Choquet type theorems and continuous domains

Klaus Keimel
Technische Universität Darmstadt
www.mathematik.tu-darmstadt.de/ keimel

March 15, 2010 Kyoto

Work in progress,
jointly with Jean Goubault-Larrecq, ENS Cachan, France

General framework

Adapt methods of measure theory and functional analysis
- to continuous domains in the sense of D. S. Scott
- to a non Hausdorff setting

Slogan:

asymmetric topology $\to$ asymmetric functional analysis

Why?

Domains as introduced by D.S. Scott for semantics of programming languages can be viewed alternatively as order or as topological structures. This point of view has been extended to more general topologies: Stably compact spaces (A. Jung et al.), qcb-spaces (A. Simpson et al.)

All these spaces are far from being Hausdorff: They subsume order through the specialisation order ($x \leq y$ iff $x \in cl(\{y\})$) which is to be viewed as an order of increasing information (which introduces 'asymmetry').

A sematics for systems comprising nondeterministic and probabilistic features requires the development of powerdomain constructions (hyperspaces, probabilistic powerdomains) for these spaces and to prove that they have the desired properties.

Some Philosophy

Continuous domains $\cong$
an order theoretical abstraction of a situation where all objects can be approximated from below by their relatively compact (finitary) parts.

Scott-continuous functions $\cong$
functions preserving approximation from below.

Effectivisation through an enumeration of a countable 'basis' which allows to approximate 'computable' objects by a recursively enumerable set of their relatively compact parts belonging to the basis.

Methods of topology and analysis based mainly on compactness arguments are likely to carry over to the non-Hausdorff situation, but not methods being based on completeness arguments.

Distribution functions

Fact: For a (positive bounded Borel) measure μ on $\mathbb{R}$ its *distribution function* $F\colon \overline{\mathbb{R}} \to \mathbb{R}_+$ defined by:

$$F(x) = \mu(]-\infty, x[)$$

has the following properties:

(i) F is strict:	$\inf_{x\in\mathbb{R}} F(x) = 0$
(ii) F is bounded:	$\sup_{x\in\mathbb{R}} F(x) < +\infty$
(iii) F is monotone:	$x \leq y \Rightarrow F(x) \leq F(y)$
(iv) F is lower semicontinuous:	$x_i \nearrow x \Rightarrow F(x_i) \nearrow F(x)$

And every function with (i), (ii), (iii), (iv) is the distribution of a unique measure on $\mathbb{R}$.

Choquet's Theorem 1954

X a locally compact Hausdorff space,
$\mathcal{O}(X)$ the lattice of all open subsets U,
$\mathcal{K}(X)$ the space of all nonempty compact subsets K with the Vietoris topology generated by

$$\Box U = \{K \mid K \subseteq U\} \text{ and } \Diamond U = \{K \mid K \cap U \neq \emptyset\}$$

For a measure μ on the hyperspace $\mathcal{K}(X)$ its distribution function $F\colon \mathcal{O}(X) \to \mathbb{R}$ defined by $F(U) = \mu(\Box U)$
has the following properties:

(i) F is strict: $F(\emptyset) = 0$
(iii) F totally monotone: .....
(iv) F lower semicontinuous: $U_i \nearrow U \Rightarrow F(U_i) \nearrow F(U)$

And every such function is the distribution of a uniquely determined measure on $\mathcal{K}(X)$.

Problem

For which spaces L can we characterize measures by their distribution functions?

There is a long paper by A. Revuz in the Annales de l'Institut Fourier 1956 dealing with this problem. The spaces that Revuz is coming up with look very much like continuous lattices.

Claim The setting of continuous lattices is appropriate to deal with the above problem.

Dcpo

= directed complete partially ordered set

= partially ordered set X in which every directed family x_i has a least upper bound $x = \sup_i x_i$; write $x_i \nearrow x$.

A map $g\colon X \to Y$ of dcpos is *Scott-continuous* if it is

(i) monotone: $x \leq y \Rightarrow f(x) \leq f(y)$

(ii) lower semicontinuous: $x_i \nearrow x \Rightarrow f(x_i) \nearrow f(x)$

This notion of continuity is equivalent to continuity with respect to the Scott topology:

A subset C of a dcpo X is *Scott-closed* if

(i) $x \leq y,\ y \in C \quad \Rightarrow \quad x \in C$

(ii) $\forall i.\ x_i \in C$ and $x_i \nearrow x \quad \Rightarrow \quad x \in C$.

Continuous dcpos

For elements u and v of a dcpo we say
$u \ll v$ (u is *way-below* v or u is *relatively compact* in v)
if $x_i \nearrow x \geq v$ implies $u \leq x_i$ for some i.

A dcpo is said to be *continuous* if for every v, there is a directed family $u_i \ll v$ such that $u_i \nearrow v$.

Example: $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a continuous dcpo;
$r \ll s$ iff $r < s$;
the Scott-open sets are the intervals $]r, +\infty]$.

Example: For a locally compact space X,
$\mathcal{O}(X)$, the lattice of open subsets, is a continuous dcpo,
$U \ll V$ iff there is a compact set K such that $U \subseteq K \subseteq V$.

Measures and Valuations

Measure: $m : \mathcal{B} \to R_+$
defined on a Boolean algebra $\mathcal{B}$ (of subsets of a set X)
finitely additive: $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$
countably additive: $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$
in addition to finite additivity provided $\mathcal{B}$ is a σ-algebra.

Valuation: $m : \mathcal{L} \to \overline{R}_+$
defined on a lattice $\mathcal{L}$ (of subsets of a set X or more generally a distributive lattice with a least element)

strict:	$m(\emptyset) = 0$
monotone:	$A \subseteq B \Rightarrow m(A) \le m(B)$
modular:	$m(A \cup B) + m(A \cap B) = m(A) + m(B)$

m is Scott-continuous if $A_i \nearrow A \Rightarrow m(A_i) \nearrow m(A)$.
provided $\mathcal{L}$ is directed complete.

Measures and Valuations (ctd)

For Boolean algebras, finitely additive measures and valuations agree.

The notion of a valuation goes back to G. Birkhoff (1939) for arbitrary lattices. Valuations and their relation to measures have been considered by G. Choquet (1955). As a substitute for measures they are a standard tool in Geometric Probability Theory (Schneider, McMullen, Klain, Rota).

Valuations and Borel Measures

For a topological space X, let $\mathcal{O}(X)$ denote the lattice of open subsets. The restriction of a Borel measure on X to the open sets is a valuation on $\mathcal{O}(X)$, but not necessarily Scott-continuous, only countably continuous (which is equivalent to Scott-continuity if $\mathcal{O}(X)$ has a countable basis).

Conversely, one may ask, whether a given Scott-continuous valuation on $\mathcal{O}(X)$ can be extended to a Borel mesure.

For example, on a locally compact Hausdorff space every regular Borel measure restricts to a Scott-continuous valuation and, on a locally compact sober space, every Scott-continuous valuation can be extended to a Borel measure (Keimel-Lawson 2005).

Choquet Domain Theoretically

Let L be a continuous dcpo which is a $\wedge$-semilattice, $\mathcal{H}(L)$ the lattice of Scott-closed subsets. For every Scott-continuous valuation m on $\mathcal{H}(L)$, its distribution function

$F\colon L \to \mathbb{R}_+$ defined by $F(x) = m(cl(\{x\}))$

has the following properties:

(i) F is strict:	$\inf_{x \in L} F(x) = 0$
(ii) F is bounded:	$\sup_{x \in L} F(x) < +\infty$
(iii) F is totally monotone:	....
(iv) F is Scott-continuous:	$x_i \nearrow x \Rightarrow F(x_i) \nearrow F(x)$

And every F with these properties is the distribution of a uniquely defined Scott-continuous valuation on $\mathcal{H}(X)$.

Proof: Discrete step

The closures of finite subsets $E \subseteq L$ form a lattice $\mathcal{B}$: $cl(E) \cup cl(E') = cl(E \cup E')$, $cl(E) \cap cl(E') = cl(E \wedge E')$

Define $m\Big(cl(\{u\})\Big) = F(u)$

$m\Big(cl(\{u_1, \ldots . u_n\})\Big) =$

$$\sum_i F(u_i) - \sum_{i<j} F(u_i \wedge u_j) + \sum_{i<j<k} F(u_i \wedge u_j \wedge u_k) -+ \ldots$$

Then m is a strict modular map on $\mathcal{B}$. It is monotone, hence, a valuation if and only if F is *totally monotone*, that is, iff

$$u \geq u_1, \ldots, u_n \implies F(u) \geq m\Big(cl(\{u_1, \ldots, u_n\})\Big)$$

Proof of Modularity

Let $\mathcal{X}_{cl(E)}$ denote the characteristic function of $cl(E)$, V the real vector space generated by the characteristic functions $\mathcal{X}_{cl(\{x\})}$, $x \in L$. These characteristic functions are linearly independent, hence a basis of the vector space V. The function $cl(\{x\}) \mapsto F(x)$ has a unique linear extension $F^*\colon V \to \mathbb{R}$.

$cl(\{u_1, \ldots, u_n\}) = \bigcup_i cl(\{u_i\}) = X \setminus \bigcap_i (X \setminus cl(\{u_i\})$, so

$$\begin{aligned}\mathcal{X}_{cl(F)} &= 1 - \textstyle\prod_i (1 - \mathcal{X}_{cl(\{u_i\})}) \\ &= \textstyle\sum_i \mathcal{X}_{cl(\{u_i\})} - \sum_{i<j} \mathcal{X}_{cl(\{u_i \wedge u_j\})} + - \ldots\end{aligned}$$

Thus, $\mathcal{X}_{cl(E)} \in V$. Define $m(cl(E)) = F^*(\mathcal{X}_{cl(E)})$. Then m is a valuation on the lattice $\mathcal{B}$ (because F^* is linear), and

$$m(cl(E)) = F^*(\mathcal{X}_{cl(E)}) = \textstyle\sum_i F(u_i) - \sum_{i<j} F(u_i \wedge u_j) + - \ldots.$$

Proof: Continuous extension

Basis of a continuous dcpo L: a subset B such that, for every $v \in L$, there is a directed family $(u_i)_i$ in B such that $u_i \ll v$ for all i and $u_i \nearrow v$.

(Note. Such a family $(u_i)_i$ is a 'Meister': If $w_j \nearrow v$ then, for every j, there is an i such that $u_i \leq v_j$.)

Basis Lemma For any monotone map $m \colon B \to \overline{\mathbb{R}}$, the map $m^* \colon L \to \overline{\mathbb{R}}$ defined by

$$m^*(v) = \sup_i m(u_i) \text{ where } u_i \in B, u_i \ll v, u_i \nearrow v$$

is Scott-continuous, and it is the greatest Scott-continuous map $\leq m$. If $m^*(u) = m(u)$ for all $u \in B$, then m^* is the unique Scott-continuous extension of m.

Proof: Continuous extension (ctd)

Lemma For a continuous dcpo L the set $\mathcal{H}(L)$ of all closed subsets is a continuous lattice, the closures of finite subsets E form a basis $\mathcal{B}$ and
$cl(E) \ll C$ iff for each $u \in E$ there is a $v \in C$ with $u \ll v$.

Applying the Basis Lemma we obtain a Scott-continuous $m^*\colon \mathcal{H}(L) \to \mathbb{R}$ by putting

$$m^*(C) = \sup\{m(cl(E)) \mid E \subseteq L \text{ finite }, cl(E) \ll C\}.$$

A continuitity argument shows that m^* is a valuation.

Technical Lemma If $F\colon L \to \mathbb{R}$ is Scott-continuous, then $m(cl(E)) = m^*(cl(E))$ for every finite subset E of L.

This finishes the proof.

Back to Choquet

We may choose for L the lattice $\mathcal{O}(X)$ of open subsets of a locally compact space X. Our theorem yields a bijection between
strict, totally monotone Scott-continuous $F\colon \mathcal{O}(X) \to \mathbb{R}$ and
Scott-continuous valuations on $\mathcal{H}(\mathcal{O}(X)) \cong \mathcal{O}(\mathcal{K}(X))$ as is not difficult to see.

But our result includes locally compact spaces that need not be Hausdorff provided one chooses the appropriate definition of local compactness (every point has a neighborhood basis of compact neighborhoods) and the appropriate class of compact sets (compact saturated sets).

Conclusion

I must have convinced you that domain theoretical ideas are useful to deal with classical arguments in analysis as far as they are based on compactness arguments and to extend them to non Hausdorff asituations.

We are just writing down the Choquet type theorems for the usual powerdomains in semantics: demonic, angelic, erratic.

The construction contains more potential to be exploited:

- The continuous domains need not be directed complete.

- Integrals of lower semicontinuous functions should be directly definable by a completion of the vector space V generated by the characteristic functions.

Lower SemiContinuous Functions

A Scott-continuous valuation m on $O(X)$ will simply be called a valuation on X. We denote by $V(X)$ the set of all valuations on X.

$f : X \to \overline{R}_+$ is lower semicontinuous (lsc, for short) if $[f(x) > r] = \{x \in X | f(x) > r\}$ is open for all r. Denote by $LSC(X)$ the set of all these lsc functions.

$f, f' \in LSC(X) \Rightarrow f + f' \in LSC(X),\ rf \in LSC(X)$ for $r \geq 0$,
$m, m' \in V(X) \Rightarrow m + m' \in V(X),\ rm \in V(X)$ for $r \geq 0$.

We say the V(X) and $LSC(X)$ are cones. With respect to the pointwise defined orders $LSC(X)$ and $V(X)$ are dcpos.

The Choquet Integral

For $\in LSC(X)$ and $m \in V(X)$, the function $r \mapsto m([f(x) > r]) : \overline{R}_+ \to \overline{R}_+$ is monotione decreasing and, hence, has an (improper) Rieman integral and we define

$$\int f dm = \int_0^\infty m([f(x) > r]) dr$$

Riesz Representation Theorem For every fixed $m \in V(X)$, the map $f \mapsto \int f dm : LSC(X) \to \overline{R}_+$ is linear and Scott-continuous. For every linear Scott-continuous map $M : LSC(X) \to \overline{R}_+$, there is an $m \in V(X)$ such that $M(f) = \int f dm$ for all $f \in LSC(X)$.

Theorem For every fixed $f \in LSC(X)$, the map $f \mapsto \int f dm : V(X) \to \overline{R}_+$ is linear and Scott-continuous. For every linear Scott-continuous map $F : V(X) \to \overline{R}_+$, there is an $f \in LSC(X)$ such that $F(m) = \int f dm$ for all $m \in V(X)$, provided that X is a continuous dcpo with the Scott topology.

The second partof the previous theorem is not true for arbitrary spaces. We have to endow $V(X)$ with a coarser topology, the weak*topology, which is the coarsest topology such that the maps $m \mapsto \int f dm : V(X) \to \overline{R}_+$ are continuous for every $f \in LSC(X)$:

Schröder-Simpson-Theorem For any topological space X endow $V(X)$ with the weak*topology. Then, for every lower semicontinuous linear functional $F : V(X) \to \overline{R}_+$, there is an $f \in LSC(X)$ such that $F(m) = \int f dm$ for all $m \in V(X)$.