

Asymmetric topology and analysis in denotational semantics

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Asymmetric topology:

T_0 -spaces combining topology and order

Approximation from below by elements of finite type

Semicontinuous functions

Asymmetric functional analysis:

T_0 -topological cones

\cong topological ordered vector spaces without negatives

Semicontinuous linear functionals and operators

Forty years ago

Dana S. Scott

introduced **certain lattices as domains for semantics** in
an **Outline of an mathematical theory of computation**

he produced his models for the untyped λ -calculus,
and he coined the notion of a **continuous lattice**.

Forty years ago

Yuri L. Eršov

in his investigations on the **Theory of Numerations** and his study of **Computable functionals of higher type** came up with the notion of an **f -space**,

and he introduced **A -spaces**,
topological spaces with **good** approximations,
a topological version of continuous lattices without completeness.

Origins in computability theory in the 1950ies

$[\mathbb{N} \rightarrow \mathbb{N}]$ set of partial functions $f: \mathbb{N} \rightarrow \mathbb{N}$

- naturally **ordered** by

$$f \sqsubseteq g \iff \text{dom } f \subseteq \text{dom } g \text{ and } f = g|_{\text{dom } f}$$

- a notion of **approximation from below**

$$f_i \nearrow g \iff g \subseteq \bigcup_i f_i$$

- natural **topology** associated with this notion of approximation

$$C \subseteq [\mathbb{N} \rightarrow \mathbb{N}] \text{ **closed** if } x_i \nearrow y, x_i \in C \implies y \in C$$

- There are good approximations: Every g can be approximated from below by **finitary objects**: functions with finite domain

- the natural topology has a basis

$$\uparrow f = \{g \in [\mathbb{N} \rightarrow \mathbb{N}] \mid f \sqsubseteq g\}, f \text{ finitary}$$

- **Baire space**: The topology restricted to the set $\mathbb{N}^{\mathbb{N}}$ of total functions is the product topology.

Origins in computability theory in the 1950ies

$[\mathbb{N} \xrightarrow{\text{comp}} \mathbb{N}]$ subset of computable (= recursive) partial $f: \mathbb{N} \rightarrow \mathbb{N}$.

There is an effective surjection $\zeta: \mathbb{N} \rightarrow [\mathbb{N} \xrightarrow{\text{comp}} \mathbb{N}]$

Functionals of higher type:

$F: [\mathbb{N} \xrightarrow{\text{comp}} \mathbb{N}] \rightarrow [\mathbb{N} \xrightarrow{\text{comp}} \mathbb{N}]$ computable if there is a total computable $h: \mathbb{N} \rightarrow \mathbb{N}$ such that the following diagram commutes:

$$\begin{array}{ccc} [\mathbb{N} \xrightarrow{\text{comp}} \mathbb{N}] & \xrightarrow{F} & [\mathbb{N} \xrightarrow{\text{comp}} \mathbb{N}] \\ \uparrow \zeta & & \uparrow \zeta \\ \mathbb{N} & \xrightarrow{h} & \mathbb{N} \end{array}$$

Myhill-Shepardson Theorem

Every computable $F: [\mathbb{N} \xrightarrow{\text{comp}} \mathbb{N}] \rightarrow [\mathbb{N} \xrightarrow{\text{comp}} \mathbb{N}]$ has a unique extension to a continuous $\bar{F}: [\mathbb{N} \rightarrow \mathbb{N}] \rightarrow [\mathbb{N} \rightarrow \mathbb{N}]$.

Topological problems

$\mathbb{N}^{\mathbb{N}}$ is zero dimensional.

How about $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}$, ...

Questions of this type are of interest for computability over the reals (see D. Normann). Some answers by M. Schröder.

Semantic Domains

- Directed **c**omplete **p**artially **o**rdered set continuous dcpos (Scott, Plotkin).
- f -spaces, A -spaces, α -spaces, (Eršov).
- Stably compact spaces (Jung).
- **Q**uotients of **c**ountably **b**ased spaces (Schröder, Simpson).

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All these spaces are T_0 but far from being Hausdorff. Order and topology are linked. Continuous maps are order preserving. Turning the order upside down is not continuous. Slogan: **asymmetric topology**.

Two approaches: Order and Topology

Posets

Set P with a partial order \leq (**information order**)

T_0 -spaces

Set X with a topology $\mathcal{O}X$ such that points can be distinguished by open sets (Open sets $U \in \mathcal{O}X$ **observable properties**).

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Order \rightarrow Topology: **Scott topology**

Topology \rightarrow Order: **Specialization order**

$$x \leq y \iff (x \in U \implies y \in U) \iff x \in \text{closure}(y)$$

Directed **c**omplete **p**artially **o**rdered set

- = partially ordered set P in which every directed family x_i has a least upper bound
- Approximation from below: $x_i \nearrow x$ if $(x_i)_i$ is a directed family of elements such that $x \leq \bigvee_i x_i$
- Scott topology: $C \subseteq P$ is closed if $x_i \nearrow x, x_i \in C \implies x \in C$
- Lower semicontinuous (lsc) functions $f: P \rightarrow P'$ f continuous for the respective Scott topologies equivalently, $x_i \nearrow x \implies f(x_i) \nearrow f(x)$

Continuous domains order theoretically

Given a dcpo P and $u, v \in P$ we say

- $u \ll v$ (u is **way-below** v or u is **relatively compact** in v) if
 $x_i \nearrow v \implies \exists i. x_i \geq u$

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- $u \ll v$ (u is **way-below** v or u is **relatively compact** in v) if $x_i \nearrow v \implies \exists i. x_i \geq u$
- A **good approximation** of v is a directed family $x_i \ll v$ such that $x_i \nearrow v$
- $B \subseteq P$ is a **basis** if every $v \in P$ has a good approximation by a directed family of elements in B .
- A dcpo P is called **continuous**, if it has a basis.

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Example: $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a continuous dcpo;
 $r \ll s$ iff $r < s$; the rational numbers form a basis;
the Scott-open sets are the intervals $]r, +\infty]$.

Example: Locally compact spaces

X a locally compact T_0 -space

(every point has a neighborhood basis of compact neighborhoods)

$\mathcal{O}X$, the lattice of open subsets, is a continuous dcpo,

$U \ll V$ iff there is a compact set K such that $U \subseteq K \subseteq V$.

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Attention for non T_2 -spaces:

Most compact subsets are not closed.

The **good** compact subsets: **saturated** compact sets

(A set is saturated if it is an intersection of opens.)

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Hofmann Mislove Theorem

In a sober space the intersection of any filter basis of nonempty compact saturated sets is nonempty.

In a locally compact sober space, the collection $\mathcal{K}X$ of nonempty saturated compact sets is a dcpo for the order \supseteq , even a continuous one; $K \gg K'$ iff $\text{interior}(K) \supseteq K'$.

Continuous domains topologically

In a T_0 -space B define the **topological way-below relation** $u \prec v$ if $\uparrow u = \{x \in B \mid u \leq x\}$ is a neighborhood of v .

B is a **C-space** if, for each of its points v and each neighborhood U of v , there is an element $u \in U$ such that $u \prec v$.

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- In a C-space, every element v has a neighborhood basis of sets of the form $\uparrow u$.
- Every C-space is locally compact as the sets $\uparrow u$ are compact.
- Every continuous dcpo is a C-space for its Scott topology, the topological way-below relation \prec and the order theoretical \ll agree.

The interplay

Theorem (Eršov, Erné)

When B is a basis of a continuous dcpo P , then B is a C-space for the topology induced by the Scott topology on P .

Conversely, every C-space B is the basis of a uniquely determined continuous dcpo $P = \tilde{B}$ in such a way that the topological way-below relation on B is induced by the way-below relation on P

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The Basic Extension Lemma

Let B be a C-space (or a basis of a continuous dcpo P),
 $f: B \rightarrow P'$ a lower semicontinuous function into a dcpo P' .

Then f has a unique lower semicontinuous extension
 $\tilde{f}: P = \tilde{B} \rightarrow P'$ defined by

$$\tilde{f}(v) = \sup_{u \in B, u \ll v} f(u)$$

Case study I: Semantics of Probabilistic Features

In semantics of programming languages one interprets objects of (data) type σ (or τ) by elements of a semantic domain S (or T). A program P with objects of type σ as inputs and objects of type τ as outputs is interpreted by a continuous function $f: S \rightarrow T$.

Case study I: Semantics of Probabilistic Features

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If the language admits probabilistic choice, the output will be described by a probability distribution depending on the input, i.e., a function from S to the set of probability distributions on T .

We need a construction of a 'domain of probabilities (more generally measures)' over our semantic domains.

Measures and Valuations

Borel measures on Hausdorff spaces X : $m: \mathcal{B}X \rightarrow \overline{\mathbb{R}}_+$ defined on the σ -algebra $\mathcal{B}X$ generated by the the collection $\mathcal{O}X$ of open sets satisfying

finite additivity: $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$

countable continuity: $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$

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For **non-Hausdorff spaces** replace Borel measures by the simpler notion of **valuations**: $m: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$ satisfying

strictness $m(\emptyset) = 0$

modularity $m(A \cup B) + m(A \cap B) = m(A) + m(B)$

monotonicity $A \subseteq B \Rightarrow m(A) \leq m(B)$

that are

lower semicontinuous $A_i \nearrow A \Rightarrow m(A_i) \nearrow m(A)$

Measures and Valuations

The restriction of a Borel measure on X to the open sets is a valuation on $\mathcal{O}(X)$, but not necessarily lower semicontinuous, only countably lsc.

But: A Radon measure on a locally compact space restricts to a lsc valuation on the opens.

(A Radon measure is a Borel measure such that $m(K_i) \nearrow m(U)$ when the K_i range over the saturated compact subsets of the open set U .)

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Conversely, on a locally compact sober space, every lsc valuation can be extended to a Radon measure (Keimel-Lawson 2005).

The Domain of lsc Valuations

X a T_0 -space,

$\mathcal{V}X$ the set of all lsc valuations

$\mathcal{V}X$ ordered by $\mu \leq \nu$ iff $\mu(U) \leq \nu(U)$ for all $U \in \mathcal{O}(X)$ is a dcpo.

Addition $\mu + \nu$ and multiplication $r\mu$ by reals $r \geq 0$ are lower semicontinuous. We say: $\mathcal{V}X$ is a **dcpo-cone**.

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X may be considered to be a subset of $\mathcal{V}X$ by identifying $x \in X$ with the Dirac measure δ_x which is a valuation.

Theorem

If X is a C-space or continuous domain with the Scott topology, then $\mathcal{V}X$ is a **continuous** dcpo-cone. It is the free dcpo-cone over X : For every dcpo-cone D and every lsc map $f: X \rightarrow D$, there is a unique lsc **linear** map $\tilde{f}: \mathcal{V}(X) \rightarrow D$ extending f .

Sketch of Proof

The **simple** valuations $\sum_{i=1}^n r_i \delta_{x_i}$ with $r_i \in \mathbb{R}_+$, $x_i \in X$ form a basis of $\mathcal{V}(X)$.

Given $f: X \rightarrow D$ lsc, extend it linearly to the basis of simple valuations: $\tilde{f}(\sum_{i=1}^n r_i \delta_{x_i}) = \sum_{i=1}^n r_i f(x_i)$, then use the Basic Extension Lemma to extend it to all of $\mathcal{V}(X)$.

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Corollary

Let X be a C-space or continuous dcpo. For every lower semicontinuous $f: X \rightarrow \overline{\mathbb{R}}_+$, there is a unique lower semicontinuous **linear** map $\tilde{f}: \mathcal{V}(X) \rightarrow \overline{\mathbb{R}}_+$ extending f .

$$\tilde{f}(\mu) =: \int f \, d\mu.$$

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Slogan: dcpo-cones are **asymmetric vector spaces** and 'asymmetric topology' leads to **asymmetric analysis**.

Duality for dcpo-cones

For any space X , consider $\mathcal{V}X$ and $\mathcal{L}X$ the dcpo-cone of all lsc functions $f: X \rightarrow \overline{\mathbb{R}}_+$

Integration (Choquet Integral) for $f \in \mathcal{L}X$, $\mu \in \mathcal{V}X$:

$$\int f \, d\mu = \int_0^\infty \mu(\{x \in X \mid f(x) > r\}) \, dr$$

$(\mu, f) \mapsto \int f \, d\mu: \mathcal{V}X \times \mathcal{L}X \rightarrow \overline{\mathbb{R}}_+$ is bilinear and lsc.

Theorem

For every lsc linear function $M: \mathcal{L}X \rightarrow \overline{\mathbb{R}}_+$ there is a $\mu \in \mathcal{V}X$ such that $M(f) = \int f \, d\mu$ for all $f \in \mathcal{L}X$. (**Riesz Repr. Theorem**)

If X is a C-space, for every lsc linear function $\varphi: \mathcal{V}X \rightarrow \overline{\mathbb{R}}_+$ there is an $f \in \mathcal{L}X$ such that $\varphi(\mu) = \int f \, d\mu$ for all $\mu \in \mathcal{V}X$.

The Schröder-Simpson Theorem

Let X be any space. Endow $\mathcal{V}X$ with the weak*-Scott-topology, the weakest topology such that $\mu \mapsto \int f \, d\mu: \mathcal{V}X \rightarrow \overline{\mathbb{R}}_+$ is lsc for every lsc $f: X \rightarrow \overline{\mathbb{R}}_+$.

Theorem

For every lsc linear functional $\varphi: \mathcal{V}X \rightarrow \overline{\mathbb{R}}_+$ there is a lsc $f: X \rightarrow \overline{\mathbb{R}}_+$ such that $\varphi(\mu) = \int f \, d\mu$ for every $\mu \in \mathcal{V}X$.

In this general setting the Riesz Representation Theorem is easy to prove, whilst the Schröder-Simpson Theorem is difficult. This is opposite for what we know for the classical situation.

Case Study II: Distribution functions

Fact: For a (positive bounded Borel) measure μ on \mathbb{R} its **distribution function** $F: \mathbb{R} \rightarrow \mathbb{R}_+$ defined by:

$$F(x) = \mu(] - \infty, x[)$$

has the following properties:

- 1 F is strict: $\inf_{x \in \mathbb{R}} F(x) = 0$ and bounded
- 2 F is monotone: $x \leq y \Rightarrow F(x) \leq F(y)$
- 3 F is lower semicontinuous: $x_i \nearrow x \Rightarrow F(x_i) \nearrow F(x)$

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And every function with (1), (2), (3) is the distribution of a unique measure on \mathbb{R} .

Usefulness: Riemann Stieltjes integral

$$\int f d\mu = \int f dF = \lim_{\Delta x \rightarrow 0} \sum_i f(x_i)(F(x_i + \Delta x) - F(x_i))$$

Choquet's Theorem 1954

X a locally compact Hausdorff space,
 $\mathcal{O}(X)$ the lattice of all open subsets $U \subseteq X$,
 $\mathcal{K}(X)$ the space of all nonempty compact subsets $K \subseteq X$
with the Vietoris topology generated by
 $\square U = \{K \mid K \subseteq U\}$ and $\diamond U = \{K \mid K \cap U \neq \emptyset\}$

For a measure μ on the hyperspace $\mathcal{K}(X)$ its **distribution function**

$$F: \mathcal{O}(X) \rightarrow \mathbb{R} \text{ defined by } F(U) = \mu(\square U)$$

has the following properties:

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And every such function is the distribution of a uniquely determined measure on $\mathcal{K}(X)$.

Problem

For which spaces L can we characterize measures by something like **distribution functions**?

There is a long paper by A. Revuz in the Annales de l'Institut Fourier 1956 dealing with this problem. The spaces that Revuz is coming up with – *cum grano salis* – turn out to be continuous dcpos which are \wedge -semilattices.

Claim The setting of continuous lattices is appropriate to deal with the above problem.

This is part of joint work with Jean Goubault-Larrecq, ENS Cachan, France.

Choquet's Theorem Domain Theoretically

B a C-space which is a \wedge -semilattice for the specialization order,
 $\mathcal{H}(B)$ the lattice of Scott-closed subsets.

$m: \mathcal{H}(B) \rightarrow \mathbb{R}_+$ a Scott-continuous valuation.

The distribution function $F: B \rightarrow \mathbb{R}_+$ defined by

$$F(x) = m(\text{closure}(\{x\}))$$

has the following properties:

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And every F with these properties is the distribution of a uniquely determined valuation on $\mathcal{H}(B)$.

Proof: 1. Discrete step

The closures of finite subsets $E \subseteq B$ form a lattice \mathcal{E} :
 $cl(E) \cup cl(E') = cl(E \cup E')$, $cl(E) \cap cl(E') = cl(E \wedge E')$

Given $F: B \rightarrow \mathbb{R}$ define $m: \mathcal{E} \rightarrow \mathbb{R}$ by:

$$m(cl(\{u\})) = F(u)$$

$$m(cl(\{u_1, \dots, u_n\})) =$$

$$\sum_i F(u_i) - \sum_{i < j} F(u_i \wedge u_j) + \sum_{i < j < k} F(u_i \wedge u_j \wedge u_k) - + \dots$$

Then m is a strict and modular; m is monotone if and only if F is **totally monotone**, that is, iff

$$u \geq u_1, \dots, u_n \implies F(u) \geq m(cl(\{u_1, \dots, u_n\}))$$

Proof: 1. Discrete step

χ_A denotes the characteristic function of $A \subseteq B$,
 V the real vector space generated by the $\chi_{cl(\{x\})}$, $x \in B$. These characteristic functions are linearly independent, hence a basis of the vector space V . Claim: $\chi_{cl(E)} \in V$:

$$\begin{aligned} cl(E) &= \bigcup_{u \in E} cl(\{u\}) = X \setminus \bigcap_{u \in E} (X \setminus cl(\{u\})) \\ \chi_{cl(E)} &= 1 - \prod_{u \in E} (1 - \chi_{cl(\{u\})}) \\ &= \sum_i \chi_{cl(\{u_i\})} - \sum_{i < j} \chi_{cl(\{u_i \wedge u_j\})} + \dots \end{aligned}$$

The function $x \mapsto F(x)$ has a unique linear extension $F^*: V \rightarrow \mathbb{R}$. Define $m(cl(E)) = F^*(\chi_{cl(E)})$. Then m is a strict, modular and monotone on the lattice \mathcal{E} (because F^* is linear), and

$$m(cl(E)) = F^*(\chi_{cl(E)}) = \sum_i F(u_i) - \sum_{i < j} F(u_i \wedge u_j) + \dots$$

Proof: 2. Continuous step

Lemma

For a C-space B the set $\mathcal{H}(B)$ of all closed subsets is a continuous lattice; the lattice \mathcal{E} of closures $cl(E)$ of finite subsets E is a basis.

Proof: 2. Continuous step

Lemma

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Technical Lemma

If $F: B \rightarrow \mathbb{R}_+$ is lsc, then its extension $m: \mathcal{E} \rightarrow \mathbb{R}_+$ is lsc, too.

Applying the Basic Extension Lemma we obtain a lower semicontinuous $m: \mathcal{H}(B) \rightarrow \mathbb{R}_+$ by defining for every $C \in \mathcal{H}B$:

$$m(C) = \sup\{m(cl(E)) \mid E \subseteq B \text{ finite}, cl(E) \ll C\}$$

A continuity argument shows that m is a valuation.
This finishes the proof.

Back to classical Choquet

X a locally compact space,
 $B = \mathcal{O}(X)$ is a continuous \cap -semilattice. Our theorem yields a bijection between
strict, totally monotone, lsc $F: \mathcal{O}(X) \rightarrow \mathbb{R}_+$ and
lsc valuations on $\mathcal{H}(\mathcal{O}(X))$.

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As $\mathcal{H}(\mathcal{O}(X)) \cong \mathcal{O}(\mathcal{K}(X))$ (not difficult to see), we get back
Choquet's theorem, but more generally for locally compact spaces
that need not be Hausdorff.

Conclusion

I am sure that I have convinced you that domain theoretical ideas going back to Scott and Eršov forty years ago are useful to deal with classical arguments in analysis as far as they are based on compactness arguments and to extend them to non Hausdorff situations as they occur in semantics and in the theory of computability.

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