= \frac{1}{n} \cdot \left( \frac{1}{n} \sum_{i=1}^{n} \left| f_e(x_i) - Y_i \right|^2 - \min_{g : \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \left| g(X_i) - Y_i \right|^2 \right).
\]

Lemma 3 Define $c^{(t)}$ by

\[
c^{(t+1)} = c^{(t)} - \lambda_n \cdot (\nabla_c F_n)(c^{(t)}) \quad (t = 0, \ldots, t_n - 1)
\]

for some fixed $c^{(0)}$ and

\[
\lambda_n = \frac{1}{L_n}.
\]
Assume that (11) holds for $F = F_n$ and all $a_1 = c^{(t)}$ and $a_2 = c^{(t+1)}$ and any $t \in \{0,1,\ldots,t_n-1\}$. Furthermore assume that (12) holds for all $c = c^{(t)} (t \in \{0,1,\ldots,t_n-1\})$. Then we have for any $n \geq 5$

$$F_n(c^{(t_n)}) - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2$$

$$\leq \left(1 - \frac{1}{2 \cdot n \cdot L_n}\right)^{t_n} \left( F_n(c^{(0)}) - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right).$$

**Proof.** Application of Lemma 1 and Lemma 2 implies for any $t \in \{0,\ldots,t_n-1\}$

$$F_n(c^{(t+1)}) - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2$$

$$\leq F_n(c^{(t)}) - \frac{1}{2 \cdot L_n} \cdot \|\nabla_c F_n(c^{(t)})\|^2 - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2$$

$$\leq \left(1 - \frac{1}{2 \cdot n \cdot L_n}\right) \cdot \left( F_n(c^{(t)}) - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right).$$

From this we can conclude

$$F_n(c^{(t_n)}) - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2$$

$$\leq \left(1 - \frac{1}{2 \cdot n \cdot L_n}\right) \cdot \left( F_n(c^{(t_n-1)}) - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right)$$

$$\leq \left(1 - \frac{1}{2 \cdot n \cdot L_n}\right)^2 \cdot \left( F_n(c^{(t_n-2)}) - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right)$$

$$\leq \ldots$$

$$\leq \left(1 - \frac{1}{2 \cdot n \cdot L_n}\right)^{t_n} \cdot \left( F_n(c^{(0)}) - \min_{g:R^{d}\rightarrow R} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right).$$

\[\square\]

**Lemma 4** Let $F : R^K \rightarrow R_+$ be differentiable, let $t_n \in \mathbb{N}$, and let $L_n > 0$ be such that

$$\|\nabla F(a)\|_\infty \leq L_n \cdot c_3 \cdot n^{c_4} \quad \text{holds for all } a \text{ with } \|a\|_\infty \leq 2 \cdot c_3 \cdot n^{c_4} \tag{13}$$

and

$$\|\nabla F(a_1) - \nabla F(a_2)\| \leq L_n \cdot \|a_1 - a_2\| \tag{14}$$

holds for all $a_1, a_2$ with $\|a_1\|_\infty \leq 3 \cdot c_3 \cdot n^{c_4}$ and $\|a_2\|_\infty \leq 3 \cdot c_3 \cdot n^{c_4}$. Let $a^{(0)}$ be such that

$$\|a^{(0)}\| \leq c_3 \cdot n^{c_4} \tag{15}$$
and
\[ \sqrt{\frac{2 \cdot t_n}{L_n}} \cdot F(a(0)) \leq c_3 \cdot n^{c_4}, \] (16)
and set
\[ a^{(t+1)} = a^{(t)} - \lambda_n \cdot (\nabla_a F)(a^{(t)}) \quad (t \in \{0, 1, \ldots, t_n - 1\}), \]
where
\[ \lambda_n = \frac{1}{L_n}. \]

Then we have
\[ \|a^{(t)}\|_\infty \leq 2 \cdot c_3 \cdot n^{c_4} \quad (t \in \{0, 1, \ldots, t_n\}). \]

**Proof.** We show
\[ \|a^{(s)}\|_\infty \leq 2 \cdot c_3 \cdot n^{c_4} \quad (s \in \{0, \ldots, t\}) \] (17)
for all \( t \in \{0, 1, \ldots, t_n\} \) by induction.

For \( t = 0 \) the assertion follows from (15). So assume that (17) holds for some \( t \in \{0, 1, \ldots, t_n - 1\} \). Then this together with (13) implies that we have
\[ \|a^{(t+1)}\|_\infty \leq \|a^{(t)}\|_\infty + \frac{1}{L_n} \cdot \|\nabla_a F(a^{(t)})\|_\infty \leq 3 \cdot c_3 \cdot n^{c_4}. \]

From this, the induction hypothesis and Lemma 1 we can conclude
\[ 0 \leq F(a^{(s)}) \leq F(a^{(s-1)}) - \frac{1}{2 \cdot L_n} \cdot \|\nabla_a F(a^{(s-1)})\|^2 \]
for all \( s \in \{0, \ldots, t+1\} \) which implies
\[ 0 \leq F(a^{(t+1)}) \leq F(a^{(0)}) - \sum_{s=1}^{t+1} \frac{1}{2 \cdot L_n} \cdot \|\nabla_a F(a^{(s-1)})\|^2. \]

Consequently we have
\[ \sum_{s=1}^{t+1} \frac{1}{2 \cdot L_n} \cdot \|\nabla_a F(a^{(s-1)})\|^2 \leq F(a^{(0)}), \]
which implies
\[ \|a^{(t+1)}\|_\infty \leq \|a^{(t+1)}\| \]
\[ \leq \|a^{(0)}\| + \sum_{s=1}^{t+1} \frac{1}{L_n} \cdot \|\nabla_a F(a^{(s-1)})\| \]
\[ \leq \|a^{(0)}\| + \sqrt{\frac{t+1}{L_n}} \cdot \sqrt{\sum_{s=1}^{t+1} \frac{1}{L_n} \cdot \|\nabla_a F(a^{(s-1)})\|^2} \]
\[ \leq \|a^{(0)}\| + \sqrt{\frac{t+1}{L_n}} \cdot 2 \cdot F(a^{(0)}) \]
\[ \leq 2 \cdot c_3 \cdot n^{c_4}, \]
where the last inequality followed from (15) and (16). \( \square \)
**Lemma 5** Let $\sigma$ be the logistic squasher. Let $k_n \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ with $2 \cdot k_0 \geq d$. Let $c = (c_{k,i,j}^{(s)})_{k,i,j,s}$ and $\tilde{c} = (\tilde{c}_{k,i,j}^{(s)})_{k,i,j,s}$ be weight vectors and define $f_c$ and $\tilde{f}_c$ by

$$f_c(x) = \sum_{i=1}^{k_n} c_{1,i,1}^{(L)} \cdot f_{k,i}^{(L)}(x) + c_{1,1,0}^{(L)}$$

and

$$f_c(x) = \sum_{i=1}^{k_n} c_{1,i,1}^{(L)} \cdot \tilde{f}_{k,i}^{(L)}(x) + \tilde{c}_{1,1,0}^{(L)}$$

(18)

for $f_{k,i}^{(L)}$’s and $\tilde{f}_{k,i}^{(L)}$’s recursively defined by

$$f^{(r)}_{k,i}(x) = \sigma \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f^{(r-1)}_{k,j}(x) + c_{k,i,0}^{(r-1)} \right)$$

(19)

($r = 2, \ldots, L$) and

$$\tilde{f}^{(r)}_{k,i}(x) = \sigma \left( \sum_{j=1}^{k_0} \tilde{c}_{k,i,j}^{(r-1)} \cdot \tilde{f}^{(r-1)}_{k,j}(x) + \tilde{c}_{k,i,0}^{(r-1)} \right)$$

(20)

($r = 2, \ldots, L$) and

$$f^{(1)}_{k,i}(x) = \sigma \left( \sum_{j=1}^{d} c_{k,i,j}^{(0)} \cdot x^{(j)} + c_{k,i,0}^{(0)} \right)$$

and

$$\tilde{f}^{(1)}_{k,i}(x) = \sigma \left( \sum_{j=1}^{d} \tilde{c}_{k,i,j}^{(0)} \cdot x^{(j)} + \tilde{c}_{k,i,0}^{(0)} \right).$$

(21)

**a)** For any $k \in \{1, \ldots, k_n\}$ and any $x \in \mathbb{R}^d$ we have

$$|f^{(L)}_{k,k}(x) - \tilde{f}^{(L)}_{k,k}(x)| \leq (2 \cdot k_0 + 1)^L \cdot (\max\{\|c\|_{\infty}, \|x\|_{\infty}, 1\})^L \cdot \max_{i,j,s,s' \leq L} |c_{k,i,j}^{(s)} - \tilde{c}_{k,i,j}^{(s')}|.$$

**b)** For any $x \in \mathbb{R}^d$ we have

$$|f_c(x) - f_{\tilde{c}}(x)| \leq (2 \cdot k_0 + 1) \cdot (2 \cdot k_0 + 1)^L \cdot (\max\{\|c\|_{\infty}, \|x\|_{\infty}, 1\})^{L+1} \cdot \|c - \tilde{c}\|_{\infty}.$$

**Proof.** a) We show by induction

$$|f^{(l)}_{r,k}(x) - \tilde{f}^{(l)}_{r,k}(x)| \leq (2 \cdot k_0 + 1)^l \cdot (\max\{\|c\|_{\infty}, \|x\|_{\infty}, 1\})^l \cdot \max_{i,j,s,s' \leq L} |c_{k,i,j}^{(s)} - \tilde{c}_{k,i,j}^{(s')}|$$

(22)

($l \in \{1, \ldots, L\}$). The logistic squasher satisfies $|\sigma'(x)| = |\sigma(x) \cdot (1 - \sigma(x))| \leq 1$, hence it is Lipschitz continuous with Lipschitz constant one. This implies

$$|f^{(1)}_{k,i}(x) - \tilde{f}^{(1)}_{k,i}(x)| \leq \sum_{j=1}^{d} |c_{k,i,j}^{(0)} - \tilde{c}_{k,i,j}^{(0)}| \cdot |x^{(j)}| + |c_{k,i,0}^{(0)} - \tilde{c}_{k,i,0}^{(0)}|$$

$$\leq (2 \cdot k_0 + 1) \cdot \max\{\|x\|_{\infty}, 1\} \cdot \max_{i,j,s,s' \leq L} |c_{k,i,j}^{(s)} - \tilde{c}_{k,i,j}^{(s')}|.$$

Assume now that (22) holds for some $r - 1$, where $r \in \{2, \ldots, L\}$. Then

$$|f^{(r)}_{k,i}(x) - \tilde{f}^{(r)}_{k,i}(x)|$$
Lemma 6 Let $\sigma$ be the logistic squasher. Define $f_e$ by (6)-(8) and set

$$F_n(c) = \frac{1}{n} \sum_{i=1}^{n} |f_e(X_i) - Y_i|^2.$$  

Let $c_3, c_4 \geq 1$. Assume $\|c_1\|_{\infty} \leq c_3 \cdot n^{c_4}$, $\|c_2\|_{\infty} \leq c_3 \cdot n^{c_4}$ and

$$\max_{i=1,\ldots,n} |X_i|_{\infty} \leq c_3 \cdot n^{c_4} \quad \text{and} \quad \max_{i=1,\ldots,n} |Y_i| \leq c_3 \cdot n^{c_4}.$$  

Set

$$L_n = 45 \cdot L \cdot 3^{L} \cdot \left(\max\{k_0, L, d\}\right)^{3/2} \cdot k_0^{2L} \cdot k_n^{3/2} \cdot (c_3 \cdot n^{c_4})^{4L+1}.$$  

Then we have

$$\|\nabla_c F_n(c_1)\|_{\infty} \leq L_n \cdot c_3 \cdot n^{c_4} \quad (23)$$

and

$$\|\nabla_c F_n(c_1) - (\nabla_c F_n(c_2))\| \leq L_n \cdot \|c_1 - c_2\|. \quad (24)$$

Proof. In the first step of the proof we compute the partial derivatives of $F_n(c)$. We have

$$\frac{\partial}{\partial c_{k,i,j}^{(r)}} F_n(c) = \frac{2}{n} \sum_{i=1}^{n} \left(f_e(X_i) - Y_i\right) \cdot \frac{\partial f_e}{\partial c_{k,i,j}^{(r)}}(X_i).$$
The recursive definition of $f_c$ together with the chain rule imply

$$\frac{\partial f_c}{\partial c_{1,1,i}}(X_i) = f_{1,i}^{(L)}(X_i)$$

(where we have set $f_{0,0}^{(L)}(x) = 1$) and in case $\bar{r} < L$

$$\frac{\partial f_c}{\partial c_{k,i,j}}^{(r)}(X_i) = \sum_{i=1}^{k_0} c_{1,1,i}^{(r)} \cdot \frac{\partial f_{1,i}^{(L)}}{\partial c_{k,i,j}^{(r)}}(X_i) = c_{1,1,k}^{(L)} \cdot \frac{\partial f_{k,k}^{(L)}}{\partial c_{k,i,j}^{(r)}}(X_i).$$

In case $0 \leq \bar{r} < r$ and $r > 1$ we have

$$\frac{\partial f_{k,i,j}^{(r)}}{\partial c_{k,i,j}^{(r)}}(X_i) = \sigma' \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f_{k,j}^{(r-1)}(X_i) + c_{k,i,0}^{(r-1)} \right) \cdot \frac{\partial}{\partial c_{k,i,j}^{(r)}} \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f_{k,j}^{(r-1)}(X_i) + c_{k,i,0}^{(r-1)} \right) \cdot \left( 1 - \sigma \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f_{k,j}^{(r-1)}(X_i) + c_{k,i,0}^{(r-1)} \right) \right) \cdot \frac{\partial}{\partial c_{k,i,j}^{(r)}} \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f_{k,j}^{(r-1)}(X_i) + c_{k,i,0}^{(r-1)} \right),$$

Next we explain how we can compute

$$\frac{\partial}{\partial c_{k,i,j}^{(r-1)}} \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f_{k,j}^{(r-1)}(X_i) + c_{k,i,0}^{(r-1)} \right).$$

In case $\bar{r} = r - 1 > 0$ we have

$$\frac{\partial}{\partial c_{k,i,j}^{(r-1)}} \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f_{k,j}^{(r-1)}(X_i) + c_{k,i,0}^{(r-1)} \right) = f_{k,j}^{(r-1)}(X_i) \cdot 1_{\{i = i\}}$$

(where we have set $f_{k,0}^{(r-1)}(x) = 1$), and in case $\bar{r} < r - 1$ we get

$$\frac{\partial}{\partial c_{k,i,j}^{(r)}} \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot f_{k,j}^{(r-1)}(X_i) + c_{k,i,0}^{(r-1)} \right) = \sum_{j=1}^{k_0} c_{k,i,j}^{(r-1)} \cdot \frac{\partial}{\partial c_{k,i,j}^{(r)}} f_{k,j}^{(r-1)}(X_i).$$
And in case \( r = 2 \) and \( \bar{r} = 0 \) we have

\[
\frac{\partial f_{k,i}^{(1)}}{\partial c_{k,i,j}^{(0)}}(X_l)
\]

\[
= \sigma' \left( \sum_{j=1}^{d} c_{k,i,j}^{(0)} \cdot X_l^{(j)} + c_{k,i,0}^{(0)} \right) \cdot X_l^{(j)} \cdot 1_{i=i}
\]

\[
= \sigma \left( \sum_{j=1}^{d} c_{k,i,j}^{(0)} \cdot X_l^{(j)} + c_{k,i,0}^{(0)} \right) \cdot \left( 1 - \sigma \left( \sum_{j=1}^{d} c_{k,i,j}^{(0)} \cdot X_l^{(j)} + c_{k,i,0}^{(0)} \right) \right) \cdot X_l^{(j)} \cdot 1_{i=i},
\]

where we have set \( X_l^{(0)} = 1 \).

In the second step of the proof we show for \( x \in \mathbb{R}^d \) with \( \|x\|_\infty \leq c_3 \cdot n^{c_4} \) and \( c, c_1, c_2 \) with \( \|c\|_\infty \leq c_3 \cdot n^{c_4} \), \( \|c_1\|_\infty \leq c_3 \cdot n^{c_4} \) and \( \|c_2\|_\infty \leq c_3 \cdot n^{c_4} \),

\[
\left| \frac{\partial f_c(x)}{\partial c_{k,i,j}^{(r)}} \right| \leq k_0^r \cdot (c_3 \cdot n^{c_4})^{L+1}
\]

and

\[
\left| \frac{\partial f_{c_1}(x)}{\partial c_{k,i,j}^{(r)}} - \frac{\partial f_{c_2}(x)}{\partial c_{k,i,j}^{(r)}} \right| \leq \tilde{L}_n \cdot \|c_1 - c_2\|_\infty,
\]

where

\[
\tilde{L}_n = 4L \cdot 3^L \cdot k_0^{2L-2} \cdot (c_3 \cdot n^{c_4})^{4L}.
\]

It is easy to see that the first step of the proof implies

\[
\frac{\partial f_c(x)}{\partial c_{k,i,j}^{(r)}} = \sum_{s_{r+1}=1}^{k_0} \cdots \sum_{s_{L-2}=1}^{k_0} f_{k,j}^{(r)}(x) \cdot f_{k,j}^{(r+1)}(x) \cdot (1 - f_{k,j}^{(r+1)}(x))
\]

\[
\cdot c_{k,s_{r+1},i}^{(r+2)} \cdot f_{k,s_{r+2}}^{(r+2)}(x) \cdot (1 - f_{k,s_{r+2}}^{(r+2)}(x)) \cdot c_{k,s_{r+1},s_{r+2},i}^{(r+3)} \cdot f_{k,s_{r+2}}^{(r+3)}(x) \cdot (1 - f_{k,s_{r+2}}^{(r+3)}(x))
\]

\[
\cdots c_{k,s_{L-2},s_{L-1} \cdot s_{L-3}}^{(L-2)} \cdot f_{k,s_{L-2}}^{(L-1)}(x) \cdot (1 - f_{k,s_{L-2}}^{(L-1)}(x)) \cdot c_{k,s_{L-2},s_{L-1},s_{L-3}}^{(L-1)} \cdot f_{k,s_{L-2}}^{(L)}(x) \cdot (1 - f_{k,s_{L-2}}^{(L)}(x))
\]

\[
\cdot c_{1,1,k}^{(L)}
\]

where we have used the abbreviations

\[
f_{k,j}^{(0)}(x) = \begin{cases} x^{(j)} & \text{if } j \in \{1, \ldots, d\} \\ 1 & \text{if } j = 0 \end{cases}
\]

and

\[
f_{k,0}^{(r)}(x) = 1.
\]

Because of

\[
f_{k,i}^{(r)}(x) \in [0,1] \text{ if } r > 0
\]
and

\[ |f_{k,i}^{(0)}(x)| \leq c_3 \cdot n^{c_4} \]

and

\[ \|c\|_\infty \leq c_3 \cdot n^{c_4} \]

this implies (25).

Next we prove (26). The right-hand side of (27) is a sum of at most \( k_0^{L-2} \) products, where each product contains at most \( 3L + 1 \) factors. In the worst case from these \( 3L + 1 \) factors \( L \) are Lipschitz continuous functions with Lipschitz constant bounded by one, which are bounded in absolute value by \( c_3 \cdot n^{c_4} \). And according to the proof of Lemma 5 (cf., (22)) the remaining \( 2L + 1 \) factors are Lipschitz continuous functions with Lipschitz constant bounded by

\[ (2k_0 + 1)^L \cdot (c_3 \cdot n^{c_4})^L, \]

which are bounded in absolute value by \( c_3 \cdot n^{c_4} \).

If \( g_1, \ldots, g_s : \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous functions with Lipschitz constants \( L_{g_1}, \ldots, L_{g_s} \), then

\[ \prod_{l=1}^{s} g_l \quad \text{and} \quad \sum_{l=1}^{s} g_l \]

are Lipschitz continuous functions with Lipschitz constant bounded by

\[ \sum_{l=1}^{s} L_{g_l} \cdot \prod_{k \in \{1, \ldots, s\} \setminus \{l\}} \|g_k\|_\infty \leq s \cdot \max_l L_{g_l} \cdot (\max_k \|g_k\|_\infty)^{s-1} \]

and by

\[ \sum_{l=1}^{s} L_{g_l} \leq s \cdot \max_l L_{g_l}, \]

respectively. This implies that (27) is Lipschitz continuous with Lipschitz constant bounded by

\[ k_0^{L-2} \cdot (3L + 1) \cdot (2k_0 + 1)^L \cdot (c_3 n^{c_4})^L \cdot (c_3 n^{c_4})^{3L}. \]

In the third step of the proof we show (23). We have

\[ \| (\nabla c F_n)(c) \|_\infty = \max_{k,i,j,r} \left| \frac{2}{n} \sum_{l=1}^{n} \left( f_{c}(X_l) - Y_l \right) \cdot \frac{\partial f_{c}}{\partial c_{k,i,j}}(X_l) \right| \]

\[ \leq 2 \cdot \left( (k_n + 1) \cdot \|c\|_\infty + \max_{i=1,\ldots,n} |Y_i| \right) \cdot \max_{l,k,i,j,r} \left| \frac{\partial f_{c}}{\partial c_{k,i,j}}(X_l) \right| \]

\[ \leq 6 \cdot k_n \cdot c_3 n^{c_4} \cdot \max_{l,k,i,j,r} \left| \frac{\partial f_{c}}{\partial c_{k,i,j}}(X_l) \right|. \]

From this the result follows by (25).
In the fourth step of the proof we show (24). Because of

\[ \| (\nabla_c F_n)(c_1) - (\nabla_c F_n)(c_2) \| = \left( \sum_{k,i,j,r} \left| \frac{\partial F_n}{\partial c_{k,i,j}^{(r)}}(c_1) - \frac{\partial F_n}{\partial c_{k,i,j}^{(r)}}(c_2) \right|^2 \right)^{1/2} \]

and

\[ \frac{\partial F_n}{\partial c_{k,i,j}^{(r)}}(c) = \frac{2}{n} \sum_{i=1}^{n} (f_c(X_i) - Y_i) \cdot \frac{\partial f_c}{\partial c_{k,i,j}^{(r)}}(X_i) \]

we have

\[ \| (\nabla_c F_n)(c_1) - (\nabla_c F_n)(c_2) \| \leq \sqrt{k_n \cdot (k_0 + 1 + (L - 2) \cdot (k_0^2 + k_0) + k_0 \cdot (d + 1) + k_n + 1} \cdot 2 \cdot \max_{k,i,j,r,l} \left| (f_c(X_i) - Y_i) \cdot \frac{\partial f_c}{\partial c_{k,i,j}^{(r)}}(X_i) - (f_c(X_i) - Y_i) \cdot \frac{\partial f_c}{\partial c_{k,i,j}^{(r)}}(X_i) \right|. \] (28)

By Lemma 5 we know

\[ |f_c(X_i) - f_c(X_j)| \leq (2k_n + 1) \cdot (2k_0 + 1)^L \cdot (c_3n^{c_4})^{L+1} \cdot \| c_1 - c_2 \|_{\infty}. \] (29)

Trivially,

\[ |f_c(X_i) - Y_i| \leq (k_n + 1) \cdot c_3n^{c_4} + c_3n^{c_4} = (k_n + 2) \cdot c_3n^{c_4}. \] (30)

If \( g_i \) are Lipschitz continuous functions with Lipschitz constants \( L_{g_i} \), then \( g_1 \cdot g_2 \) is Lipschitz continuous with Lipschitz constant

\[ \| g_1 \|_{\infty} \cdot L_{g_2} + \| g_2 \|_{\infty} \cdot L_{g_1}. \]

Combining this with (25), (26), (29) and (30) we get that

\[ c \mapsto (f_c(X_i) - Y_i) \cdot \frac{\partial f_c}{\partial c_{k,i,j}^{(r)}}(X_i) \]

is Lipschitz continuous with Lipschitz constant bounded by

\[ (k_n + 2) \cdot c_3n^{c_4} \cdot 4L \cdot 3L \cdot 2L^{-2} \cdot (c_3 \cdot n^{c_4})^{4L} \]

\[ + k_0^L \cdot (c_3n^{c_4})^{L+1} \cdot (2k_n + 1) \cdot (2k_0 + 1)^L \cdot (c_3n^{c_4})^{L+1} \]

\[ \leq 15 \cdot k_n \cdot L \cdot 3L \cdot 2L \cdot (c_3n^{c_4})^{4L+1}. \]

This together with (28) implies the assertion. \( \square \)

**Lemma 7** Let \( \sigma \) be the logistic squasher and let \( n, d, k_0, L \in \mathbb{N} \) with \( k_0 \geq 2 \cdot d \) and \( L \geq 2 \).

Define \( f_{1,1}^{(L)} : \mathbb{R} \to \mathbb{R} \) recursively by

\[ f_{1,k}^{(r)}(x) = \sigma \left( \sum_{j=1}^{k_0} c^{(r-1)}_{1,k,j} \cdot f_{1,j}^{(r-1)}(x) + c^{(r-1)}_{1,k,0} \right) \]

18
for some \( c_{1,k,0}^{(r-1)}, \ldots, c_{1,k,k_0}^{(r-1)} \in \mathbb{R} \) (\( r = 2, \ldots, L \)) and
\[
 f_{1,k}^{(1)}(x) = \sigma \left( \sum_{j=1}^{d} c_{1,k,j}^{(0)} \cdot x^{(j)} + c_{1,k,0}^{(0)} \right)
\]
for some \( c_{1,k,0}^{(0)}, \ldots, c_{1,k,d}^{(0)} \in \mathbb{R} \). Let \( \delta > 0 \) and let \( a, b \in \mathbb{R}^d \) such that
\[
b^{(l)} - a^{(l)} \geq 2 \cdot \delta \quad \text{for all } l \in \{1, \ldots, d\}.
\]
Assume
\[
c_{1,1,1}^{(L-1)} \leq -4 \cdot (n + 1), \quad (31)
\]
\[
|c_{1,1,j}^{(L-1)} - c_{1,1,1}^{(L-1)}| \leq \frac{1}{2k_0} \quad \text{for } j = 2, \ldots, d, \quad (32)
\]
\[
|c_{1,1,j}^{(L-1)}| \leq \frac{1}{2k_0} \quad \text{for } j = 2d + 1, \ldots, k_0, \quad (33)
\]
\[
|c_{1,k,0}^{(L-1)} + \frac{1}{2} \cdot c_{1,1,1}^{(L-1)}| \leq \frac{1}{2} \quad \text{for } k \in \{1, \ldots, 2d\}, \quad (34)
\]
\[
c_{1,k,k}^{(r-1)} \geq 8 \cdot \log(8d - 1) \quad \text{for } k \in \{1, \ldots, 2d\} \text{ and } r \in \{2, \ldots, L-1\}, \quad (35)
\]
\[
|c_{1,k,0}^{(r-1)} + \frac{1}{2} \cdot c_{1,k,k}^{(r-1)}| \leq \frac{\log(8d - 1)}{k_0} \quad \text{for } k \in \{1, \ldots, 2d\} \text{ and } r \in \{2, \ldots, L-1\}, \quad (36)
\]
\[
|c_{1,k,j}^{(r-1)}| \leq \frac{\log(8d - 1)}{k_0} \quad \text{for } j \in \{1, \ldots, k_0\} \setminus \{k\}, k \in \{1, \ldots, 2d\}, r \in \{2, \ldots, L-1\}, \quad (37)
\]
\[
c_{1,k,k}^{(0)} \leq -\frac{2}{\delta} \cdot \log(8d - 1) \quad \text{for } k \in \{1, \ldots, d\}, \quad (38)
\]
\[
|c_{1,k,0}^{(0)} + a^{(k)} \cdot c_{1,k,k}^{(0)}| \leq \frac{\log(8d - 1)}{d} \quad \text{for } k \in \{1, \ldots, d\}, \quad (39)
\]
\[
|c_{1,k,j}^{(0)}| \leq \frac{\log(8d - 1)}{d} \quad \text{for } k \in \{1, \ldots, d\}, j \in \{1, \ldots, d\} \setminus \{k\} \quad (40)
\]
\[
c_{1,d+k,k}^{(0)} \geq \frac{2}{\delta} \cdot \log(8d - 1) \quad \text{for } k \in \{1, \ldots, d\}, \quad (41)
\]
\[
|c_{1,d+k,0}^{(0)} + b^{(k)} \cdot c_{1,d+k,k}^{(0)}| \leq \frac{\log(8d - 1)}{d} \quad \text{for } k \in \{1, \ldots, d\} \quad (42)
\]
and
\[
|c_{1,d+k,j}^{(0)}| \leq \frac{\log(8d - 1)}{d} \quad \text{for } k \in \{1, \ldots, d\}, j \in \{1, \ldots, d\} \setminus \{k\}. \quad (43)
\]
Then \( f_{1,1}^{(L)} \) satisfies for any \( x \in [-1,1]^d \)
\[
f_{1,1}^{(L)}(x) \geq 1 - e^{-n} \quad \text{if } x \in [a^{(1)} + \delta, b^{(1)} - \delta] \times \cdots \times [a^{(d)} + \delta, b^{(d)} - \delta] \quad (44)
\]
and
\[
f_{1,1}^{(L)}(x) \leq e^{-n} \quad \text{if } x \notin [a^{(1)} - \delta, b^{(1)} + \delta] \times \cdots \times [a^{(d)} - \delta, b^{(d)} + \delta]. \quad (45)
\]
Proof. Let \( x \in [a^{(1)} + \delta, b^{(1)} - \delta] \times \cdots \times [a^{(d)} + \delta, b^{(d)} - \delta] \cap [-1, 1]^d \). Then we get for any \( k \in \{1, \ldots, d\} \) by (38), (39) and (40)

\[
\sum_{j=1}^d c_{1,k,j}^{(0)} \cdot x^{(j)} + c_{1,k,0}^{(0)} = c_{1,k,k}^{(0)} \cdot (x^{(k)} - a^{(k)}) + c_{1,k,0}^{(0)} + c_{1,k,k}^{(0)} \cdot a^{(k)} + \sum_{j \in \{1, \ldots, d\} \setminus \{k\}} c_{1,k,j}^{(0)} \cdot x^{(j)} \\
\leq -2 \cdot \log(8d - 1) + |c_{1,k,0}^{(0)} + c_{1,k,k}^{(0)} \cdot a^{(k)}| + \sum_{j \in \{1, \ldots, d\} \setminus \{k\}} |c_{1,k,j}^{(0)}| \\
\leq -\log(8d - 1).
\]

And by (41), (42) and (43) we get for any \( k \in \{1, \ldots, d\} \)

\[
\sum_{j=1}^d c_{1,d+k,j}^{(0)} \cdot x^{(j)} + c_{1,d+k,0}^{(0)} = -c_{1,d+k,k}^{(0)} \cdot (b^{(k)} - x^{(k)}) + c_{1,d+k,0}^{(0)} + c_{1,d+k,k}^{(0)} \cdot b^{(k)} + \sum_{j \in \{1, \ldots, d\} \setminus \{k\}} c_{1,d+k,j}^{(0)} \cdot x^{(j)} \\
\leq -2 \cdot \log(8d - 1) + |c_{1,d+k,0}^{(0)} + c_{1,d+k,k}^{(0)} \cdot b^{(k)}| + \sum_{j \in \{1, \ldots, d\} \setminus \{k\}} |c_{1,d+k,j}^{(0)}| \\
\leq -\log(8d - 1).
\]

It is easy to see that the logistic squasher satisfies

\[
\sigma(x) \geq 1 - \kappa \text{ if } x \geq \log \left( \frac{1}{\kappa} - 1 \right) \quad \text{and} \quad \sigma(x) \leq \kappa \text{ if } x \leq -\log \left( \frac{1}{\kappa} - 1 \right). \tag{46}
\]

Using this we get for any \( k \in \{1, \ldots, 2d\} \)

\[
f_{1,k}^{(1)}(x) \leq \sigma(- \log(8d - 1)) = \sigma \left( -\log \left( \frac{1}{1/(8d)} - 1 \right) \right) \leq \frac{1}{8d} \leq \frac{1}{4}.
\]

Using (35), (36) and (37), we can recursively conclude for \( r = 2, \ldots, L - 1 \) that we have for any \( k \in \{1, \ldots, 2d\} \)

\[
\sum_{j=1}^{k_0} c_{1,k,j}^{(r-1)} \cdot f_{1,j}^{(r-1)}(x) + c_{1,k,0}^{(r-1)} \\
= c_{1,k,k}^{(r-1)} \cdot f_{1,k}^{(r-1)}(x) - \frac{1}{2} + c_{1,k,k}^{(r-1)} + \frac{1}{2} + c_{1,k,0}^{(r-1)} + \sum_{j \in \{1, \ldots, k_0\} \setminus \{k\}} c_{1,k,j}^{(r-1)} \cdot f_{1,j}^{(r-1)}(x) \\
\leq -2 \cdot \log(8d - 1) + |c_{1,k,k}^{(r-1)} + \frac{1}{2} + c_{1,k,0}^{(r-1)}| + \sum_{j \in \{1, \ldots, k_0\} \setminus \{k\}} |c_{1,k,j}^{(r-1)}| \\
\leq -\log(8d - 1)
\]

20
and
\[ f^{(r)}_{1,k}(x) \leq \sigma(-\log(8d-1)) \leq \frac{1}{8d} \leq \frac{1}{4}. \]

From this together with (31), (32), (33) and (34) we conclude
\[
\sum_{j=1}^{k_0} c_{1,1,j}^{(L-1)} \cdot f^{(L-1)}_{1,j}(x) + c_{1,1,0}^{(L-1)}
= c_{1,1,1}^{(L-1)} \cdot \left( \sum_{j=1}^{2d} f^{(L-1)}_{1,j}(x) - \frac{1}{2} \right) + c_{1,1,0}^{(L-1)} + \frac{1}{2} \cdot c_{1,1,1}^{(L-1)}
+ \sum_{j=1}^{2d} \left( c_{1,1,j}^{(L-1)} - c_{1,1,1}^{(L-1)} \right) \cdot f^{(L-1)}_{1,j}(x) + \sum_{j=2d+1}^{k_0} c_{1,1,j}^{(L-1)} \cdot f^{(L-1)}_{1,j}(x)
\geq c_{1,1,1}^{(L-1)} \cdot \left( \sum_{j=1}^{2d} f^{(L-1)}_{1,j}(x) - \frac{1}{2} \right) - |c_{1,k,0}^{(L-1)}| + \frac{1}{2} \cdot c_{1,1,1}^{(L-1)}
- 2d \cdot |c_{1,1,j}^{(L-1)} - c_{1,1,1}^{(L-1)}| - \sum_{j=2d+1}^{k_0} |c_{1,1,j}^{(L-1)}|
\geq -4 \cdot (n+1) \cdot (2d \cdot \frac{1}{8d} - \frac{1}{2}) - \frac{1}{2} - \sum_{j=1}^{2d} \frac{1}{2k_0} - \sum_{j=2d+1}^{k_0} \frac{1}{2k_0}
\geq n \geq \log(1/e^{-n} - 1),
\]
which implies (44).

In order to prove (45) we assume that \( x \in [-1, 1]^d \) satisfies \( x^{(k)} \notin [a^{(k)} - \delta, b^{(k)} + \delta] \) for some \( k \in \{1, \ldots, d\} \). In case \( x^{(k)} < a^{(k)} - \delta \) we can argue similarly as above and conclude recursively from (46) and (31)-(43)
\[
\sum_{j=1}^{d} c_{1,k,j}^{(0)} \cdot x^{(j)} + c_{1,k,0}^{(0)}
= c_{1,k,k}^{(0)} \cdot (x^{(k)} - a^{(k)}) + c_{1,k,0}^{(0)} + c_{1,k,k}^{(0)} \cdot a^{(k)} + \sum_{j \in \{1, \ldots, d\} \setminus \{k\}} c_{1,k,j}^{(0)} \cdot x^{(j)}
\geq 2 \cdot \log(8d-1) - |c_{1,k,0}^{(0)} + c_{1,k,k}^{(0)} \cdot a^{(k)}| - \sum_{j \in \{1, \ldots, d\} \setminus \{k\}} |c_{1,k,j}^{(0)}|
\geq \log(8d-1),
\]
which implies
\[ f^{(1)}_{1,k}(x) \geq \sigma(\log(8d-1)) = \sigma(\log(1/(1/(8d))) - 1)) \geq 1 - \frac{1}{8d} \geq \frac{3}{4}. \]

Recursively we can conclude for \( r = 2, \ldots, L-1 \)
\[
\sum_{j=1}^{k_0} c_{1,k,j}^{(r-1)} \cdot f^{(r-1)}_{1,j}(x) + c_{1,k,0}^{(r-1)}
\]
\[ \begin{align*}
&= c_{1,k,k}^{(r-1)} \cdot \left( f_{1,k}^{(r-1)}(x) - \frac{1}{2} \right) + c_{1,k,0}^{(r-1)} \cdot \frac{1}{2} + c_{1,k,0}^{(r-1)} + \sum_{j \in \{1, \ldots, k_0\} \setminus \{k\}} c_{1,k,j}^{(r-1)} \cdot f_{1,j}^{(r-1)}(x) \\
&\geq 2 \cdot \log(8d - 1) - |c_{1,k,k}^{(r-1)} \cdot \frac{1}{2} + c_{1,k,0}^{(r-1)}| - \sum_{j \in \{1, \ldots, k_0\} \setminus \{k\}} |c_{1,k,j}^{(r-1)}| \\
&\geq \log(8d - 1)
\end{align*} \]

and
\[ f_{1,k}^{(r)}(x) \geq \sigma(\log(8d - 1)) \geq 1 - \frac{1}{8d} \geq \frac{3}{4}. \]

This yields
\[ \sum_{j=1}^{k_0} c_{1,1,j}^{(L-1)} \cdot f_{1,j}^{(L-1)}(x) + c_{1,1,0}^{(L-1)} \]
\[= c_{1,1,1}^{(L-1)} \cdot \left( \sum_{j=1}^{2d} f_{1,j}^{(L-1)}(x) - \frac{1}{2} \right) + c_{1,1,0}^{(L-1)} + \frac{1}{2} \cdot c_{1,1,1}^{(L-1)} \\
+ \sum_{j=1}^{2d} (c_{1,1,j}^{(L-1)} - c_{1,1,1}^{(L-1)}) \cdot f_{1,j}^{(L-1)}(x) + \sum_{j=1}^{k_0} c_{1,1,j}^{(L-1)} \cdot f_{1,j}^{(L-1)}(x) \\
\leq c_{1,1,1}^{(L-1)} \cdot \left( \sum_{j=1}^{2d} f_{1,j}^{(L-1)}(x) - \frac{1}{2} \right) + |c_{1,1,0}^{(L-1)} + \frac{1}{2} \cdot c_{1,1,1}^{(L-1)}| \\
+ \sum_{j=1}^{2d} |c_{1,1,j}^{(L-1)} - c_{1,1,1}^{(L-1)}| + \sum_{j=2d+1}^{k_0} |c_{1,1,j}^{(L-1)}| \\
\leq -4 \cdot (n + 1) \left( \frac{3}{4} - \frac{1}{2} \right) + \frac{1}{2} + \sum_{j=1}^{2d} \frac{1}{2k_0} + \sum_{j=2d+1}^{k_0} \frac{1}{2k_0} \\
\leq -n \leq -\log(1/e^{-n} - 1),
\]

which implies (45).

In the same way we get the assertion in case \( x^{(k)} > b^{(k)} + \delta \). \( \square \)

**Remark 3.** It is easy to see that the number of weights of the neural network \( f_{1,1}^{(L)} \) is given by
\[ (L - 2) \cdot (k_0^2 + k_0) + k_0 \cdot (d + 2) + 1. \]

**Lemma 8** Let \( \sigma \) be the logistic squasher. Let \( f_e \) be defined by (6)-(8), let \( k \in \{1, \ldots, k_n\} \) and assume that
\[ \max_{t \in \{1, \ldots, n\}} f_{k,k}^{(L)}(X_t) \cdot (1 - f_{k,k}^{(L)}(X_t)) \leq e^{-n} \quad (47) \]
holds. Assume furthermore \( ||c||_\infty \leq c_3 \cdot n^{c_4} \) and
\[ \max_{j=1,\ldots,n} |X_j|_\infty \leq c_3 \cdot n^{c_4} \quad \text{and} \quad \max_{j=1,\ldots,n} |Y_j| \leq c_3 \cdot n^{c_4}. \quad (48) \]
Set

\[ F_n(c) = \frac{1}{n} \sum_{i=1}^{n} |f_c(X_i) - Y_i|^2. \]

Then we have for all \( r < L \) and all \( i, j \)

\[ \left| \frac{\partial F_n}{\partial c_{k,i,j}^{(r)}}(c) \right| \leq 2 \cdot \sqrt{F_n(c)} \cdot k_0^L \cdot (c_3 \cdot n^{c_4})^{L+1} \cdot e^{-n} \]

**Proof.** By the Cauchy-Schwarz inequality we get

\[ \left| \frac{\partial}{\partial c_{k,i,j}^{(r)}} F_n(c) \right| = \left| \frac{2}{n} \sum_{l=1}^{n} (f_c(X_l) - Y_l) \cdot \frac{\partial f_c}{\partial c_{k,i,j}^{(r)}}(X_l) \right| \]
\[ \leq 2 \cdot \sqrt{F_n(c)} \cdot \max_{l=1, \ldots, n} \left| \frac{\partial f_c}{\partial c_{k,i,j}^{(r)}}(X_l) \right|. \]

Using the recursive definition of \( f_c \) together with (47), \( r < L \) and \( \sigma'(x) = \sigma(x) \cdot (1 - \sigma(x)) \) we get

\[ \left| \frac{\partial f_c}{\partial c_{k,i,j}^{(r)}}(X_l) \right| \]
\[ = \left| \sum_{i=1}^{k_n} c_{1,1,i}^{(L)} \cdot \frac{\partial f_{L}^{(i)}}{\partial c_{k,i,j}^{(r)}}(X_l) \right| \]
\[ = |c_{1,1,k}^{(L)}| \cdot \left| \frac{\partial f_{k,k}^{(L)}}{\partial c_{k,i,j}^{(r)}}(X_l) \right| \]
\[ = |c_{1,1,k}^{(L)}| \cdot f_{k,k}^{(L)}(X_l) \cdot (1 - f_{k,k}^{(L)}(X_l)) \cdot \left| \frac{\partial}{\partial c_{k,i,j}^{(r)}} \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(L-1)} \cdot f_{k,j}^{(L-1)}(X_l) + c_{k,i,0}^{(L-1)} \right) \right| \]
\[ \leq |c_{1,1,k}^{(L)}| \cdot e^{-n} \cdot \left| \frac{\partial}{\partial c_{k,i,j}^{(r)}} \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(L-1)} \cdot f_{k,j}^{(L-1)}(X_l) + c_{k,i,0}^{(L-1)} \right) \right|. \]

As in the proof of Lemma 6 (cf., proof of (25)) it is possible to show

\[ |c_{1,1,k}^{(L)}| \cdot \left| \frac{\partial}{\partial c_{k,i,j}^{(r)}} \left( \sum_{j=1}^{k_0} c_{k,i,j}^{(L-1)} \cdot f_{k,j}^{(L-1)}(X_l) + c_{k,i,0}^{(L-1)} \right) \right| \leq k_0^L \cdot (c_3 \cdot n^{c_4})^{L+1}, \]

which implies the assertion. \( \square \)

**Proof of Theorem 1.** The proof is divided into six steps.
In the first step of the proof we show that for every \( t \in \{1, \ldots, n\} \) there exist (random) 

\[
(c^{(r)}_{1,i,j}),_{i,j,r,r<L} \in \left[-n^4, n^4\right]^{(L-2)(k_0^2+k_0)+(d+2)+1}
\]
such that for any \((c^{(r)}_{1,i,j}),_{i,j,r,r<L}\) with 

\[
\max_{i,j,r,r<L} \left| c^{(r)}_{1,i,j} - \tilde{c}^{(r)}_{1,i,j}\right| < \min\left\{ 2(n+1), \frac{1}{16k_0}, \frac{1}{16}, \log(8d-1), \frac{1}{24k_0}\right\}
\]  \hspace{1cm} (49)

we have that any function \( f^{(L)}_{1,1} \) corresponding to any \((\tilde{c}^{(r)}_{1,i,j}),_{i,j,r,r<L}\) with 

\[
\max_{i,j,r,r<L} \left| \tilde{c}^{(r)}_{1,i,j} - c^{(r)}_{1,i,j}\right| < \min\left\{ 2(n+1), \frac{1}{16k_0}, \frac{1}{16}, \log(8d-1), \frac{1}{24k_0}\right\}
\]  \hspace{1cm} (50)
satisfies in case \( \min\left\{ \|X_i - X_j\|_\infty : 1 \leq i, j \leq n, X_i \neq X_j\right\} \geq 1/(n+1)^3 \)

\[
f^{(L)}_{1,1}(X_t) \geq 1 - e^{-n} \quad \text{and} \quad \max_{t \in \{1, \ldots, n\}, X_t \neq X_t} f^{(L)}_{1,1}(X_t) \leq e^{-n}.
\]  \hspace{1cm} (51)

Set \( \delta_n = 1/(n+1)^3 \) and \( a^{(i)} = X^{(i)}_t - \frac{\delta_n}{2} \) and \( b^{(i)} = X^{(i)}_t + \frac{\delta_n}{2} \) \((i = 1, \ldots, d)\). Then we have 

\[
X_t \in \left[ a^{(1)} + \frac{\delta_n}{4}, b^{(1)} - \frac{\delta_n}{4}\right] \times \ldots \times \left[ a^{(d)} + \frac{\delta_n}{4}, b^{(d)} - \frac{\delta_n}{4}\right],
\]

and

\[
\min\left\{ \|X_i - X_j\|_\infty : 1 \leq i, j \leq n, X_i \neq X_j\right\} \geq 1/(n+1)^3 \]

imply that we also have

\[
X_t \notin \left[ a^{(1)} - \frac{\delta_n}{4}, b^{(1)} + \frac{\delta_n}{4}\right] \times \ldots \times \left[ a^{(d)} - \frac{\delta_n}{4}, b^{(d)} + \frac{\delta_n}{4}\right]
\]

for all \( t \in \{1, \ldots, n\} \) with \( X_t \neq X_t \). If \((\tilde{c}^{(r)}_{1,i,j}),_{i,j,r,r<L}\) satisfies 

\[
\tilde{c}^{(L-1)}_{1,1,1} \leq -8 \cdot (n+1),
\]

\[
|\tilde{c}^{(L-1)}_{1,1,j} - \tilde{c}^{(L-1)}_{1,1,1}| \leq \frac{1}{4k_0} \quad \text{for} \quad j = 2, \ldots, d,
\]

\[
|\tilde{c}^{(L-1)}_{1,1,j} - \tilde{c}^{(L-1)}_{1,1,1}| \leq \frac{1}{4k_0} \quad \text{for} \quad j = 2d+1, \ldots, k_0,
\]

\[
|\tilde{c}^{(L-1)}_{1,1,j} - \tilde{c}^{(L-1)}_{1,1,1}| \leq \frac{1}{4} \quad \text{for} \quad j = 2d+1, \ldots, k_0,
\]

\[
\tilde{c}^{(r-1)}_{1,k,0} \geq 16 \cdot \log(8d-1) \quad \text{for} \quad k \in \{1, \ldots, 2d\}, \quad r \in \{2, \ldots, L-1\},
\]

\[
|\tilde{c}^{(r-1)}_{1,k,0} + \frac{1}{2} \cdot \tilde{c}^{(r-1)}_{1,k,k}| \leq \frac{\log(8d-1)}{2k_0} \quad \text{for} \quad k \in \{1, \ldots, 2d\} \quad \text{and} \quad r \in \{2, \ldots, L-1\},
\]
\[ |c_{1,k,j}^{(r-1)}| \leq \frac{\log(8d-1)}{2k_0} \quad \text{for} \ j \in \{1, \ldots, k_0\} \setminus \{k\}, \ k \in \{1, \ldots, 2d\}, \ r \in \{2, \ldots, L-1\}, \]
\[ c_{1,k,k}^{(0)} \leq -\frac{4}{\delta_n} \cdot \log(8d-1) \quad \text{for} \ k \in \{1, \ldots, d\}, \]
\[ |\tilde{c}_{1,k,j}^{(0)} + a^{(k)} \cdot c_{1,k,k}^{(0)}| \leq \frac{\log(8d-1)}{2d} \quad \text{for} \ k \in \{1, \ldots, d\}, \]
\[ c_{1,d+k,k}^{(0)} \geq \frac{4}{\delta_n} \cdot \log(8d-1) \quad \text{for} \ k \in \{1, \ldots, d\}, \]
\[ |\tilde{c}_{1,d+k,0}^{(0)} + b^{(k)} \cdot c_{1,d+k,k}^{(0)}| \leq \frac{\log(8d-1)}{2d} \quad \text{for} \ k \in \{1, \ldots, d\} \]
and
\[ |\tilde{c}_{1,d+k,j}^{(0)}| \leq \frac{\log(8d-1)}{2d} \quad \text{for} \ k \in \{1, \ldots, d\}, \ j \in \{1, \ldots, d\} \setminus \{k\}, \]
then it is easy to see that for any \((c_{1,i,j}^{(r)})_{i,j,r:L}\) which satisfies (49) we have that any \((\tilde{c}_{1,i,j}^{(r)})_{i,j,r:L}\) which satisfies (50) also satisfies (31)-(43). Application of Lemma 7 yields (51).

In the second step of the proof we show that for \(n\) sufficiently large with probability at least \(1 - n \cdot e^{-n}\) the weights in the random initialization of the weights are chosen such that for each \(l \in \{1, \ldots, n\}\) the weights for some index \(k_l\) satisfy (49) (and hence all functions with weights satisfying (50) satisfy (51)). We assume in the sequel that \(n\) is sufficiently large. If we sample the weight vector from the uniform distribution on
\[-n^4, n^4]^{(L-2)-(k_0^2+k_0)+k_0(d+2)+1},\]
then condition (49) is satisfied for a weight vector \(\tilde{c}\) corresponding to \(X_1\) with probability at least
\[
\left(\frac{1}{n^5}\right)^{(L-2)-(k_0^2+k_0)+k_0(d+2)+1} = \frac{1}{n^5(L-2)-(k_0^2+k_0)+5-k_0(d+2)+5} =: \eta_n.
\]
Hence after \(r_n = n \cdot \left[ \frac{1}{\eta_n} \right]\) of such independent choices (49) is never satisfied with probability less than or equal to
\[
(1 - \eta_n)^{r_n} \leq \left(1 - \frac{n}{r_n}\right)^{r_n} \leq \exp\left(-\frac{n}{r_n} \cdot r_n\right) = e^{-n}.
\]
Now we consider \(n\)-times successively \(r_n\) choices of the weights, i.e.,
\[
k_n = n^2 \cdot \left[ \frac{1}{\eta_n} \right] = n^5(L-2)-(k_0^2+k_0)+5-k_0(d+2)+7
\]
and
\[
|c_{1,k,j}^{(r-1)}| \leq \frac{\log(8d-1)}{2k_0} \quad \text{for} \ j \in \{1, \ldots, k_0\} \setminus \{k\}, \ k \in \{1, \ldots, 2d\}, \ r \in \{2, \ldots, L-1\},
\]
\[ c_{1,k,k}^{(0)} \leq -\frac{4}{\delta_n} \cdot \log(8d-1) \quad \text{for} \ k \in \{1, \ldots, d\}, \]
\[ |\tilde{c}_{1,k,j}^{(0)} + a^{(k)} \cdot c_{1,k,k}^{(0)}| \leq \frac{\log(8d-1)}{2d} \quad \text{for} \ k \in \{1, \ldots, d\}, \]
\[ c_{1,d+k,k}^{(0)} \geq \frac{4}{\delta_n} \cdot \log(8d-1) \quad \text{for} \ k \in \{1, \ldots, d\}, \]
\[ |\tilde{c}_{1,d+k,0}^{(0)} + b^{(k)} \cdot c_{1,d+k,k}^{(0)}| \leq \frac{\log(8d-1)}{2d} \quad \text{for} \ k \in \{1, \ldots, d\} \]
and
\[ |\tilde{c}_{1,d+k,j}^{(0)}| \leq \frac{\log(8d-1)}{2d} \quad \text{for} \ k \in \{1, \ldots, d\}, \ j \in \{1, \ldots, d\} \setminus \{k\}, \]
then it is easy to see that for any \((c_{1,i,j}^{(r)})_{i,j,r:L}\) which satisfies (49) we have that any \((\tilde{c}_{1,i,j}^{(r)})_{i,j,r:L}\) which satisfies (50) also satisfies (31)-(43). Application of Lemma 7 yields (51).

In the second step of the proof we show that for \(n\) sufficiently large with probability at least \(1 - n \cdot e^{-n}\) the weights in the random initialization of the weights are chosen such that for each \(l \in \{1, \ldots, n\}\) the weights for some index \(k_l\) satisfy (49) (and hence all functions with weights satisfying (50) satisfy (51)). We assume in the sequel that \(n\) is sufficiently large. If we sample the weight vector from the uniform distribution on
\[-n^4, n^4]^{(L-2)-(k_0^2+k_0)+k_0(d+2)+1},\]
then condition (49) is satisfied for a weight vector \(\tilde{c}\) corresponding to \(X_1\) with probability at least
\[
\left(\frac{1}{n^5}\right)^{(L-2)-(k_0^2+k_0)+k_0(d+2)+1} = \frac{1}{n^5(L-2)-(k_0^2+k_0)+5-k_0(d+2)+5} =: \eta_n.
\]
Hence after \(r_n = n \cdot \left[ \frac{1}{\eta_n} \right]\) of such independent choices (49) is never satisfied with probability less than or equal to
\[
(1 - \eta_n)^{r_n} \leq \left(1 - \frac{n}{r_n}\right)^{r_n} \leq \exp\left(-\frac{n}{r_n} \cdot r_n\right) = e^{-n}.
\]
Now we consider \(n\)-times successively \(r_n\) choices of the weights, i.e.,
\[
k_n = n^2 \cdot \left[ \frac{1}{\eta_n} \right] = n^5(L-2)-(k_0^2+k_0)+5-k_0(d+2)+7
\]
such choices. Then the probability that in the first series of weights there are no weights corresponding to $X_1$ chosen, or in the second no weights corresponding to $X_2$, ..., or in the $n$-th no weights corresponding to $X_n$ is bounded from above by

$$
\sum_{i=1}^{n} e^{-n} = n \cdot e^{-n}.
$$

Set

$$
L_n = n^{8(L-2)-(k_0^2+k_0)+8k_0(d+2)+16L+15}.
$$

In the third step of the proof we show that we have for $n$ sufficiently large

$$
\|c^{(t)}\|_\infty \leq 2 \cdot n^4 \quad \text{for} \quad t = 0, 1, \ldots, t_n
$$

and

$$
\| (\nabla c F_n)(c) - (\nabla c F_n)(c^{(t)}) \| \leq L_n \cdot \|c - c^{(t)}\|
$$

for all $c = c^{(t)} + s \cdot (c^{(t+1)} - c^{(t)})$ and all $s \in [0, 1]$, for all $t = 0, 1, \ldots, t_n - 1$.

By Lemma 6 we know that for $n$ sufficiently large (13) and (14) hold for $c_3 = 1$ and $c_4 = 4$. The initial choice of our weights implies furthermore (15) and (16) for $n$ sufficiently large. Application of Lemma 4 yields (52). And (52) together with another application of Lemma 6 implies (53).

In the fourth step of the proof we show for $n$ sufficiently large

$$
F_n(c^{(t)}) \leq n^4 \quad \text{for} \quad t = 0, 1, \ldots, t_n.
$$

Because of (53) we can conclude from Lemma 1 that we have for $n$ sufficiently large

$$
F_n(c^{(t+1)}) \leq F_n(c^{(t)}) \quad \text{for} \quad t = 0, 1, \ldots, t_n - 1.
$$

But the initial choice of the weights implies

$$
F_n(c^{(0)}) = \frac{1}{n} \sum_{i=1}^{n} Y_i^2 \leq n^4.
$$

In the fifth step of the proof we show that for $n$ sufficiently large and with probability at least $1 - n \cdot e^{-n} \cdot (12)$ holds for all $c = c^{(t)}$ ($t = 0, 1, \ldots, t_n - 1$). Because of the first and the second step of the proof it suffices to show

$$
|c_{j,k,l}^{(r)} - c_{j,k,l}^{(r)}| \leq \frac{1}{n}
$$

for all $i \in \{1, \ldots, n\}$ and all $k, l, r$ with $r < L$, where $c_{j,k,l}^{(r)}$ and $c_{j,k,l}^{(r)}$ are the corresponding components of $c^{(t)}$ and $c^{(0)}$. Here $j_i$ is chosen such that

$$
f_{j_i,j_i}(X_i) \geq 1 - e^{-n} \quad \text{and} \quad \max_{t \in \{1, \ldots, n\}, X_t \neq X_i} f_{j_i,j_i}(X_t) \leq e^{-n}.
$$

(55)
By Lemma 8 and the result of the fourth step of the proof we can successively conclude for $n$ sufficiently large that we have for $t = 0, 1, \ldots, t_n - 1$
\[
\left| \frac{\partial}{\partial t_j,k,l} F_n(c(t)) \right| \leq 2 \cdot n^2 \cdot k_0 \cdot (n^4)^{L+1} \cdot e^{-n} \leq \frac{1}{2n^3} = \frac{1}{t_n} \cdot \lambda_n \cdot \frac{1}{n}
\]
and that consequently (55) holds for $c(t)$.

In the sixth step of the proof we show the assertion of Theorem 1. By the results of the third and the fifth step of the proof we know that the assumptions of Lemma 3 are satisfied. Application of Lemma 3 yields
\[
F_n(c(t)) - \min_{g: \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \\
\leq \left( 1 - \frac{1}{2 \cdot n \cdot L_n} \right)^{t_n} \cdot \left( F_n(c(0)) - \min_{g: \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right) \\
\leq \exp \left( - \frac{t_n}{2 \cdot n \cdot L_n} \right) \cdot \left( F_n(c(0)) - \min_{g: \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right) \\
= \exp(-n) \cdot \left( F_n(c(0)) - \min_{g: \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(X_i) - Y_i|^2 \right).
\]
With
\[
F_n(c(0)) \leq n^4
\]
we get the assertion. □

4.2 Proof of Theorem 2

Lemma 9 Let $n \in \mathbb{N}$, $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^d \times \mathbb{R}$, $f: \mathbb{R}^d \to \mathbb{R}$, $\kappa_n > 0$ and assume
\[
\frac{1}{n} \sum_{i=1}^{n} |f(x_i) - y_i|^2 \leq \min_{g: \mathbb{R}^d \to \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |g(x_i) - y_i|^2 + \kappa_n.
\]
(56)

Set
\[
\bar{m}_n(x) = \frac{\sum_{i=1}^{n} y_i \cdot I_{\{x_i=x\}}}{\sum_{i=1}^{n} I_{\{x_i=x\}}} \quad (x \in \mathbb{R}^d),
\]
where we use the convention $0/0 = 0$. Then we have for any $i \in \{1, \ldots, n\}$
\[
|f(x_i) - \bar{m}_n(x_i)| \leq \sqrt{n \cdot \kappa_n}.
\]

Proof. We have
\[
\frac{1}{n} \sum_{i=1}^{n} |f(x_i) - y_i|^2 = \frac{1}{n} \sum_{i=1}^{n} |f(x_i) - \bar{m}_n(x_i)|^2 + \frac{1}{n} \sum_{i=1}^{n} |\bar{m}_n(x_i) - y_i|^2,
\]

27
since
\[
\frac{1}{n} \sum_{i=1}^{n} (f(x_i) - \bar{m}_n(x_i)) \cdot (\bar{m}_n(x_i) - y_i) = 0.
\]

Application of (56) yields
\[
\frac{1}{n} \sum_{i=1}^{n} |f(x_i) - \bar{m}_n(x_i)|^2 \leq \kappa_n,
\]
which implies the assertion.

Proof of Theorem 2. Set
\[
p_k = \frac{1}{n} \quad (k \in \{1, \ldots, n\})
\]
and set \( p_k = 0 \) for \( k > n \). Set \( x_k = (k/n, 0, \ldots, 0)^T \) and define the distribution of \((X,Y)\) by

1. \( P[X = x_k] = p_k \quad (k \in \mathbb{N}) \),
2. \( Y = m(X) + \epsilon \) where \( X, \epsilon \) are independent and \( m : \mathbb{R}^d \to \mathbb{R} \),
3. \( P\{\epsilon = -1\} = \frac{1}{2} = P\{\epsilon = 1\} \),
4. \( m(x) = 0 \quad (x \in \mathbb{R}^d) \).

Then \( m \) is the regression function of \((X,Y)\) and the distribution of \((X,Y)\) satisfies the assumptions of Theorem 2.

Set
\[
\bar{m}_n(x) = \frac{\sum_{i=1}^{n} Y_i \cdot I\{X_i = x\}}{\sum_{i=1}^{n} I\{X_i = x\}} \quad (x \in \mathbb{R}^d).
\]

Using
\[
|\bar{m}_n(x)|^2 \leq 2 \cdot |m_n(x_k)|^2 + 2 \cdot |m_n(x_k) - \bar{m}_n(x_k)|^2
\]
together with Lemma 9 we get
\[
E \int |m_n(x) - m(x)|^2 P_X(dx)
\]
\[
\geq E \left\{ \sum_{k=1}^{n} |m_n(x_k)|^2 \cdot p_k \cdot I\{\sum_{i=1}^{n} I\{X_i = x_k\} > 0\} \cdot I\{U \in C_n\} \right\}
\]
\[
\geq E \left\{ \sum_{k=1}^{n} \left( \frac{1}{2} |\bar{m}_n(x_k)|^2 - |m_n(x_k) - \bar{m}_n(x_k)|^2 \right) \cdot p_k \cdot I\{\sum_{i=1}^{n} I\{X_i = x_k\} > 0\} \cdot I\{U \in C_n\} \right\}
\]
28
Using the fact that

The definition of \( \bar{m}_n \) implies the assertion.

Putting together the above results implies the assertion.
References


