Appendix B

Refinement calculus with probability or fuzziness

Preliminary Report
Work during a research visit*
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1 Introduction

In their unpublished paper *Refinement calculus with fuzziness*, Hengyang Wu and Yixiang Chen extend the refinement calculus of Back and Wright [] to a kind of fuzzy situation. They work with fuzzy predicates, that is, predicates with values in the unit interval $\mathbb{I} := [0, 1]$. The logical connectives on $\nvDash := \{0, 1\}$ are extended to the unit interval according to Wangs proposal []. Formally there results ressemble quite exactly exactly to those by Mingsheng Ying **??** who extended the refinement calculus to probabilistic predicates, the main difference being that Ying uses other extensons of the logical connectives to the unit interval. As a result of the discussions during the research visit of Keimel to China in May/June 2008 we propose an abstract setting which works allows a unified treatment of the refinement calculus for the probabilistic and fuzzy logics.

The central idea stems from the fact that that most of the proofs in the papers mentionned above rely on the following observation: The logical operations $\wedge$ and $\rightarrow$ extending conjunction and implication to the unit interval are adjoint in the following sense: For all $a, b \in \mathbb{I}$,

$$\begin{aligned} a \rightarrow b &= \max\{x \mid a \wedge x \leq b\} \\ a \wedge b &= \min\{y \mid a \rightarrow x \geq b\} \end{aligned}$$

We intend to show in the following that most developments in Yings and Wu/Chens paper can be carried through just using the adjointness property of conjunction and implication. There is one exception. The duality between demonic and angelic updates requires a stronger logical property which holds in Wu and Chens fuzzy setting but not in the probabilistic one of Ying.

In our presentation we restrict the values of probabilistic predicates to the unit interval, whilst Ying admits arbitrary nonnegative real values in general. Our restriction is well motivated and the theory becomes smoother.

Instead of the unit interval, we then admit more general quantales L as value domains for probabilistic/fuzzy predicates. This is motivated by the fact that our interval valued probabilistic/fuzzy predicates on a space form again a value quantale and that binary probabilistic/fuzzy relations between S and T can alternatively been viewed as unary probabilistic/fuzzy predicates on S with values in the quantale of unary predicates on T. This is a probabilistic generalisation of the fact that binary (Boolean) relations can alternatively be seen as subsets of the direct product $S \times T$ or as set-valued functions from S with values in the powerset of T.

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2 Residuation

Let us recall some facts about adjoint pairs of functions between posets P and Q which one can find in text books on ordered sets and latices.

Definition . *A pair of maps $g^*: P \to Q$ and $g_*: Q \to P$ is called an* adjoint pair *if*

1. *g^* and g_* are order preserving and*
2. *$g^*(x) \le y \iff x \le g_*(y)$ for all $x \in P$ and all $y \in Q$*

In this situation g^* is called the upper adjoint *und g_* the* lower adjoint.

A map $g: P \to Q$ has a lower adjoint g_* if and only if g is order preserving and if, for each $y \in Q$, the set $\{x \in P \mid g(x) \le y\}$ has a greatest element; the lower adjoint is then given by

$$g_*(y) = \max\{x \in P \mid g(x) \le y\} \tag{LA}$$

Conversely, a map $d: Q \to P$ has an upper adjoint if and only if it is order preserving and if, for all $x \in P$, the set $\{y \in Q \mid x \le d(y)\}$ has a least element; the upper adjoint is then given by

$$d^*(x) = \min\{y \in Q \mid x \le d(y)\} \tag{UA}$$

We note the following:

Lemma 2.1. *(a) The identity map $\mathrm{id}: P \to P$ is equal to its lower and upper adjoint, i.e.,*

$$\mathrm{id}^* = \mathrm{id}_* = \mathrm{id}$$

(b) If $g: P \to Q$ and $h: Q \to R$ both have a lower adjoint, then their composition $h \circ g: P \to R$ has a lower adjoint and

$$(h \circ g)_* = g_* \circ h_*$$

and similarly for maps having upper adjoints.
(c) If (g^*, g_*) and (h^*, h_*) are two adjoint pairs between P and Q, then

$$h^* \le g^* \iff h_* \ge g_*$$

Lemma 2.2. *If P and Q are complete lattices, a map $g: P \to Q$ has a lower adjoint if and only if*

$$g(\sup_i x_i) = \sup_i g(x_i) \text{ for every family } (x_i) \text{ of elements of } P$$

and $d: Q \to P$ has an upper adjoint if and only if

$$d(\inf_i y_i) = \inf_i d(y_i) \text{ for every family of elements } (y_i) \text{ in } Q$$

Now consider a poset L with two binary operations

$$(a,b) \mapsto a \otimes b: L \times L \to L$$

$$(a,b) \mapsto a \to b: L \times L \to L$$

For every $a \in L$, we consider the left translations

$$\{a\} = x \mapsto (a \otimes x) \text{ and } [a] = x \mapsto (a \to x)$$

which are maps from L into itself.

Definition . *We say that* $\otimes$ *is the* upper residual *of* $\to$ *and that* $\to$ *is the* lower residual *of* $\otimes$*, for every element* $a \in L$*, the maps* $\{a\}$ *and* $[a]$ *from* L *into itself form an adjoint pair, that is, if* $\{a\} = [a]^*$ *and* $[a] = \{a\}_*$*. In other words: For all* $a \in L$*,*

1. *the maps* $\{a\}$ *and* $[a]$ *are order preserving and*
2. $a \otimes x \leq y \iff x \leq a \to y$ *for all* $x, y \in L$*.*

An operation $\otimes$ has a lower residual if and only if, for all $a \in L$,

1. the left translation $\{a\}$ is order preserving and
2. the set $\{x \in L \mid a \otimes x \leq b\}$ has a greatest element;

the lower residual is then given by

$$\text{(LR)} \qquad a \to b = \max\{x \in L \mid a \otimes x \leq b\}$$

Conversely, an operation $\to$ has an upper residual if and only if, for all $a \in L$,

1. the left translation $[a]$ is order preserving and
2. the set $\{y \in L \mid b \leq a \to y\}$ has a least element;

the upper residual is then given by

$$\text{(UR)} \qquad a \otimes b = \min\{y \in L \mid b \leq a \to y\}$$

If L is a complete lattice then, by 2.2 an operation $\otimes$ has a lower residual if and only if

$$a \otimes \sup_i b_i = \sup_i (a \otimes b_i)$$

and an operation $\to$ has an upper residual if and only if

$$a \to \inf_i b_i = \inf_i (a \to b_i)$$

for every $a \in L$ and every family of elements $(b_i)_i$ in L.

Suppose that $\otimes$ and $\to$ are residuals in the above sense. From Lemma 2.1 we deduce:

Lemma 2.3. *(a) For an element* $u \in L$*, we have* $\{u\} = \mathrm{id}$ *if and only if* $[u] = \mathrm{id}$*.*
(b) For $a, b \in L$*, we have* $(\{a\} \circ \{b\})_* = [b] \circ [a]$ *and* $([b] \circ [a])^* = \{a\} \circ \{b\}$*.*
(c) $\{a\} \leq \{b\}$ *iff* $[a] \geq [b]$*.*

We now want to consider L to be a set of truth values extending the Booleans $0, 1$. For this we require L firstly to be a poset with a greatest element 1 and a least element 0. Secondly we require L to be endowed with two binary operations $\otimes$ and $\to$ which should extend conjunction and implication. As for the Boolean truth values $0, 1$, we require $\otimes$ to be the upper residual of $\to$ (or, equivalently, $\to$ to be the lower residual of $\otimes$). This is expressed by the requirements:

For arbitrary elements $a, b, c \in L$,

$$b \leq c \quad \Rightarrow \quad a \otimes b \leq a \otimes c \tag{1}$$

$$b \leq c \quad \Rightarrow \quad a \to b \leq a \to c \tag{2}$$

$$a \otimes b \leq c \quad \Leftrightarrow \quad b \leq a \to c \tag{3}$$

In addition to extend the conjunction on the Booleans, conjunction should be associative, commutative and 1 should be a neutral element. We write this down in a diguised form: For arbitrary elements $a, b, c \in L$,

$$1 \otimes a = a \quad (4)$$
$$a \otimes 1 = a \quad (5)$$
$$a \otimes (b \otimes c) = b \otimes (a \otimes c) \quad (6)$$

For the implication, we require the following: For arbitrary elements $a, b, c \in L$,

$$1 \to a = a \quad (7)$$
$$a \leq b \Leftrightarrow 1 = a \to b \quad (8)$$
$$a \to (b \to c) = b \to (a \to c) \quad (9)$$

Definition . *A* value domain *is a poset L with a greatest element 1 and a smallest element 0 together with two binary operations $(a, b) \mapsto a \otimes b\colon L \times L \to L$ and $(a, b) \mapsto a \to b\colon L \times L \to L$ which satisfy the properties (1) – (9) above.*

Remark 2.4. *In the presence of the residuation requirements (1), (2), (3), the properties (4), (5), (6) are equivalent to the properties (7), (8), (9), respectively.*

Proof. Indeed, (4) and (7) mean that the left translations $\lambda(1)$ and $\mu(1)$ are both the identity map. Thus (4) and (7) are equivalent by 2.3(a). Note that 2.3(b) implies that, in the presence of residuation, the left translations $\lambda(a)$ and $\lambda(b)$ commute if and only if the left translations $\mu(a)$ and $\mu(b)$ commute. Thus (6) and (9) are equivalent, as (6) says that the left translations $\lambda(a)$ and $\lambda(b)$ commute whilst (9) says that the left translations $\mu(a)$ and $\mu(b)$ commute. It remains to show that(5) and (8) are equivalent: Indeed, let $a \to b = 1$. Then $1 \leq a \to b$ which is equivalent to $a \otimes 1 \leq b$ by (3). If we suppose (5) $a \otimes 1 = a$, we can deduce $a \leq b$. Thus (5) implies (8). Conversely, $a \otimes 1 \leq x$ iff $1 \leq a \to x$ by (3), which is the same as $1 = a \to x$, as 1 is the greatest element. By (8), the latter is equivalent to $a \leq x$. Hence $a \otimes 1 = a$. □

Because of this remark we may equivalently define a value domain in the following way:

Corollary 2.5. *A value domain is a poset with an operation $\to$ on which satisfies the properties (7), (8), (9) and which has an upper residual $\otimes$, which is then defined by (UR). Equivalently, we may begin with an operation $\otimes$ which satisfies (4), (5), (6) and has a lower residual $\to$, which is then defined by (UR).*

Example 2.6. *In probabilistic and fuzzy logic one usually chooses the truth values in the unit interval $\mathbb{I} = [0, 1]$. For the implication quite a number of variants have been proposed. We list a number of them together with the upper residual if it exists:*

1. *The Lukasiewicz operator:*

$$a \to_L b := \min(1 - a + b, 1), \quad a \otimes_L b := \max(a + b - 1, 0)$$

2. *The Wang operator:*

$$a \to_W b := \begin{cases} 1 & \text{if } a \leq b \\ \max(1 - a, b) & \text{otherwise} \end{cases}, \quad a \otimes_W b := \begin{cases} \min(a, b) & \text{if } a + b > 1 \\ 0 & \text{otherwise} \end{cases}$$

3. *The Gödel operator:*

$$a \to_G b := \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise,} \end{cases}, \quad a \otimes_G b = \min(a, b)$$

4. *The Goguen operator:*

$$a \rightarrow_{Go} b := \begin{cases} 1 & \text{if } a = 0 \\ \min(\dfrac{b}{a}, 1) & \text{otherwise} \end{cases}, \quad a \otimes_{Go} b = a \cdot b$$

5. *The Reichenbach operator:*

$$a \rightarrow_R b = 1 - a + ab, \quad a \otimes_R b := \begin{cases} \min(\frac{a+b-1}{a}, 1) & \text{if } a > 0 \\ 0 & \text{otherwise} \end{cases}$$

6. *The Gaines-Rescher operator:*

$$a \rightarrow_{GR} b := \begin{cases} 1 & \text{if} a \leq b \\ 0 & \text{otherwise,} \end{cases}, \quad a \otimes_{GR} b := \begin{cases} a & \text{if} b \neq 0 \\ 0 & \text{if } b = 0 \end{cases}$$

7. *The Kleene-Dienes operator:*

$$a \rightarrow_{KD} b := \max(1 - a, b), \quad a \otimes_{KD} b := \begin{cases} 0 & \text{if} a + b \leq 1 \\ b & \text{otherwise} \end{cases}$$

8. *The Zadeh operator:*

$$a \rightarrow_Z b := \max(1 - a, \min(a, b))$$

In the first four cases, we have indeed a value domain as defined above, but not in the other four cases. The Reichenbach operator and the Kleene-Dienes operator do not satisfy (5) and (5). The Gaines-Rescher operator does not satisfy (4) and (7). The Zadeh operator does not have an upper residual.

In **??**, *Ying chooses Goguen's setting for his probabilistic logic, and Wu/Chen* **??** *work with Wang's logical operators.*

Let us derive now properties of value domains.

Proposition 2.7. *In a value domain L, conjunction is commutative and associative, i.e., for all $a, b, c \in L$,*

$$a \otimes b = b \otimes a \tag{10}$$
$$a \otimes (b \otimes c) = a \otimes b) \otimes c \tag{11}$$

Commutativity follows from (6) for $c = 1$ using (5). Using commutativity, (6) implies $(b \otimes c) \otimes a = a \otimes (b \otimes c) = b \otimes (a \otimes c) = b \otimes (c \otimes a)$, that is, we have associativity. Note that, conversely, commmutativity and associativity together imply (6).

Let us look now for the dependence of $\rightarrow$ on the first argument:

Proposition 2.8.

$$a \otimes b \rightarrow c = a \rightarrow (b \rightarrow c) \tag{12}$$
$$a \leq a' \Rightarrow a \rightarrow c \geq a' \rightarrow c \tag{13}$$

Proof. We have $x \leq a \otimes b \rightarrow c$ iff $(a \otimes b) \otimes x \leq c$ iff $a \otimes (b \otimes x) \leq c$ (by associativity) iff $bx \leq a \rightarrow c$ iff $x \leq b \rightarrow (a \rightarrow c)$ iff $x \leq a \rightarrow (b \rightarrow c)$ by (9). This proves the first claim.

For the second, let $a \leq a'$. Then $a \otimes x \leq a + \otimes x$, by commutativity and (1). Thus $x \leq a' \rightarrow b \iff a' \otimes x \leq b \implies a \otimes x \leq b \iff x \leq a \implies b$ which implies $a' \rightarrow b \leq a \rightarrow b$. □

Proposition 2.9. *If the value domain L is complete as a lattice, we have:*

$$a \otimes \sup_i b_i = \sup_i (a \otimes b_i) \tag{14}$$

$$(\sup_i a_i) \otimes b = \sup_i (a_i \otimes b) \tag{15}$$

$$a \to \inf_i b_i = \inf_i (a \to b_i) \tag{16}$$

$$(\sup_i a_i) \to b = \inf_i (a_i \to b) \tag{17}$$

In a complete lattice the first and the third identity come from Lemma 2.2 using the fact that the left translations $x \mapsto a \otimes x$ and $x \mapsto a \to x$ form an adjoint pair. The second identity follows from the first, as $\otimes$ is commutative. For the last equation we notice that $x \leq \sup_i a_i \to b$ iff $(\sup_i a_i) \otimes x \leq b$ iff $\sup_i (a_i \otimes x) \leq b$ iff $a_i \otimes x \leq b$ for all i iff $x \leq a_i \to b$ for all i iff $x \leq \inf_i (a_i \to b)$.

Definition . *A map $F \colon L \to L'$ of value domains is called a* homomorphism, *if F preserves multiplication and implication* .

3 Predicates with values in a value domain

Let L be a value domain. We consider L-valued predicates on a set S. We call *L-valued predicate* (or *L-predicate*, for short) on S every function $\rho \colon S \to L$; we denote by $\mathcal{L}(S, L)$ the set of these L-valued predicates.

$\mathcal{L}(S, L)$ is a poset with respect to the pointwise defined order $\rho \leq \sigma$ iff $\rho(x) \leq \sigma(x)$ for all $x \in S$. We may define multiplication and implication for L-valued predicates pointwise for all $x \in S$ by

$$(\rho \otimes \sigma)(x) = \rho(x) \otimes \sigma(x)$$

$$(\rho \to \sigma)(x) = \rho(x) \to \sigma(x)$$

These operations on predicates satisfy again the laws (1) – (9). Thus, the L-valued predicates form again a value domain $\mathcal{L}(S, L)$. If L is a complete lattice, the same holds for $\mathcal{L}(S, L)$; suprema and infima are formed pointwise.

For every $a \in L$, we denote by a also the predicate which is the constant function on S with value a, thus obtaining an injection $j \colon L \to \mathcal{L}(S, L)$. This injection is an order embedding nad a homomorphism of value domains. The top element 1 of L is mapped to the top element of $\mathcal{L}(S, L)$, namely the constant function 1.

Suppose that L is a complete lattice. Then the injection j preserves arbitrary suprema and infima, too. Hence it has a lower and an upper adjoint which we interpret as quantification:

$$\exists x.\rho(x) := \min\{a \in L \mid \rho \leq a\} = \sup_{x \in S} \rho(x)$$

$$\forall x.\rho(x) := \max\{a \in L \mid a \leq \rho\} = \inf_{x \in S} \rho(x)$$

Thus, $\rho \mapsto \exists x.\rho(x)$ and $\rho \mapsto \forall x.\rho(x)$ map L-valued predicates to elements in L. These maps preserve arbitrary suprema and infima, respectively.

We may continue this procedure and consider $\mathcal{L}(S, L)$-valued predicates on a set T and form the value domain $\mathcal{L}(T, \mathcal{L}(S, L))$ of these predicates, etc.

We define an *L-predicate transformer* to be an arbitrary map $t \colon \mathcal{L}(S, L) \to \mathcal{L}(T, L)$, that is, a $\mathcal{L}(T, L)$-valued predicate on $\mathcal{L}(S, L)$; the predicate transformers form again a value domain $\mathcal{L}(\mathcal{L}(S, L), \mathcal{L}(T, L))$.

4 Statement updates

We consider some special predicate transformers.

For a map $f\colon T \to S$, we define the *functional update*

$$\mathcal{L}(f,L)\colon \mathcal{L}(S,L) \to \mathcal{L}(T,L) \text{ by } \mathcal{L}(f,L)(\rho) := \rho \circ f$$

On easily verifies that $\mathcal{L}(f,L)$ is a homomorphism of value domains. Thus, $\mathcal{L}(-,L)$ is a contravariant functor from the category of sets and maps to the category of value domains and homomorphisms, that is,
(1) $\mathcal{L}(\mathrm{id}_S, L) = \mathrm{id}_S$: the identity map on S induces the identical predicate transformer $skip$, and
(2) $\mathcal{L}(f \circ g, L) = \mathcal{L}(g,L) \circ \mathcal{L}(f,L)$.
If L is a complete lattice, then $\mathcal{L}(f,L)$ preserves arbitrary suprema and infima, too.

For $\alpha \in \mathcal{L}(S,L)$ the *assertion update* $\{\alpha\}\colon \mathcal{L}(S,L) \to \mathcal{L}(S,L)$ and the *assumption updates* $[\alpha]\colon \mathcal{L}(S,L) \to \mathcal{L}(S,L)$ are defined by $\{\alpha\}(\rho) = \alpha \otimes \rho$ and $[\alpha](\rho) = \alpha \to \rho$, respectively.

As the predicated form a value domain, we have the following properties:

1. $\{\alpha\}$ and $[\alpha]$ are adjoints, i.e., they are order preserving and $\{\alpha\}(\rho) \leq \sigma$ iff $\rho \geq [\alpha](\sigma)$.
(These are just the properties (1), (2) and (3).)

2. If L is a complete lattice, $\{\alpha\}$ preserves arbitrary suprema and $[\alpha]$ arbitrary infima.
(These are just the properties (16) and (17).)

3. $\{\alpha\}(1) = \alpha,\ [\alpha](1) = 1$.
(Indeed, this results from (5) and (8).)

4. $\{1\} = \mathrm{id} = [1]$.
(This comes from (4) and (7).)

5. $\{\alpha\} \leq \mathrm{id} \leq [\alpha]$.
(Indeed, for all ρ we have $\{\alpha\}(\rho) = \alpha \otimes \rho \leq \rho$, whence $\rho \leq \alpha \to \rho = [\alpha](\rho)$.)

6. $[\alpha] \circ \{\alpha\} \geq \mathrm{id} \geq \{\alpha\} \circ [\alpha]$.
(Indeed, for all ρ, $[\alpha](\{\alpha\}(\rho)) = \alpha \to \alpha \otimes \rho \geq \rho \geq \alpha \otimes (\alpha \to \rho) = \{\alpha\}([\alpha](\rho)$.)

7. $\{0\} = 0$ and $[0] = 1$.
(Indeed, $0 \otimes \alpha = 0$ and $0 \to \alpha = 1$.)

8. $\{\alpha\} \leq \{\beta\}$ iff $[\beta] \leq [\alpha]$ iff $\alpha \leq \beta$.
(Indeed, the first equivalence comes from 2.1. For the second, let $\{\alpha\} \leq \{\beta\}$; then $\alpha = \{\alpha\}(1) \leq \{\beta\}(1) = \beta$. Conversely, if $\alpha \leq \beta$, then $\{\alpha\}(\rho) = \alpha \otimes \rho \leq \beta \otimes \rho = \{\beta\}(\rho)$ for all ρ, whence $\{\alpha\} \leq \{\beta\}$.)

9. If L is a complete lattice, then $\{\sup_i \alpha_i\} = \sup_i \{\alpha_i\}$ and $[\sup_i \alpha_i] = \inf_i [\alpha_i]$.
(Indeed, for all ρ, we have $(\sup_i \{\alpha_i\})(\rho) = \sup_i \{\alpha_i\}(\rho) = \sup_i (\alpha_i \otimes \rho) = (\sup_i \alpha_i) \otimes \rho = \{\sup_i \alpha_i\}(\rho)$, where we have used commutativity (11) and (16). Similarly, $[\sup_i \alpha_i](\rho) = (\sup_i \alpha_i) \to \rho = \inf_i (\alpha_i \to \rho) = \inf_i [\alpha_i](\rho)$ by (18).)

10. $\{\alpha \otimes \beta\} = \{\alpha\} \circ \{\beta\}$.
(This is equivalent to associativity (12) of $\otimes$.)

11. $[\alpha \otimes \beta] = [\alpha] \circ [\beta]$.
(Indeed, for all ρ, we have $[\alpha \otimes \beta](\rho) = \alpha \otimes \beta \to \rho = \alpha \to [\beta \to \rho) = ([\alpha] \circ [\beta])(\rho)$, where we have used (13).)

12. $\{\alpha\}(\sigma \otimes \rho) = \sigma \otimes \{\alpha\}(\rho)$ and $[\alpha](\sigma \to \rho) = \sigma \to [\alpha](\rho)$ (These are just rewritings of the properties (6) and (9) which are equivalent by residuation.)

We note that the requirements (7) and (9) have only been used for the last three items only. Let us add two properties for the composition of our updates with arbitrary predicate transformers:

1. $\{\alpha\} \circ t \leq t' \iff t \leq [\alpha] \circ t'$.
(Indeed, $\{\alpha\} \circ t \leq t'$ iff, for all ρ, $\{\alpha\}(t(\rho)) = \alpha \otimes t(\rho) \leq t'(\rho)$ iff, for all ρ, $t(\rho) \leq \alpha \to t'(\rho)$ iff $t \leq [\alpha] \circ t'$.)

2. $t \leq t' \circ \{\alpha\}$ iff $t \circ [\alpha] \leq t'$, provided that t and t' are order preserving.
(Indeed, if $t \leq t' \circ \{\alpha\}$, then $t \circ [\alpha] \leq t' \circ \{\alpha\} \circ [\alpha] \leq t' \circ \mathrm{id} = t'$ by (5) and by the monotonicity of t'. Conversely, if $t \circ [\alpha] \leq t'$, then $t = t \circ \mathrm{id} \leq t \circ [\alpha] \circ \{\alpha\} \leq t' \circ \{\alpha\}$ also by (5) and by the monotonicity of t.)

5 Relations

Through this section let R, S and T be sets and L a value domain which is complete as a lattice.

We consider L-valued relations between T and S to be functions ρ from the product $T \times S$ to L. That is, L-valued relations between T and S are simply L-valued predicates on $T \times S$. The logical operations for relations are defined as for predicates. Thus, the L-valued relations between T and S form a value domain $\mathcal{L}(S \times T, L)$.

An alternative approach to an L-valued relations between T and S is that of $\mathcal{L}(S, L)$-valued predicates on T:

There is a canonical bijection between $\mathcal{L}(T \times S, L)$ and $\mathcal{L}(T, \mathcal{L}(S, L))$. To every L-valued relation $\rho\colon T \times S \to L$ we associate the $\mathcal{L}(S, L)$-valued predicate $\overline{\rho}\colon T \to \mathcal{L}(S, L)$ defined by

$$\overline{\rho}(y)(x) = \rho(y, x) \text{ for all } y \in T, x \in S$$

This bijection is an order isomorphism and an isomorpism of value domains.

We may define quantification $\exists y.\rho$ and $\forall y.\rho$ by

$$(\exists y.\rho) := \sup_y \overline{\rho}(y)$$

$$(\forall y.\rho) := \inf_y \overline{\rho}(y)$$

Thus existential quantification $\exists y$ maps L-valued relations between T and S to L-valued predicates on S; and this map from relations to predicates preserves arbitrary suprema. Universal quantification $\forall y$ maps L-valued relations between T and S to L-valued predicates on S; and this map from relations to predicates preserves arbitrary infima.

Altogether we see that the logic of binary L-valued relations between T and S is the same as the logic of $\mathcal{L}(S, L)$-valued predicates on T.

We denote by id_S the identity relation on S, i.e., $\mathrm{id}_S(x, y) = 1$ if $x = y$ and 0 else.

Classically the relational product of two relations ρ between T and S and σ between S and R is expressed by:

$$(t, r) \in \rho \circ \sigma \iff \exists s.(t, s) \in \rho \wedge (s, r) \in \sigma$$

Translating this into L-valued logic we obtain:

Given L-valued relations $\rho \in \mathcal{L}(T \times S, L)$ and $\sigma \in \mathcal{L}(S \times R, L)$, their relational product is the relation $\rho \circ \sigma\colon T \times R \to L$ defined by

$$(\rho \circ \sigma)(s, r) = \sup_{s \in S} \rho(t, s) \otimes \sigma(s, r)$$

As in classical relational algebra we have:

Proposition 5.1.

$$\mathrm{id} \circ \rho = \rho = \rho \circ \mathrm{id} \tag{18}$$

$$\rho \circ (\sigma \circ \tau) = (\rho \circ \sigma) \circ \tau \tag{19}$$

$$\rho \circ (\sup_i \sigma_i) = \sup_i (\rho \circ \sigma_i), \quad (\sup_i \rho_i) \circ \sigma = \sup_i (\rho_i \circ \sigma) \tag{20}$$

Proof. $(\mathrm{id} \circ \rho)(z, x) = \sup_y \mathrm{id}(z, y) \otimes \rho(y, x) = \rho(z, x)$, as $\mathrm{id}(z, z) = 1$ and $\mathrm{id}(z, y) = 0$ for $z \neq y$.

$$\begin{array}{rcll} (\rho \circ (\sigma \circ \tau))(z, u) & = & \sup_y \rho(x, y) \otimes (\sup_z \sigma(y, z) \otimes \tau(z, u)) & \text{by definition} \\ & = & \sup_y \sup_z \rho(x, y) \otimes (\sigma(y, z) \otimes \tau(z, u)) & \text{using (15)} \\ & = & \sup_z \sup_y (\rho(x, y) \otimes \sigma(y, z)) \otimes \tau(z, u) & \text{using (12)} \\ & = & \sup_z (\sup_y \rho(x, y) \otimes \sigma(y, z)) \otimes \tau(z, u) & \text{using (15)} \\ & = & ((\rho \circ \sigma) \circ \tau)(z, u) & \text{by definition} \end{array}$$

$$\begin{array}{rcl} (\rho \circ (\sup_i \sigma_i))(z, x) & = & \sup_y (\rho(z, y) \otimes (\sup_i \sigma_i)(y, x)) \\ & = & \sup_y (\rho(z, y) \otimes \sup_i \sigma_i(y, x)) \\ & = & \sup_y \sup_i (\rho(z, y) \otimes \sigma_i(y, x)) \\ & = & \sup_i \sup_y (\rho(z, y) \otimes \sigma_i(y, x)) \\ & = & \sup_i (\rho \circ \sigma)(z, x) \end{array}$$

□

For a binary L-valued relation ρ on S, we may form its powers $\rho^0 := \mathrm{id}, \rho^1 := \rho, \rho^2 := \rho \circ \rho, \ldots, \rho^{n+1} := \rho^n \circ \rho$. The reflexive transitive hull of ρ is given by

$$\rho^{trans} := \sup_{n \geq 0} \rho^n$$

It satisfies indeed $\rho^{trans} \circ \rho^{trans} = \sup_n \rho^n \circ \sup_m \rho^m = \sup_{n,m} \rho^n \circ \rho^m = \sup_{n,m} \rho^{n+m} = \rho^{trans}$, where we have used (20) and (19).

Every L-valued relation ρ between T and S has an obvious converse ρ^{-1} defined by $\rho^{-1}(s, t) = \rho(t, s)$ which is an L-valued relation between S and T. A binary L-valued relation ρ on S is symmetric, if $\rho = \rho^{-1}$. The symmetrisation ρ^{symm} of such a relation ρ, classically given by the statement $(t, s) \in \rho \wedge (t, s) \in \rho^{-1}$ translates into $\rho^{symm}(t, s) = \rho(t, s) \otimes \rho(s, t)$.

6 Relation updates

Let S and T be two sets and ρ an L-valued probabilistic relation between T and S. We simply write $\mathcal{L}S$ instead of $\mathcal{L}(S, L)$, etc.

We define *domain* and *range* of ρ to be the probabilistic predicates on T and S, respectively, given by

$$(\mathrm{dom}\, \rho)(t) := \sup_s \rho(t, s), \quad (\mathrm{range}\, \rho)(s) = \sup_t \rho(t, s)$$

Again these definitions are modelled according to the classical ones

$$t \in \mathrm{dom}\, \rho \iff \exists s.(t, s) \in \rho, \quad s \in \mathrm{range}\, \rho \iff \exists t.(t, s) \in \rho$$

Let us turn to the *angelic* and *demonic updates* $\{\rho\}$ and $[\rho]$ associated with an L-valued relation ρ between T and S. These will transform predicates on S to predicates on T. Classically, for a postcondition β on S, the angelic and demonic updates of a relation are defined by

$$t \in \{\rho\}(\beta) \iff \exists s.(t, s) \in \rho \wedge s \in \beta$$

$$t \in [\rho](\beta) \iff \forall s.(t,s) \in \rho \to s \in \beta$$

(In other contexts, these are just the two ways to define the inverse image of of a subset under a relation.) Translating this into L-valued logic, we arrive at the definitions

$$\{\rho\}(\beta)(t) := \sup_s \rho(t,s) \otimes \beta(s), \quad [\rho](\beta)(t) := \inf_{s \in S} (\rho(t,s) \to \beta(s))$$

Proposition 6.1.

$$\{\rho \circ \sigma\} = \{\rho\} \circ \{\sigma\}, \quad [\rho \circ \sigma] = [\rho] \circ [\sigma]$$

Proof. We prove the second equation; the proof of the first one is similar.

$$\begin{array}{rcll} ([\rho] \circ [\sigma])(\gamma)(t) & = & [\rho]([\sigma](\gamma))(t) & \\ & = & \inf_s(\rho(t,s) \to \inf_r(\sigma(s,r) \to \gamma(r))) & \\ & = & \inf_s \inf_r(\rho(t,s) \to (\sigma(s,r) \to \gamma(r))) & \text{by (16)} \\ & = & \inf_s \inf_r(\rho(t,s) \otimes \sigma(s,r) \to \gamma(r)) & \text{by (13)} \\ & = & \inf_r \inf_s(\rho(t,s) \otimes \sigma(s,r) \to \gamma(r)) & \\ & = & \inf_r((\sup_s(\rho(t,s) \otimes \sigma(s,r)) \to \gamma(r)) & \text{by (17)} \\ & = & \inf_r((\rho \circ \sigma)(t,r) \to \gamma(r)) & \text{by the definition of } \circ \\ & = & [\rho \circ \sigma](\gamma)(t) & \end{array}$$

□

Recall the obvious embedding j of L into $\mathfrak{L}S$ mapping every $a \in L$ to the constant function with value a. The lower adjoint $\alpha \mapsto \forall x.\alpha := \max\{a \in L \mid a \leq \alpha\} = \inf_x \alpha(x)$ preserves arbitrary infima.

Definition . *For two L-valued predicates α and β on S, the* implication strength *is defined to be*

$$str(\alpha \to \beta) := \forall x.(\alpha \to \beta)$$

Eqivalently, $str(\alpha \to \beta) = \max\{a \in L \mid a \otimes \alpha \leq \beta\}$. In particular, $str(\alpha \to \beta) = 1$ if and only if $\alpha \leq \beta$.

Definition . *A predicate transformer $t \colon \mathfrak{L}S \to \mathfrak{L}T$ will be called* monotone *if $\alpha \leq \beta \implies t(\alpha) \leq t(\beta)$ and* strongly monotone *if $str(\alpha \to \beta) \leq str(t(\alpha) \to t(\beta))$. Further, t will be called* $\otimes$-homogeneous *if*

$$t(a \otimes \alpha) = a \otimes t(\alpha) \textit{ for all } a \in L, \alpha \in \mathfrak{L}(S,L)$$

and $\to$-homogeneous *if*

$$t(a \to \alpha) = a \to t(\alpha) \textit{ for all } a \in L, \alpha \in \mathfrak{L}(S,L)$$

Lemma 6.2. *Every strongly monotone predicate transformer t is monotone. If t is monotone and $\otimes$-homogeneous or $\to$-homogeneous, then t is strongly monotone.*

Proof. If $\alpha \leq \beta$ then $str(\alpha \to \beta) = 1$; hence, if t is strongly monotone, $str(\alpha \to \beta) \leq str(t(\alpha) \to t(\beta))$ implies $str(t(\alpha) \to t(\beta)) = 1$, whence $t(\alpha) \leq t(\beta)$. Thus t is monotone. For the converse, let $a \leq str(\alpha \to \beta)$ which is equivalent to $a \leq \alpha \to \beta$. Then $a \otimes \alpha \leq \beta$ and $\alpha \leq a \to \beta$. Using the monotonicity of t, we conclude that $t(a \otimes \alpha) \leq t(\beta)$ and $t(\alpha) \leq t(a \to \beta$. Using the homogeneity of t, we obtain $a \otimes t(\alpha) \leq t(\beta)$ and $t(alpha) \leq a \to t(beta)$. Each of these inequalities implies that $a \leq t(\alpha) \to t(\beta)$, whence $a \leq str(t(\alpha) \to t(\beta))$. We conclude that $str(\alpha \to \beta)) \leq str(t(\alpha) \to t(\beta))$ □

Lemma 6.3. *For an L-valued relation ρ between T and S, the angelic update $\{\rho\}$ is $\otimes$-homogeneous and the demonic update $[\rho]$ is $\to$-homogeneous.*

Proof. Let $\alpha \in \mathfrak{L}S$ and $a \in L$. Then

$$\begin{array}{rcll} \{\rho\}(a \otimes \alpha)(t) & = & \sup_s \rho(t,s) \otimes ((a \otimes \alpha)(s)) & \\ & = & \sup_s \rho(t,s) \otimes (a \otimes \alpha(s)) & \\ & = & \sup_s a \otimes (\{\rho\}(t,s) \otimes \alpha(s)) & by (6) \\ & = & a \otimes \sup_s \{\rho\}(t,s) \otimes \alpha(s) & by () \\ & = & a \otimes \{\rho\}(\alpha)(t) & \end{array}$$

This shows that the angelic update is $\otimes$-homogeneous. The proof for the demonic update is similar. □

Lemma 6.4. *(a) For an L-valued relation ρ between T and S, the angelic and the demonic updates are strongly monotone.*
(b) If (t_i) is a family of strongly monotone predicate transformers, then $\sup_i t_i$ and $\inf_i t_i$ are also strongly monotone.
(c) The composition of strongly monotone predicated transformers is strongly monotone.

Proof. (a) follows from the two previous lemmas. The proof of (b) and (c) is straightforward. □

Theorem 6.5. *Suppose that L is a value domain and S, T are sets. A predicate transformer $t\colon \mathfrak{L}S \to \mathfrak{L}T$ is strongly monotone if and only if it has a decomposition $t = [\sigma] \circ \{\rho\}$ for the angelic update of some relation ρ and the demonic update of some relation σ.*

Proof. By the previous lemma, the composition $t = [\sigma] \circ \{\rho\}$ of the angelic update of some relation ρ and the demonic update of some relation σ is strongly monotonic. Let us prove the converse.

Given a strongly monotonic t, define a relation ρ between T and $\mathfrak{L}S$ and a relation σ beween $\mathfrak{L}S$ and S by

$$\rho(y, \alpha) := t(\alpha)(y)$$

$$\sigma(\alpha, x) := \alpha(x)$$

Let us make explicit the demonic update $[\sigma]\colon \mathfrak{L}S \to \mathfrak{L}\mathfrak{L}S$: For $\alpha \in \mathfrak{L}S$, $[\sigma](\alpha)$ is the function $\beta \mapsto \inf_x(\sigma(\beta, x) \to \alpha(x)) = \inf_x(\beta(x) \to \alpha(x))$ which is $\beta \mapsto str(\beta \to \alpha)$.

Let us make explicit the angelic update of $\{\rho\}\colon \mathfrak{L}\mathfrak{L}S \to \mathfrak{L}T$: For $A \in \mathfrak{L}\mathfrak{L}S$, $\{\rho\}(A)$ is a map from T to L. For every $y \in T$ we have: $\{\rho\}(A)(y) = \sup_\beta \rho(y, \beta) \otimes A(\beta) = \sup_\beta t(\beta)(y) \otimes A(\beta)$, whence $\{\rho\}(A) = \sup_\beta t(\beta) \otimes A(\beta)$.

Altogether we obtain $\{\rho\}([\sigma](\alpha)) = \sup_\beta t(\beta) \otimes str(\beta \to \alpha)$. Considering the special case $\alpha = \beta$ we obtain $\sup_\beta t(\beta) \otimes str(\beta \to \alpha) \geq t(\alpha) \cdot str(\alpha \to \alpha) = t(\alpha) \otimes 1 = t(\alpha)$. Conversely, by our hypothesis of strong monotonicity, $str(\beta \to \alpha) \leq str(t(\beta) \to t(\alpha))$, whence $\sup_\beta t(\beta) \otimes str(\beta \to \alpha) \leq \sup_\beta t(\beta) \otimes str(t(\beta) \to t(\alpha)) \leq t(\alpha)$ by the definition of $str(t(\beta) \to t(\alpha))$. This proves that $\{\rho\}([\sigma](\alpha)) = t(\beta)$, whence $\{\rho\} \circ [\sigma] = t$ □

7 Negation and duality

It is not so clear how to deal with negation in probabilistic/fuzzy logic. Firstly, one may consider negation as a derived operation:

$$\sim a := a \to 0$$

In most cases this negation will not satisfy $\sim\sim a = a$. In the examples 2.6 on the unit interval, negation is often defined to be

$$\neg a := 1 - a$$

Then clearly $\neg\neg a = a$ holds.In the first two examples in 2.6 and in the last two, both of these negation operators coincide, but not in the others. In Boolean algebra we also have $a \implies b = \neg(a \wedge \neg b$. This law remains true in the first two examples, but not in the others.

Accordingly, we have several ways to define disjunction:

$$a \vee b :=\sim (\sim a \otimes \sim b)$$

$$a \text{ or } b := \neg(\neg a \otimes \neg b)$$

Ying **??** uses the Goguen operator and the negation $\neg a = 1 - a$ for his probabilistic logic. Wu and Chen use the Wang operator for their fuzzy setting.

Definition . *A value domain L which carries in addition a unary operation $a \mapsto \neg a \colon L \to L$ which is an involutive order anti-isomorphism is said to be a* value domain with negation*. We say that $\neg$ is a* classical negation *if the following law holds:*

$$a \to b = \neg(a \otimes \neg b) \tag{21}$$

If one has a value domain L with negation, the negation of L-valued predicates $\neg\alpha$ is defined pointwise

$$(\neg\alpha)(x) = \neg\alpha(x)$$

Negation is again an order anti-isomorphism on the value domain $\mathcal{L}(S, L)$. The negation on $\mathcal{L}(S, L)$ is classical iff it is classical on L.

As L-valued relations and predicate transformers can be viewed as predicates, a (classical) negation on L yields a (classical) negation on relations and on predicate transformers.

The dual $t^\circ : \mathcal{L}(S, L) \to \mathcal{L}(T, L)$ of a predicate transformer $t \colon \mathcal{L}(S, L) \to \mathcal{L}(T, S)$ is defined by

$$t^\circ(\rho) = \neg t(\neg\rho)$$

Clearly, $t^{\circ\circ} = t$ and $t_1 \leq t_2 \iff t_1^\circ \geq t_2^\circ$, i.e., duality establishes an involutive order anti-isomorphism on the set of predicate transformers different from negation.

If L has negation, then this negation is classical if and only if $\{\alpha\}^\circ = [\alpha]$, i.e., if and only if the assertion and the assumption updates are dual to one another.

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