Projective Topology on Bifinite Domains and Applications

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Abstract

We revisit extension results from continuous valuations to Radon measures for bifinite domains. In the framework of bifinite domains, the Prokhorov theorem (existence of projective limits of Radon measures) appears as a natural tool, and helps building a bridge between Measure theory and Domain theory. The study we present also fills a gap in the literature concerning the coincidence between projective and Lawson topology for bifinite domains. Motivated by probabilistic considerations, we study the extension of measures in order to define Borel measures on the space of maximal elements of a bifinite domain.

Key words: Bifinite domains, Continuous valuations, Radon measures, Maximal elements

1 Introduction

A recent research area concerns so-called *probabilistic concurrent systems* [16,1]. The main problem is to describe and study, a random behavior of systems with concurrency properties. Engineering applications of this topic are found in the study of large distributed systems, such as telecommunication networks [3].

Probabilistic extensions have been developed for some models from Concurrency theory, in particular for Winskel's event structures and for 1-bounded Petri nets. The domain of configurations of an event structure represents the

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different processes that can occur in the system modeled by the event structure. In turn, the maximal elements of the domain represent the complete histories, or *runs* of the system. According to the usual concepts from stochastic processes theory, a probabilistic event structure, seen as a model of concurrent probabilistic system, is thus specified by a probability measure on the space of runs of the system, i.e., on the space of maximal elements of its domain of configurations. It is understood that the σ-algebra that equips the space of maximal configurations is the Borel σ-algebra related to some topology of the domain, for instance, the Borel σ-algebra associated with the Lawson topology. This setting encompasses of course systems without concurrency, such as discrete Markov chains, where the maximal elements of the domain are the infinite sequences of states of the chain.

Hence, the notion of concurrent probabilistic system is conceptually not very different from other "classical" probabilistic systems, and is not even particular to event structures. A general concurrent probabilistic system can be defined as a *probability measure on the space of maximal elements of some* Dcpo. We will explain below why this definition suffers from too much generality to be useful in practice.

The next step in the theory of probabilistic concurrent systems is to explicitly specify a probability measure on the space of maximal elements of a Dcpo. This is usually decomposed, at least for classical stochastic processes, in two steps:

(1) Specify a probability for finite processes of the system, if possible in an incremental fashion (for instance, the chain rule for discrete Markov chains); this is the central job of probability theory [1].
(2) "Extend" the probabilistic behavior of finite processes to a probability measure on the space of maximal elements; this requires a measure-theoretic argument.

It turns out that, for concurrent systems, both steps 1 and 2 above are more difficult than for non-concurrent systems, such as Markov chains. The issues encountered when dealing with concurrency models have led one of the authors to study a restricted class of event structures, in particular for step 1, the so-called *locally finite event structures* [1]. Other authors have studied the even more restrictive class of *confusion-free* event structures [16]. In the study of locally finite event structures, the extension measure-theoretic argument used was Prokhorov extension theorem for projective systems of probabilities. This solution has several advantages: besides its simplicity and elegance, it provides an effective way to describe the probability measure on the space of maximal configurations by means of a (countable) collection of *finite probability measures*. It is therefore very attractive to extend this method to models more general than event structures.

A natural class of domains that could be used for extension results of this kind is the class of *bifinite domains* thanks to their representation as projective limits of finite posets. Bifinite domains have been introduced by Plotkin [15] in the countably based case as projective limits of sequences of finite posets and by Gunter [7] and Jung [10] in the general case. The class of bifinite domains encompasses the domains of configurations of Winskel's event structures. Bifinite domains have encountered a particular interest since their category is Cartesian closed [6].

This paper aims to present extension results for bifinite domains. We do not restrict ourselves to the extension problem on the space of maximal elements of bifinite domains, but also revisit the problem of the extension of a continuous valuation on the domain to a Radon measure on the domain. For this, we propose a self-contained study of bifinite domains exclusively based on their projective representation. The extension of measures on the space of maximal elements appears as a byproduct of this study, although it was one of our original motivations.

More specifically, we prove the following results: the projective topology of the domain coincides with its Lawson topology (Theorem 1); there is a one-to-one correspondence between continuous valuations on a bifinite domain, and Radon measures on the domain equipped with the Borel-Lawson σ-algebra (Theorem 2); the space of maximal elements of a bifinite domain can be represented as a projective limit of finite sets if and only if the space is compact for the Lawson topology (Theorem 3). Theorem 1 is certainly known by specialists, although we are not aware of its explicit formulation in the literature. On the one hand, Theorem 2 is known for more general cases than for bifinite domains [2,12]. On the other hand the proof we give here is new; it uses the Prokhorov theorem on projective limits of measures; the proof is more direct than in [2], and makes clearer the use of the measure-theoretic argument. The problem of extension of continuous valuations to Borel measures has been popularized by Lawson [13]. Finally, Theorem 3 gives a fundamental limitation for the representation of a measure on the space of maximal elements of a bifinite domain as a projective limit of measures of *finite* sets.

The paper is organized as follows: Section 2 collects the needed background on projective limits and bifinite domains. To keep the paper self-contained we have given proofs of most of the results, that are usually presented as corollaries of results in more general frameworks than bifinite domains. We also there state the coincidence between the projective and the Lawson topologies on bifinite domains. Then we apply this result to the extension of continuous valuations to Radon measures in Section 3, and study the representation of the space of maximal elements as a projective limit of finite sets in Section 4.

2 Background

2.1 Dcpos

We recall some elements from Domain theory (see [6]). The aim is to quickly arrive to the definition of bifinite domains, that will constitute our main model.

We assume basic knowledge on posets (Partially Ordered SETS). If X is a subset of a poset $(L, \leq)$, we denote by $\sup X$ the least upper bound (l.u.b.) of X in L, if it exists. We most usually denote a poset $(L, \leq)$ simply by L when no confusion occurs on the ordering relation involved. For a poset L, the *downward closure* $\downarrow a$ and the *upward closure* $\uparrow a$ of an element $a \in L$ are defined to be:

$$\downarrow a = \{x \in L \,:\, x \leq a\}, \qquad \uparrow a = \{x \in L \,:\, a \leq x\}.$$

Let L be a poset. A subset $D \subseteq L$ is said to be *directed*, if it is nonempty and if any two elements $x, y \in D$ have a common upper bound in D. A DCPO (Directed Complete POset) is a poset L every directed subset of which has a l.u.b. in L.

Let L, M be two posets. A mapping $f : L \to M$ is said to be *order preserving* if $f(x) \leq f(x')$ for any two elements $x \leq x'$ in L. If M is a DCPO, the image $f(D)$ of any directed set D under such a mapping f is a directed set. This makes the following definition meaningful: if L, M are two DCPOs, a mapping $f : L \to M$ is said to be *Scott-continuous* if f is order preserving, and if $\sup(f(D)) = f(\sup(D))$ for any directed set $D \subseteq L$.

2.2 Projective Limits of Sets, Posets and Spaces

Let $(L_i)_{i \in I}$ be a family of sets. We denote by:

$$L = \prod_{i \in I} L_i \,,$$

the product of the the family $(L_i)_{i \in I}$, the elements of which are all families $(l_i)_{i \in I}$ such that $l_i \in L_i$ for each $i \in I$. We denote by $\pi_i : L \to L_i$ the canonical projections.

Assume that I is equipped with some ordering $\leq$, such that I is directed. We assume that, for each pair $i \leq j$ in I, we are given mapping $g_{ij} : L_j \to L_i$,

such that the following equalities hold:

$$\forall i, j, k \in I, \quad g_{ii} = \mathrm{Id}_{L_i}\,, \qquad i \leq j \leq k \Longrightarrow g_{ik} = g_{ij} \circ g_{jk}\,.$$

The data $\Big((L_i)_{i\in I}, (g_{ij})_{i\leq j \text{ in } I}\Big)$ is called a *projective system*. The mappings g_{ij} are called the *bonding maps*. The family $(g_{ij})_{i\leq j \text{ in } I}$ is most usually understood, so that we denote the projective system simply by $(L_i)_{i\in I}$. The *projective limit* of the projective system $(L_i)_{i\in I}$ is defined to be the following subset D of L:

$$D = \Big\{(l_i)_{i\in I} \in L \,:\, \forall i \leq j \text{ in } I, \quad l_i = g_{ij}(l_j)\Big\}.$$

We denote by $g_i : D \to L_i$ the restriction to D of the projection $\pi_i : L \to L_i$, for $i \in I$. The following identities hold:

$$\forall i \leq j \text{ in } I, \quad g_i = g_{ij} \circ g_j\,. \tag{1}$$

Assume that each L_i is equipped with an ordering. Then the product L is equipped with the *product ordering*:

$$\forall (l, l') \in L \times L, \qquad l \leq l' \iff \forall i \in I, \quad \pi_i(l) \leq \pi_i(l').$$

Then $(L, \leq)$ is a partial order, and every projection $\pi_i : L \to L_i$ is order preserving. Moreover, if each L_i is a DCPO, then L is a DCPO and the projections $\pi_i : L \to L_i$ are Scott-continuous.

The projective limit D of a projective system built upon the family $(L_i)_{i\in I}$ with order preserving bonding maps $g_{ij}, i \leq j$, is equipped with the ordering induced from L by restriction. The maps $g_i : D \to L_i$ are then order preserving. If the L_i are DCPOs, and if the bonding maps are Scott-continuous, then D is a DCPO and the mappings $g_i : D \to L_i$ are Scott-continuous.

Finally, instead of an ordering, consider a topology τ_i on each of the sets L_i. The product L is equipped with the product topology τ. This is the coarsest topology rendering continuous all the canonical projections π_i. A subbasis for the open sets of this topology is given by the sets of the form

$$\pi_i^{-1}(U) \quad \text{where} \quad i \in I \quad \text{and} \quad U \in \tau_i\,.$$

It suffices indeed to choose the open sets U in some subbase for the topology τ_i.

The projective limit D of a projective system built upon the family $(L_i)_{i\in I}$ of topological spaces with continuous bonding maps $g_{ij}, i \leq j$, is equipped with the topology induced from the product topology on L by restriction. The maps $g_i : D \to L_i$ are then continuous.

Let L and M be two posets. Let $d : L \to M$ and $g : M \to L$ be two functions. We say that (g, d) is a *projection-embedding pair* if both g and d are order preserving, and if moreover:

$$d \circ g \leq \mathrm{Id}_M \ , \quad g \circ d = \mathrm{Id}_L \ . \tag{2}$$

g is called the *upper adjoint* and d is called the *lower adjoint*. As the lower adjoint d is uniquely determined by the upper adjoint g, if it exists [6, O-3.2], we may denote it by $d = \hat{g}$. We say that g is a *projection*, if it is order preserving and if there is a lower adjoint d such that (g, d) is a projection embedding pair. By (2), projections are surjective and their lower adjoints are order embeddings. A projection-embedding pair is a particular case of an adjunction pair (see [6, O-3.1]). We recall the following properties of adjunction pairs $(g, \widehat{g})$:

(A1) [6, O-3.1] For elements $s \in M$ and $t \in L$, one has $g(s) \geq t$ if and only if $s \geq \widehat{g}(t)$; in other words: $g^{-1}(\uparrow t) = \uparrow \widehat{g}(s)$.

(A2) [6, O-3.3] $\widehat{g}$ is Scott-continuous.

(A3) Projection-embedding pairs compose: if $(g, \widehat{g})$ and $(f, \widehat{f})$ are two projection-embedding pairs as shown in the left diagram below, then the composite $(g \circ f, \widehat{f} \circ \widehat{g})$ in the diagram at right is also a projection embedding-pair:

$$L \underset{g}{\overset{\widehat{g}}{\rightleftarrows}} M \underset{f}{\overset{\widehat{f}}{\rightleftarrows}} N \ , \qquad L \underset{g \circ f}{\overset{\widehat{f} \circ \widehat{g}}{\rightleftarrows}} N$$

We will be interested in projective systems $(L_i)_{i \in I}$ of DCPOs with Scott-continuous projections $g_{i,j}$, $i \leq j$, as bonding maps, and their projective limit D.

Note that, thanks to property (A3), the identity $g_{ik} = g_{ij} \circ g_{jk}$ for $i \leq j \leq k$ implies the contravariant identity

$$\widehat{g}_{ik} = \widehat{g}_{jk} \circ \widehat{g}_{ij} \tag{3}$$

on lower adjoints. We have the following result which, informally speaking, says that we can make $k \to \infty$ in the above equation:

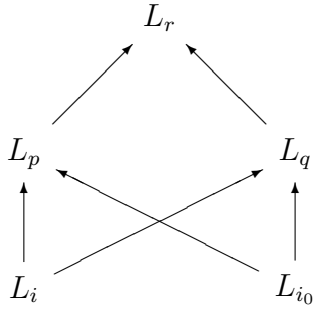
Lemma 1 *Let D be the projective limit of a projective system of* DCPO*s $(L_i)_{i \in I}$ with Scott-continuous projections g_{ij} as bonding maps. Then the following properties hold:*

(1) for each $i \in I$, the canonical map $g_i : D \to L_i$ is a Scott-continuous projection and has a lower adjoint $\widehat{g}_i$;

(2) the identity $\widehat{g}_i = \widehat{g}_j \circ \widehat{g}_{ij}$ holds for any pair $i \leq j$ in I;

(3) for each $y \in D$, the family $y_i = \widehat{g}_i \circ g_i(y)$, $i \in I$ is a directed subset of D, and $\sup_i y_i = y$.

Proof. 1. Let $i_0 \in I$, and $x \in L_{i_0}$. We define an element $(x_i)_{i\in I}$ in the product $\prod_{i\in I} L_i$ as follows: For each $i \in I$, let $p \in I$ such that $p \geq i$ and $p \geq i_0$. Such a p exists since I is directed. Then we put $x_i = g_{ip} \circ \widehat{g}_{i_0p}(x)$. We claim that x_i does not depend on the choice of p. Indeed, let $q \in I$ be another common upper bound of i, i_0, and let $x'_i = g_{iq} \circ \widehat{g}_{i_0q}(x)$. Pick $r \in I$ such that $r \geq p, q$. Such an r exists since I is directed; the following diagram represents the posets involved:



where the arrows represent the lower adjoints of bonding maps. The diagram commutes thanks to Equation (3). By definition, we have the identity $g_{pr} \circ \widehat{g}_{pr} = \mathrm{Id}_{L_p}$, whence:

$$x_i = g_{ip} \circ \widehat{g}_{i_0p}(x) = g_{ip} \circ (g_{pr} \circ \widehat{g}_{pr}) \circ \widehat{g}_{i_0p}(x) = g_{ir} \circ \widehat{g}_{i_0r}(x).$$

For the same reasons we have:

$$x'_i = g_{iq} \circ \widehat{g}_{i_0q}(x) = g_{iq} \circ (g_{qr} \circ \widehat{g}_{qr}) \circ \widehat{g}_{i_0q}(x) = g_{ir} \circ \widehat{g}_{i_0r}(x).$$

Therefore $x_i = x'_i$ as claimed. Moreover, the element $(x_i)_{i\in I}$ belongs to the projective limit D. Indeed, let $i \leq j$, we have to show:

$$g_{ij}(x_j) = x_i\,. \tag{4}$$

For this, pick $p \in I$ such that $p \geq j, i_0$. Then we also have $p \geq i$. Therefore $x_i = g_{ip} \circ \widehat{g}_{i_0p}(x)$ and $x_j = g_{jp} \circ \widehat{g}_{i_0p}(x)$. Hence:

$$g_{ij}(x_j) = (g_{ij} \circ g_{jp}) \circ \widehat{g}_{i_0p}(x) = g_{ip} \circ \widehat{g}_{i_0p}(x) = x_i\,.$$

which proves (4). We consider thus the mapping $\widehat{g}_{i_0} : L_{i_0} \to D$ defined by $x \in L_{i_0} \mapsto \widehat{g}_{i_0}(x) = (x_i)_{i\in I}$, and we prove that $(g_{i_0}, \widehat{g}_{i_0})$ is a projection-embedding pair. It is clear from the definition that $\widehat{g}_{i_0}$ is order preserving, and we already know that g_{i_0} is order preserving. It is also clear that $g_{i_0} \circ \widehat{g}_{i_0} = \mathrm{Id}_{L_{i_0}}$. It remains thus only to show:

$$\forall (x, y) \in L_{i_0} \times D, \quad x \leq g_{i_0}(y) \iff \widehat{g}_{i_0}(x) \leq y. \tag{5}$$

For this, we claim first that, if $z = (z_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ are two elements of D, then:

$$z \leq y \iff \exists k \in I : \forall i \in I, \quad i \geq k \Rightarrow z_i \leq y_i \,. \tag{6}$$

We prove this claim. The ($\Rightarrow$) part is trivial. For the converse implication, assume there exists $k \in I$ as in (6). For each $i \in I$, there is a $j \in I$ such that $j \geq i$ and $j \geq k$ since I is directed. Then $z_j \leq y_j$, and since g_{ij} is order preserving, this implies $z_i = g_{ij}(z_j) \leq g_{ij}(y_j) = y_i$. Hence $z \leq y$, and the claim is proved.

We now come back to (5). Let $x \in L_{i_0}$ and let $y = (y_i)_{i\in I}$ be an element of D. If $\widehat{g}_{i_0}(x) \leq y$, then since g_{i_0} is order preserving and by the identity $g_{i_0} \circ \widehat{g}_{i_0} = \mathrm{Id}_{L_0}$, this implies that $x \leq g_{i_0}(y)$, which proves the ($\Leftarrow$) part of (5). Conversely, assume that $x \leq g_{i_0}(y)$, and let $z = \widehat{g}_{i_0}(x)$. Then, by definition of $\widehat{g}_{i_0}$, we have $z_i = \widehat{g}_{i_0 i}(x)$ for any $i \in I$ such that $i \geq i_0$. Hence, for any $i \geq i_0$, we have:

$$z_i = \widehat{g}_{i_0 i}(x) \leq \widehat{g}_{i_0 i}(y_{i_0}) = \widehat{g}_{i_0 i} \circ g_{i_0 i}(y_i) \leq y_i \,,$$

where the last inequality comes from property (A2) above. Therefore, thanks to (6), we conclude that $z = \widehat{g}_{i_0}(x) \leq y$, which completes the proof of (5). We have obtained so far that $(g_{i_0}, \widehat{g}_{i_0})$ is a projection-embedding pair.

2. From the identity $g_i = g_{ij} \circ g_j$, valid for $i \leq j$, we obtain thanks to (A3) by taking the lower adjoints $\widehat{g}_i = \widehat{g}_j \circ \widehat{g}_{ij}$.

3. Denote $f_i = \widehat{g}_i \circ g_i$ for each $i \in I$. Fix $y \in D$. Observe first that $f_i(y) \leq f_j(y)$ if $i \leq j$. Indeed, $f_i(y) = \widehat{g}_j \circ (\widehat{g}_{ij} \circ g_{ij}) \circ g_j(y)$ by point 2 above. Since $\widehat{g}_{ij} \circ g_{ij} \leq \mathrm{Id}_{L_j}$ by property (A1) of projection-embedding pairs, it follows that $f_i(y) \leq \widehat{g}_j \circ g_j(y) = f_j(y)$, as claimed. Since I is directed, it follows that $y_i = f_i(y)$, $i \in I$, is a directed subset of D. Now we show that $\sup_i y_i$. On the one hand, $y \geq \widehat{f}_i(y)$ for all $i \in I$ by property (A1). On the other hand, if $z \in D$ is such that $z \geq f_i(y)$ for all $i \in I$, then $g_i(z) \geq g_i(y)$ for all $i \in I$ by the definition of adjunction pairs. In other words, $z \geq y$ in D. Hence, $y = \sup_y y_i$. □

It is appropriate now to recall the notions of compact elements and algebraic domains:

Definition 1 *An element k of a* Dcpo *D is called* compact*, if the following property holds: whenever X is a directed subset of D such that $\sup X \geq k$, then there is an element $x \in X$ such that $x \geq k$. A* Dcpo *D is called* algebraic*, if for each of its elements x there is a directed set K of compact elements such that $x = \sup K$.*

These properties of domains are preserved under projective limits. First we state the following lemma (see [6, Exercise I-4.34]):

Lemma 2 *Let $g\colon C \to D$ be a Scott-continuous projection map of* DCPOs *and $\widehat{g}\colon D \to C$ its lower adjoint. Then an element $k \in D$ is compact in D if and only if $\widehat{g}(k)$ is compact in C.*

We now are ready for:

Lemma 3 *Let D be the projective limit of a projective system $(L_i)_{i\in I}$ of algebraic domains with Scott-continuous projections $g_{i,j}$, $i \leq j$, as bonding maps. Let $\widehat{g}_i : L_i \to D$ be the lower adjoints defined in Lemma 1, point 1. Then D is an algebraic domain. An element $y \in D$ is* compact *if and only if $y = \widehat{g}_i(k)$ for some $i \in I$ and some compact element $k \in L_i$.*

Proof. Let $y \in D$ be such that $y = \widehat{g}_i(k)$ for some $i \in I$ and some compact element $k \in L_i$. By the preceding lemma, y is compact. Conversely, let y be a compact element of D. As y is the l.u.b. of the directed set $\widehat{g}_i \circ g_i(y)$, there is an $i \in I$ such that $y = \widehat{g}_i \circ g_i(y)$, whence $y = \widehat{g}_i(x)$ with $x = g_i(y) \in L_i$. Again by the preceding lemma, x is compact in L_i. Thus, we have proved that the compact elements y of D are of the form $y = \widehat{g}_i(k)$ for some $i \in I$ and some compact element $k \in L_i$.

In order to prove algebraicity of D, let y be an arbitrary element of D. We know that y is the l.u.b. of the directed family $\widehat{g}(x_i)$, where $x_i = g_i(y) \in L_i$. As each L_i is supposed to be algebraic, there is a directed set X_i of compact elements in L_i such that $x_i = \sup X_i$. Then the set $Y = \bigcup_i \widehat{g}(X_i)$ is a directed family of compact elements in D such that $\sup X = \sup_i \sup \widehat{g}(X_i) = \sup_i \widehat{g}(\sup X_i) = \sup_i \widehat{g}_i(x) = y$. $\square$

We now come to the main object of our paper:

Definition 2 *Assume that I is a directed poset. A projective system $(L_i)_{i\in I}$, with bonding maps g_{ij} for $i \leq j$, is called a* projective system of finite type *if all L_i are finite posets, and if each bonding map $g_{ij} : L_j \to L_i$, for $i \leq j$, is a projection with lower adjoint $\widehat{g}_{ij} : L_i \to L_j$. The projective limit of such a projective system is called a* bifinite domain.

As in a finite domain every element is compact, the preceding lemma has the following consequence:

Corollary 1 *Let D be a bifinite domain, represented as the projective limit of a projective system (L_i) of finite type with bonding maps g_{ij}. Then D is an algebraic domain. An element y of D is compact if and only if $y = \widehat{g}_i(x)$ for some $i \in I$ and some $x \in L_i$.*

Several topologies can be defined on DCPOs, and in particular on bifinite domains. This subsection describes these topologies. It is one of the aims of the paper to describe their relationships.

Recall that a topology τ on a set X is said to be *coarser* than a topology σ on X if $\tau \subseteq \sigma$. The topology *generated* by a family $\mathcal{F}$ of subsets of X is defined as the coarsest topology that contains all elements of $\mathcal{F}$ as open sets.

Scott, lower and Lawson topologies. A subset U of a DCPO D is called *Scott-open* if:

(1) U is increasing; i.e.: $\forall x \in U,\ \uparrow x \subseteq U$;
(2) (Scott condition) for any directed subset X of D, we have: $\sup X \in U \Rightarrow U \cap X \neq \emptyset$.

The collection of Scott-open sets is a topology on D called the *Scott topology*.

The *lower topology* on a DCPO D is the topology generated by the sets of the form $D \setminus \uparrow x$, with x ranging over D. Finally, the *Lawson topology* on D is the join of the Scott and of the lower topologies on D. We denote by σ, ω and λ the Scott topology, the lower topology, and the Lawson topology, respectively.

For an algebraic domain D, the sets of the form $\uparrow k$ for compact elements $k \in D$ form a basis for the open sets of the Scott topology, and their complements $D \setminus \uparrow k$ form a subbasis for the open sets of the lower topology.

On a finite set, the open sets for the Scott topology are the upper sets and the open sets for the lower topology are the lower sets. The Lawson topology is discrete.

Lemma 4 *Let C and D be* DCPO*s and $g\colon C \to D$ a Scott-continuous projection with lower adjoint $\widehat{g}\colon D \to C$. Then g is lower continuous, hence Lawson-continuous, and $\widehat{g}$ is an embedding for the respective Scott topologies.*

Proof. By property (A3) characterizing adjunctions, the inverse image $g^{-1}(\uparrow y)$ is $\uparrow \widehat{g}(y)$. Thus the inverse image of a subbasic closed set for the lower topology on D is a subbasic closed set for the lower topology on C. This shows that $\widehat{g}$ is lower continuous.

As $\widehat{g}$ is Scott-continuous, it remains to show, that for any Scott-open set V in D, there is a Scott-open set U in C such that $U \cap \widehat{g}(D) = \widehat{g}(V)$. For V Scott-open in D, let $U = \bigcup_{v \in V} \uparrow \widehat{g}(v)$. For $x \in D$ with $\widehat{g}(x) \in U$ there is a

$v \in V$ such that $\widehat{g}(x) \geq \widehat{g}(v)$ whence $x = g(\widehat{g}(x)) \geq g(\widehat{g}(v)) = v$ which implies $x \in V$. Thus $U \cap \widehat{g}(D) = \widehat{g}(V)$. It remains to show that U is Scott-open. By definition, U is an upper set. Let X be a directed subset of C such that $\sup X \in U$. Then $\sup X \geq \widehat{g}(v)$ for some $v \in V$. As g is Scott-continuous, we get $\sup g(X) = g(\sup X) \geq g(\widehat{g}(v)) = v \in V$. As V is Scott-open, we conclude that there is an $x \in X$ with $g(x) \in V$. We conclude that $\widehat{g}(g(x)) \leq x$, whence $x \in U$. $\Box$

Projective Topologies. Let D be an algebraic domain, defined as the projective limit of a projective system $(L_i)_{i \in I}$ of algebraic domains with projections g_{ij} as bonding maps. We consider our three topologies on DCPOs, the Scott topology, the lower topology and the Lawson topology, on all of the L_i. Let L be the product of the family $(L_i)_{i \in I}$. Each of the three topologies yields a product topology on L and induces a topology on the subset D. We call it the associated *projective topology*, and it is the coarsest topology on L that makes all the projections $g_i : D \to L_i$ continuous. A subbasis for the open sets of the product topology is given by the sets of the form $g_i^{-1}(U)$ where $g_i \colon D \to L_i$ is any of the canonical projections and U a basic open set in L_i.

Finally, for a bifinite domain D, we refer to the Lawson-projective topology simply with the expression *projective topology*. This is what is usually understood when talking about the projective topology of a projective limit of finite sets equipped with the discrete topology, as it is the case for bifinite domains.

The Scott topologies on the L_i yield a projective topology $\tilde{\sigma}$ on D. As a basis for the Scott-open sets of the algebraic domains L_i is given by the sets of the form $\uparrow x$, where x is a compact element of L_i, a subbasis for the open sets for the topology $\tilde{\sigma}$ on L is given by the subsets of the form

$$U = g_i^{-1}(\uparrow x), \quad i \in I, \quad x \in L_i \,. \tag{7}$$

As $g_i^{-1}(\uparrow x) = \uparrow \widehat{g}(x)$ by (A1) and as, by Corollary 1, the compact elements of D are precisely the images $\widehat{g}_i(x)$ of compact elements in the L_i, the projective topology $\tilde{\sigma}$ coincides with the intrinsic Scott topology on the projective limit D.

The lower topologies on the L_i yield a projective topology $\tilde{\omega}$ on D. As a basis for the lower closed sets of the algebraic domains L_i is given by the sets of the form $\uparrow x$, where x is a compact element of L_i, a subbasis for the closed sets for the topology $\tilde{\omega}$ on L is given by the subsets of the form

$$U = g_i^{-1}(\uparrow x), \quad i \in I, \quad x \in L_i \,. \tag{8}$$

As $g_i^{-1}(\uparrow x) = \uparrow \widehat{g}(x)$ by (A1) and as, by Corollary 1, the compact elements of D are precisely the images $\widehat{g}_i(x)$ of compact elements in the L_i, the projective

topology $\tilde{\omega}$ coincides with the intrinsic lower topology on the projective limit D.

The Lawson topologies on the L_i yield a projective topology $\tilde{\lambda}$ on D. As the Lawson topology is the join of the lower and the Scott topology, the projective topology $\tilde{\lambda}$ coincides with the intrinsic Lawson topology on D by the above.

With respect to the Lawson topology, an algebraic domain is always a Hausdorff space. Thus, on $L = \prod_i L_i$, the product of the Lawson topologies is Hausdorff, too. We claim that the projective limit domain D is a closed subset in L: Indeed D can be described as follows:

$$D = \bigcap_{\substack{i,j \in I \\ i \leq j}} \Big\{ l \in L \,:\, \pi_i(l) = g_{i,j} \circ \pi_j(l) \Big\}.$$

As the projections $g_{i,j}$ are Lawson-continuous for all $i, j \in I$, also the maps π_i and $g_{i,j} \circ \pi_j$ are continuous functions. Therefore the above equation shows that D is an intersection of closed subsets of L.

If the Lawson topologies on all the L_i are compact, their product topology on L is compact, too, by Tychonoff theorem. As a closed subset of a compact space is compact, we conclude that the Lawson topology on the projective limit D is Lawson compact, too. We conclude:

Proposition 1 *Let D be the projective limit of a projective system L_i of algebraic domains and D their projective limit. Then D is an algebraic domain, too. The intrinsic topologies on D, the Scott, lower and Lawson topology, coincide with the respective projective limit topologies. If the domains L_i are Lawson-compact, the same holds for the projective limit D.*

As finite domains are Lawson compact, in fact discrete, we obtain our first theorem for bifinite domains:

Theorem 1 *A bifinite domain is Lawson compact. Its Lawson topology coincides with its projective topology regardless which projective system of finite type is used to represent D.*

2.5 Example: Event Structures

The domain of configurations of Winskel's event structures [14] is an example of bifinite domain. Recall that an event structure is a triple $(E, \leq, \#)$, where $(E, \leq)$ is a poset at most countable and such that $\downarrow e$ is finite for every $e \in E$, and $\#$ is a binary symmetric and irreflexive relation on E such that: for all $e_1, e_2, e_3 \in E$, $e_1 \# e_2$ and $e_2 \leq e_3$ imply $e_1 \# e_3$. A *configuration* of E is any

downward closed subset $x \subseteq E$ such that $\# \cap (x \times x) = \emptyset$. Configurations are ordered by inclusion. They form a bifinite domain. Indeed, take I as the set of finite downward closed subsets of E, ordered by inclusion, and L_i is the set of configuration subsets of i, for $i \in I$. Then for $i \subseteq j$, $g_{ij} : L_j \to L_i$ is defined by the intersection $g_{ij}(x) = x \cap i$, for all $x \in L_j$. Then there is an isomorphism of posets $\Phi : D \to \mathcal{L}$, where D is the projective limit of the projective system of finite type $(L_i)_{i \in I}$, and $\mathcal{L}$ is the poset of configurations of the event structure. Take Φ defined by:

$$\forall (x_i)_{i \in I} \in D, \qquad \Phi((x_i)_{i \in I}) = \bigcup_{i \in I} x_i \,.$$

Such bifinite domains have the property of being coprime algebraic; recall that a dcpo [poset] is called coprime algebraic if it is bounded complete [a complete bounded poset] (i.e., any two bounded elements have a sup) and if each of its elements is a supremum of completely co-prime elements, where an element p is completely co-prime if $p \leq \bigvee_i x_i$ implies $p \leq x_i$ for some i. Bifinite domains are more general however; for instance, every finite poset is bifinite, whereas a finite poset is prime algebraic if and only it is a distributive meet semilattice.

3 Extension of Continuous Valuations

In this section we apply the results from the previous section to the problem of extending continuous valuations on a bifinite domain to Borel measures. This extension result is known in a much more general framework. However the proof we propose is simpler than, e.g., the proof of [2], since it makes use of the peculiar representation of a bifinite domain as a projective limit. The measure theoretic argument that we use is the Prokhorov extension theorem, that gives a (necessary and) sufficient condition for the existence of projective limits of measures.

Two subsections are devoted to the background on projective systems of measures §3.1 and on continuous valuations §3.2.

3.1 Projective Limits of Measures

σ-algebras and measures. Let Y be a set. An *algebra of sets* on Y is a collection $\mathfrak{F}$ of subsets of Y closed under complementation and under finite intersections. In particular, $\emptyset$ and Y belong to $\mathfrak{F}$. A *σ-algebra* on Y is an algebra $\mathfrak{F}$ that is closed under countable intersections. A pair $(Y, \mathfrak{F})$, where $\mathfrak{F}$ is a σ-algebra on Y, is called a *measurable space*. If the σ-algebra is understood,

a subset $A \subseteq Y$ is said to be *measurable* if $A \in \mathfrak{F}$. If $\mathcal{F}$ is a collection of subsets of Y, the algebra *generated* by $\mathcal{F}$ is the smallest algebra that contains $\mathcal{F}$; the σ-algebra *generated* by $\mathcal{F}$ is the smallest σ-algebra that contains $\mathcal{F}$.

A *measure* on an algebra $\mathfrak{F}$ is a set function $m : \mathfrak{F} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, such that $m(A) \geq 0$ for all $A \in \mathfrak{F}$ and $m(A \cup B) = m(A)+m(B)$, whenever A and B are disjoint sets belonging to $\mathfrak{F}$. A *σ-additive* measure m on a σ-algebra $\mathfrak{F}$ is a measure on $\mathfrak{F}$ such that $m(\bigcup_{n\geq 1} A_n) = \lim_{n\to\infty} m(A_n)$, whenever $(A_n)_{n\geq 1}$ is an increasing sequence of elements of $\mathfrak{F}$. Note that, implicitly, we only consider *bounded* measures, i.e., we do not allow measures to take the value $+\infty$.

If τ is the topology on the Hausdorff space Y, the *Borel σ-algebra* $\mathfrak{F}_\tau$ is the σ-algebra on Y generated by τ. A *Radon measure* is a σ-additive measure defined on $(Y, \mathfrak{F}_\tau)$ such that, for any measurable subset $A \in \mathfrak{F}_\tau$, the following holds:

$$m(A) = \sup\{m(K),\ K \text{ compact},\ K \subseteq A\} = \inf\{m(U),\ U \text{ open},\ U \subseteq A\}.$$

If Y is a finite set, equipped with its discrete topology, the associated Borel σ-algebra is simply the powerset of Y; we call it the *discrete σ-algebra*. We use then $m(x)$ as a shorthand for $m(\{x\})$, for every $x \in Y$. A measure m is then uniquely determined by the nonnegative function $m : Y \to \mathbb{R}$, and $m(A) = \sum_{x\in A} m(x)$ for every $A \subseteq Y$.

Measurable mappings. Image measure. Let $(Y, \mathfrak{F})$ and $(Z, \mathfrak{G})$ be two measurable spaces. A mapping $\varphi : Y \to Z$ is said to be *measurable* if $\varphi^{-1}(B) \in \mathfrak{F}$ for any $B \in \mathfrak{G}$. Such a measurable mapping maps a σ-additive measure m on $(Y, \mathfrak{F})$ to a σ-additive measure φm on $(Z, \mathfrak{G})$ defined by $\varphi m(B) = m\Big(\varphi^{-1}(B)\Big)$, for all $B \in \mathfrak{G}$. This is indeed a left action, i.e., whenever they are well defined, $(\varphi \circ \psi)m = \varphi(\psi m)$.

Note that, if Y and Z are two finite sets equipped with their discrete σ-algebras, then any function $Y \to Z$ is measurable.

Projective systems of measures. Let $(L_i)_{i\in I}$ be a projective system of finite sets, with surjective bonding maps g_{ij} for $i \leq j$. Let $\mathfrak{F}_i$ denote the discrete σ-algebra of L_i, for $i \in I$. Let $(m_i)_{i\in I}$ be a family of measures, such that m_i is a measure on $(L_i, \mathfrak{F}_i)$ for each $i \in I$. We say that $(m_i)_{i\in I}$ is a *projective system of measures* if the following holds:

$$\forall i \leq j \text{ in } I, \quad m_i = g_{ij} m_j\,.$$

Such a projective system of measures always satisfies the so-called Prokhorov condition, that we recall now: Let D denote the projective limit of the projective system, and let $g_i : D \to L_i$ be the canonical projections, for $i \in I$. For every $\epsilon > 0$, there exists a compact $K \subseteq D$ such that $m_i\Big(L_i \setminus g_i(K)\Big) < \epsilon$ holds for all $i \in I$. This condition is trivially satisfied since D itself is compact, hence $K = D$ matches the requirement. As a consequence we have [5]:

Prokhorov extension theorem. *Let $(m_i)_{i\in I}$ be a projective system of measures on a projective system $(L_i)_{i\in I}$ of finite sets. Let D denote the projective limit of $(L_i)_{i\in I}$, and let $\mathfrak{F}$ be the Borel σ-algebra on D associated with the projective topology on D. Then there is a unique Radon measure m on $(D, \mathfrak{F})$ such that:*

$$\forall i \in I, \quad m_i = g_i m\,.$$

The measure m is called the projective limit *of $(m_i)_{i\in I}$.*

3.2 Continuous Valuations

If D is a DCPO, with σ the Scott topology on D, a *valuation* is a set-function $\nu : \sigma \to \mathbb{R}$ such that:

(1) ν is nondecreasing, and $\nu(\emptyset) = 0$;
(2) *(modularity)* $\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B)$ for any $A, B \in \sigma$.

A valuation ν on a DCPO is said to be *continuous* (Lawson, [13]) if it satisfies the following condition:

(3) If $(U_j)_{j\in J}$ is directed in σ, then $\nu(\sup_{j\in J} U_j) = \sup_{j\in J} \nu(U_j)$.

The following key result is due to Horn and Tarski [8].

Lemma 5 *A valuation $\nu : \sigma \to \mathbb{R}$, where σ is the Scott topology of a* DCPO *D, has a unique extension to a measure μ defined on the algebra of sets generated by σ.*

We finally state this lemma:

Lemma 6 *Let D be a finite poset, and let $\nu_1, \nu_2 : \sigma \to \mathbb{R}$ be two valuations. If $\nu_1(\uparrow x) = \nu_2(\uparrow x)$ holds for every $x \in D$, then $\nu_1 = \nu_2$.*

Proof. First observe the following. Let D be a finite nonempty poset equipped with a valuation ν, and let y be any minimal element of D. Consider $D' = D \setminus \{y\}$. Then any upward closed subset U of D' is an upward closed subset of D, and the restriction ν' of ν to those subsets defines a valuation on D'.

We now proceed with the proof of the lemma, by induction on the cardinality of D. The result is obvious if D has one element. Assume it holds for any poset of cardinality $n \geq 1$, and assume that the cardinal of D is $n+1$. Pick y some minimal element of D, put $D' = D \setminus \{y\}$, and consider the two restrictions ν_1' and ν_2' from ν_1 and ν_2 respectively, associated with D' as above. Then ν_1' and ν_2' satisfy $\nu_1'(\uparrow x) = \nu_2'(\uparrow x)$ for any $x \in D'$, and therefore $\nu_1' = \nu_2'$ thanks to the induction hypothesis.

Consider the two measures μ_1 and μ_2 on D, extensions of ν_1 and ν_2 provided by the Horn-Tarski Lemma (Lemma 5), and let U be any upward closed subset of D. On the one hand, if y does not belong to U, then $U \subseteq D'$ and therefore $\nu_1(U) = \nu_1'(U) = \nu_2'(U) = \nu_2(U)$. On the other hand, assume that y belongs to U. Observe that we have:

$$\mu_1(y) = \nu_1(\uparrow y) - \nu_1(\uparrow y \setminus \{y\}) = \nu_2(\uparrow y) - \nu_2(\uparrow y \setminus \{y\}) = \mu_2(y).$$

Therefore we get:

$$\nu_1(U) = \nu_1'(U \setminus \{y\}) + \mu_1(y) = \nu_2'(U \setminus \{y\}) + \mu_2(y) = \nu_2(U \setminus \{y\}) + \mu_2(y) = \nu_2(U).$$

This completes the induction and the proof of the lemma. □

3.3 Extension of Continuous Valuations to Radon Measures

Horn-Tarski's Lemma shows that a valuation extends uniquely to a measure defined on the algebra of sets generated by the Scott topology of a DCPO D. Next, if we consider a continuous valuation, it is reasonable to expect that ν can be extended to a σ-additive measure defined on the σ-algebra generated by the Scott topology. As already mentioned, this kind of result indeed holds in fairly general cases. The proof we present here is adapted to bifinite domains and it yields an approximation of the Radon measure by simple measures, i.e., linear combinations of point measures.

Theorem 2 *Let D be a bifinite domain equipped with a continuous valuation $\nu : \sigma \to \mathbb{R}$, and let $\mathfrak{F}$ be the Borel σ-algebra associated with the Lawson topology λ on D (obviously, $\sigma \subseteq \mathfrak{F}$). Then there exists a unique Radon measure $m : \mathfrak{F} \to \mathbb{R}$ that extends ν on σ. This defines a one-to-one and onto correspondence between continuous valuations on (D, σ) and Radon measures on (D, λ).*

Proof. Let ν be a continuous valuation on D. We proceed step by step to construct a Radon measure on $(D, \mathfrak{F})$ that extends ν. We represent D as the projective limit of a projective system of finite type $(L_i)_{i \in I}$, with bonding maps g_{ij}, and we let $g_i : D \to L_i$ denote the canonical projections, with $\widehat{g}_i : L_i \to D$ their lower adjoints.

1. We define a valuation ν_i on the upper (=Scott-open) subsets of L_i by setting:

$$\nu_i(A) = \nu\Big(g_i^{-1}(A)\Big) \text{ for every upper set } A \subseteq L_i\,.$$

From the identity $g_i = g_{ij} \circ g_j$ valid for $i \leq j$, we deduce:

$$g_{ij}\nu_j(A) = \nu_j\Big(g_{ij}^{-1}(A)\Big) = \nu\Big((g_{ij}\circ g_j)^{-1}(A)\Big) = \nu_i(A) \text{ for all upper sets } A \subseteq L_i\,.$$

Let μ_i be the unique extension of ν_i to the Boolean algebra of all subsets of L_i given by the Horn-Tarski Lemma (Lemma 5). As the last identity extends to all subsets A of L_i, the family $(\mu_i)_{i\in I}$ is a projective system of measures. Therefore, by the Prokhorov theorem, there exists a unique Radon measure m on $(D, \mathfrak{F})$ such that $g_i m = \mu_i$ for all $i \in I$.

2. We now show that m extends ν. Consider first a Scott-open set U of the form $U = {\uparrow}y$, where y is a compact element of D. There are $i \in I$ and $x \in L_i$ such that $y = \widehat{g}_i(x)$. By (A3), $U = g_i^{-1}({\uparrow}x)$, where ${\uparrow}x$ denotes the upward closure of x in L_i. Therefore, we get:

$$m(U) = m\Big(g_i^{-1}({\uparrow}x)\Big) = g_i m({\uparrow}x) = \mu_i({\uparrow}x) = \nu_i({\uparrow}x) = \nu\Big(g_i^{-1}({\uparrow}x)\Big) = \nu(U).$$

Next, we claim that we have

$$m({\uparrow}y_1 \cup \cdots \cup {\uparrow}y_n) = \nu({\uparrow}y_1 \cup \cdots \cup {\uparrow}y_n) \tag{9}$$

for any sequence $y_1, \ldots, y_n$ of compact elements of D. In order to prove this claim, consider some index $i \in I$ such that there are elements $x_1, \ldots, x_n$ of L_i with $y_j = \widehat{g}_i(x_j)$ for all $j = 1, \ldots, n$. Such an index i exists since I is directed. Then we use the above, combined with Lemma 6 applied to the finite poset L_i to obtain (9).

Now we show that $m(U) = \nu(U)$ holds for any Scott-open set U. Let $\mathcal{A}$ be the family of Scott-open sets of the form $A = {\uparrow}y_1 \cup \cdots \cup {\uparrow}y_n$, $n \geq 1$, with y_i compact in D and $y_i \in U$ for all $i = 1, \ldots, n$. Then $\mathcal{A}$ is directed in σ, and $U = \bigcup \mathcal{A}$ since the sets ${\uparrow}y$, where y ranges over the compact elements of D, is a basis of the Scott topology. From (9) we have:

$$m(A) = \nu(A) \text{ for all } A \in \mathcal{A}\,. \tag{10}$$

On the one hand, we have by the continuity of the valuation ν: $\nu(U) = \sup_{A\in\mathcal{A}} \nu(A) = \sup_{A\in\mathcal{A}} m(A)$. In particular: $\nu(U) \leq m(U)$. On the other hand, since m is a Radon measure, there is for any $\epsilon > 0$ a compact subset $K \subseteq U$ such that $m(K) > m(U) - \epsilon$. By compactness of K, there is an element $A \in \mathcal{A}$ such that $K \subseteq A$. We deduce $\nu(A) = m(A) \geq m(K) > m(U) - \epsilon$. Since this holds for any $\epsilon > 0$, we get $\nu(U) = \sup_{A\in\mathcal{A}} \nu(A) \geq m(U)$, and finally $\nu(U) = m(U)$ as desired.

3. So far, we have shown that a continuous valuation ν can be extended to a Radon measure. The uniqueness of the extension comes from the uniqueness in the Prokhorov theorem and in the Horn-Tarski lemma.

Conversely, the same compactness argument that we used in point 2 shows that, for any Radon measure m on $(D, \mathfrak{F})$, the set function $\nu : \sigma \to \mathbb{R}$ defined by $\nu(U) = m(U)$ for any $U \in \sigma$, is a continuous valuation. This shows that continuous valuations and Radon measures are in a one-to-one correspondence. □

Remark 1 The proof of the previous theorem also yields an explicit approximation of the given continuous valuation ν and its extension to a Radon measure by a directed system of simple valuations: Indeed, the measure μ_i on the finite set L_i is a simple measure, i.e., a linear combination $\sum_{x \in L_i} r_x \delta_x$ of Dirac measures. Let $L'_i \subseteq D$ be the the image of the finite poset L_i under the embedding $\widehat{g}_i \colon L_i \to D$ and consider the simple valuation (= simple measure) $\mu'_i = \sum_{y \in L'_i} r_y \delta_y$ on D. For $i \in I$, these simple valuations form a directed set the least upper bound of which is the original valuation ν.

Remark 2 If the index set I has a a cofinal sequence (that is, a sequence $(i_n)_{n \geq 1}$ such that, for all $i \in I$ there exists $n \geq 1$ with $i \leq i_n$), then D is metrizable and compact. Therefore every Borel measure is Radon [4, Th. 1.1, p.7], and the above Theorem 2 states an equivalence between continuous valuations and Borel measures.

Remark 3 (Scott versus Lawson Borel σ-algebra) The above theorem deals with the Borel σ-algebra $\mathfrak{F}$ associated with the Lawson topology. Let $\mathfrak{G}$ denote the Scott Borel σ-algebra. Obviously, $\mathfrak{G} \subseteq \mathfrak{F}$. Although the inclusion may be strict, Theorem 2 also shows, through a large detour, that Radon measures on $\mathfrak{F}$ correspond exactly to Radon measures on $\mathfrak{G}$.

4 Space of Maximal Elements

From the probabilistic point of view, the space of maximal elements of a Dcpo is of particular interest, since it represents the space of histories of a system modeled by the Dcpo. It is thus of interest to know whether the technique of projective systems of measures that we used above can be applied to construct measures—and in particular, probability measures—on the space of maximal elements, by means of projective limits of *finite* measures.

As it is well known from Stone duality theory, spaces obtained as projective limits of finite sets are precisely the Stones spaces (compact, Hausdorff and completely disconnected, see [9, p. 69]). When considering the space of maxi-

mal elements of a bifinite domain, we have a natural projective representation of it in case of compactness (Theorem 3). The point here is also to observe that, in many cases, the space of maximal elements is *not* compact (Examples 1 and 2 following the theorem).

We first need a remark on sub-projective systems.

Remark on sub-projective systems. Let X be the projective limit of finite sets $(X_i)_{i\in I}$, with bonding maps $g_{ij} : X_j \to X_i$ for $i \leq j$. We say that a projective system $(Y_i)_{i\in I}$, with bonding maps $g'_{ij} : Y_j \to Y_i$ for $i \leq j$, is a *sub-projective system* of $(X_i)_{i\in I}$ if $Y_i \subseteq X_i$ for all $i \in I$, and g'_{ij} is the restriction of g_{ij} to Y_j for all i, j with $i \leq j$. In this case, there is a continuous injection $Y \to X$, where Y is the projective limit of $(Y_i)_{i\in I}$ and X is the projective limit of $(X_i)_{i\in I}$.

Maximal elements of bifinite domains and their projective representation. As a DCPO, any bifinite domain has maximal elements. We denote by M_D the set of maximal elements of a bifinite domain D. M_D is equipped with the restriction of the projective topology on D.

Let D be a bifinite domain, projective limit of a projective system of finite type $(L_i)_{i\in I}$, with bonding maps g_{ij} and canonical projections $g_i : D \to L_i$. Define $R_i = g_i(M_D)$ for $i \in I$. Then, for all $i, j \in I$ with $i \leq j$, we have $g_{ij}(R_j) \subseteq R_i$, and therefore we consider the mapping $f_{ij} : R_j \to R_i$, restriction of g_{ij} to R_j. We define by this a sub-projective system $(R_i)_{i\in I}$, with bonding maps f_{ij}. If M_D can be represented as a projective limit of finite sets, the sub-projective system $(R_i)_{i\in I}$ appears as a natural candidate. Actually, the following holds:

Lemma 7 *Let D be a bifinite domain, projective limit of a projective system $(L_i)_{i\in I}$ of finite type, and let $(R_i)_{i\in I}$ be the sub-projective system defined as above. Let R be the projective limit of $(R_i)_{i\in I}$, seen as a subset of D. Then R coincides with the closure of M_D in D, w.r.t. the projective topology.*

Proof. Let C denote the closure of M_D in D w.r.t. the projective topology. We first show that $R \subseteq C$. For this, let $\xi \in R$, let U be any open set containing ξ, and we show that $U \cap M_D \neq \emptyset$. We assume without generality that U has the form $U = g_i^{-1}(x)$, since these sets form a basis of the projective topology. Then $g_i(\xi) = x$ by construction. On the other hand, there is an elements $\zeta \in M_D$ such that $g_i(\zeta) = g_i(\xi)$ since $\xi \in R$. Therefore $g_i(\zeta) = x$, i.e., $\zeta \in U \cap \Omega$. This shows that $R \subseteq C$.

For the converse inclusion $C \subseteq R$, observe first that $M_D \subseteq R$, by definition of R being the projective limit of $(R_i)_{i\in I}$, with $R_i = g_i(M_D)$. But R is compact

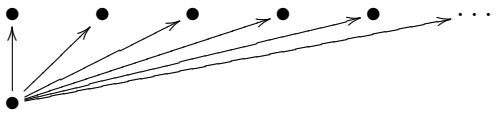


Fig. 1. *Tree for Example 1.*

as a projective limit of finite sets. Therefore R is in particular closed in D. Since $R \supseteq M_D$, this implies that R contains the closure of M_D. Hence $R = C$. □

Theorem 3 *Let D be a bifinite domain. Then the topological space M_D of maximal elements of D can be represented as a projective limit of finite sets if and only if M_D is compact. In this case, M_D is naturally represented as the projective limit of the above projective system $(R_i)_{i\in I}$.*

Proof. It is clear that, if M_D can be represented as a projective limit of finite sets, then it is compact.

Conversely, assume that M_D is compact. Then M_D is closed in D, and therefore M_D coincides with its closure. It follows from Lemma 7 that the sub-projective system $(R_i)_{i\in I}$ introduced above, which is a projective system of finite sets, has its limit R that satisfies $R = M_D$. □

The examples below show that compactness is not easy to guarantee. The two first examples show bifinite domains with non compact spaces of maximal elements. Example 3 gives a sufficient condition for an event structure (see §2.5) to have a compact space of maximal configurations.

Example 1 A first simple example of bifinite domain D which space of maximal elements is not compact is the following: take D to be the set of paths of the tree with one root, and countably many immediate successors (pictured in Figure 1). More formally, take $I = \mathbb{N}$, the set of nonnegative integers, and $L_i = \{0, 1, \ldots, i\}$ for $i \in I$, with the following ordering: $0 \leq k$ for any $k \in L_i$, and otherwise the ordering is discrete. Then take, for $i, j \in I$ with $i \leq j$, $g_{ij} : L_j \to L_i$ defined by $g_{ij}(k) = k$ if $k \leq i$ and $g_{ij}(k) = 0$ otherwise. Then g_{ij} is member of the projection-embedding pair with lower adjoint $\widehat{g}_{ij} : L_i \to L_j$ defined by $\widehat{g}_{ij}(k) = k$ for $k \in L_i$. The bifinite domain D, projective limit of $(L_i)_{i\in I}$ is given by $L = \mathbb{N}$, with the discrete ordering on $\{1, 2, \ldots\}$, and 0 as bottom element. The space of maximal elements M_D is given by $\{1, 2, \ldots\}$, every element of which is a compact element of D. M_D is thus an infinite set of isolated elements, so it is not a compact space.

Example 2 In the above example, the bottom element 0 in D has infinitely many immediate successors. From a modeling viewpoint, we could prefer that finitely many actions should be enabled at any time. Unfortunately, this is not

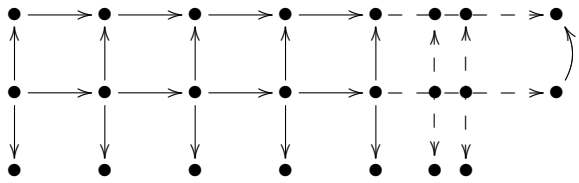


Fig. 2. *Bifinite domain for Example 2.*

enough to guarantee compactness of the space of maximal elements. We leave to the reader to check that the poset pictured in Figure 2 is a bifinite domain (it can be seen as the domain of configurations of an event structure), with the property that every element has at most 3 immediate successors, but still with a non compact space of maximal elements.

Example 3 Let $(E, \leq, \#)$ be an event structure, and let D be the domain of configurations of E. Say that a downward closed subset P of E is *intrinsic* if, for every configuration ξ which is maximal in E, the set-theoretic intersection $\xi \cap P$, which is obviously a configuration of P, is maximal in P. Then we have: *if every $e \in E$ belongs to some finite intrinsic downward closed subset of E, then the space M_D is compact.* Indeed, we check in this case that the space M_D is closed in D, and thus compact.

5 Conclusion

We have presented a self-contained study of bifinite domains based on their representation as projective limits of projective systems of finite type. We have studied the relationship between the projective topology of bifinite domains and their usual topologies that come from Domain theory, showing that the projective topology coincides with the Lawson topology. As an application, we have established for bifinite domains the one-to-one correspondence between continuous valuations and Radon measures. Finally, motivated by probabilistic considerations, we have given a concrete representation of the space of maximal elements of a bifinite domain as a projective limit of finite sets if this space is compact—which is the necessary and sufficient condition for the existence of such a representation.

Future work goes along two lines. First, it would be interesting to extend the techniques of projective systems of measures used here in frameworks more general than bifinite domains. Secondly, the probabilistic interpretation can be pushed further. Domain, and in particular bifinite domains, present a suitable framework for partially ordered stochastic processes. In particular, we expect to successfully apply to domains the theory of martingales with partially ordered, directed sets.

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