Abstract ordered compact convex sets and the algebras of the (sub-)probabilistic power domain monad over ordered compact spaces

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December 6, 2008

1 Introduction

The problem that has motivated the investigations in this paper comes from denotational semantics of programming languages. Every program has a type; with every type σ one associates a semantic domain D_σ; programs of type σ are interpreted by elements of D_σ. If one adds a feature to a programming languages, it has to be modelled by an adequate construction on the semantic domains. This construction has to be free in a certain sense so that the model does not have properties not intended in the language. It has turned out that the category theoretical notion of a *monad* captures well this requirement of freeness (see [2]). But having a free construction, one would like to know the structures for which the construction is free, that is, one would like to characterize the *(Eilenberg-Moore) algebras* of the monad.

We are concerned with languages having probabilistic features. Adding probabilistic choice to a deterministic language requires the construction of a *probabilistic powerdomain* $\mathcal{P}D$ over every semantic domain D. This powerdomain may consist of some kind of probability measures on D, but more often of subprobabilities, i.e., positive measures μ with total mass $\mu(D) \leq 1$ the difference $1 - \mu(D)$ expressing the probability of nontermination of the program denoted by μ. We denote by $\mathcal{M}_{\leq 1}D$ the *subprobabilistic powerdomain*.

Most categories used in denotational semantics are of a topological nature. One of them is the category of stably compact spaces and continuous maps. In [5], Cohen, Escardo and the author began with the investigation of the Eilenberg-Moore algebras of the extended probabilistic powerdomain monad over this category. The problem turned out to be difficult.

As a first step, in [11], the author attacked a simpler more classical problem. He considered the category of ordered compact spaces X and order-preserving continuous maps in the sense of Nachbin [15]. Then $\mathcal{P}X$ is the space of probability measures with the vague topology and the *stochastic order* introduced by Edwards [6]. The algebras were characterized to be the compact convex subsets of ordered locally convex topological vector spaces. This extends an old result by Swirszcz [21] on the algebras of the monad of probability measures over compact Hausdorff spaces (without order). The proof in [10] was inspired by a proof for Swirszcz's result reproduced by Semadeni [20] and this proof required quite some functional analytic tools.

In this paper we achieve the following: 1 - We reprove the above result from [10] and extend it to the subprobabilistic case (see Section 8). 2 - We develop topological tools which avoid the use of functional analysis (see Section 2. For this, we build on previous results by Lawson and Madison [12, 13] in the unordered case. We hope that the more topological approach may be useful for the stably compact case, as in the non-Hausdorff case functional analytic methods do not apply readily. 3 - The algebras of the (sub)probabilistic powerdomain monad inherit the barycentric operations that satisfy the same equational laws as those in vector spaces. We show that it is convenient first to embed these abstract convex sets in abstract cones which are easier to handle. 4 - Our embedding theorems for abstract ordered locally compact cones and compact convex sets in ordered topological vector spaces are of interest in themselves (see Section 6).

For the connection between the problem for stably compact spaces and the more classical problem discussed here we refer to the concluding section in [11]. Stably compact spaces and their relation to ordered compact spaces are discussed in detail in [7].

The category theoretic notions of a monad and its Eilenberg-Moore algebras are used without further explanation. The relevant background information can be found in standard books on category theory as, for example, [14].

$\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{I}$ denote the reals, the nonnegative reals and the unit interval $[0, 1]$, respectively, with their usual order and topology. Our vector spaces are always meant to be vector spaces over the reals.

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Supported by DFG and RFFI

2 A topological Lemma

It is a classical question in general topology under which conditions the quotient space $X/\cong$ of a topological space X modulo an equivalence relation $\cong$ satisfies the Hausdorff separation axiom. As a space Y is Hausdorff if and only if the diagonal Δ is closed in the product space $Y \times Y$, a necessary condition is that the graph $G_{\cong} = \{(x, y) \in X \times X \mid x \cong y\}$ of the equivalence relation is closed. If X is a compact Hausdorff space, this condition is also sufficient. In Bourbaki [3, §10, Exercise 19] one finds the following nontrivial generalisation:

Lemma 2.1. *Let X be a locally compact, σ-compact Hausdorff space. If $\cong$ is an equivalence relation the graph of which is closed in $X \times X$, then the quotient space $X/\cong$ satisfies the Hausdorff separation axiom.*

Bourbaki also provides a hint to a counterexample that shows that the lemma becomes wrong if the hypothesis of σ-compactness is omitted. We will prove a variant of Bourbaki's Lemma below.

We consider a topological space X with a *preorder* $\lesssim$, that is, a reflexive transitive relation. If the graph $G_{\lesssim} = \{(x, y) \mid x \lesssim y\}$ is closed in $X \times X$, we say that X is a *preordered topological space* according to Nachbin [15]. Associated with the preorder $\lesssim$ is the equivalence relation $x \cong y$ iff $x \lesssim y$ and $y \lesssim x$. If the graph of the preorder is closed in $X \times X$, the same holds for the graph $G_{\cong}$ of the associated equivalence relation. The quotient $X/\cong$ is partially ordered by $\tilde{x} \leq \tilde{y}$ iff $x \lesssim y$, where $\tilde{x}$ denotes the equivalence class of x. Recall that a partial order is an antisymmetric preorder. We ask the question whether the quotient space $X/\cong$ with the quotient order $\leq$ is an *ordered topological space*, i.e., whether the graph $G_{\leq}$ of the partial order is closed in $X/\cong \times X/\cong$. Before giving an answer we need some preparations.

The following lemma has been proved by Nachbin for spaces with a closed partial order [15, Proposition 4 and Theorem 4]. His proof carries over to arbitrary closed binary relations:

Lemma 2.2. *Let X be a topological space with a binary relation the graph G of which is closed.*

(a) For any compact subset K, the lower set *and the* upper set

$$\downarrow K =_{\rm def} \{x \in X \mid (x, b) \in G \textit{ for some } b \in K\}$$

$$\uparrow K =_{\rm def} \{x \in X \mid (b, x) \in G \textit{ for some } b \in K\}$$

generated by K are closed in X.

(b) If X is a compact Hausdorff space and if A and B are closed subsets of X such that $(A \times B) \cap G = \emptyset$, then there are closed neighbourhoods U and V of A and B, respectively, such that $(U \times V) \cap G = \emptyset$.

Proof. (a) Suppose $a \notin \downarrow K$. Then, $(a, b) \notin G$ for all $b \in K$. As G is closed, for every $b \in K$, there are open neighbourhoods U_b and V_b for a and b, respectively, such that $U_b \times V_b \cap G = \emptyset$. As the compact set K is contained in the union of the open sets V_b, $b \in K$, there is a finite subset F of K such that K is contained in the union V of the V_b, $b \in F$. The intersection U of the finitely many open sets U_b, $b \in F$, is an open neighbourhood of a again. Moreover, $U \times V \cap G = \emptyset$ which implies $U \times K \cap G = \emptyset$. We have found an open neighborhood U of a disjoint from $\downarrow K$. As this can be done for all $a \notin \downarrow K$, we have shown that $\downarrow K$ is closed.

(b) Suppose that, for every closed neighbourhood U of A and every closed neighborhood V of B, one has $U \times V \cap G \neq \emptyset$, then the sets of this form constitute a filter basis of nonempty closed sets which, in the compact space $X \times X$, has a nonempty intersection. Let (x, y) be an element in this intersection. Then x belongs to every closed neighbourhood U of A. As A is the intersection of its closed neighbourhoods, we infer $x \in A$. Similarly one shows that $y \in B$ whence $(x, y) \in A \times B \cap G$ which contradicts the hypothesis. □

The conclusion in the preceding lemma can be strengthened if the relation G is a preorder:

Corollary 2.3. *Let X be a compact Hausdorff space with a preorder $\lesssim$ the graph of which is closed in $X \times X$. Let A and B be closed subsets of X such that $a \not\lesssim b$ for all $a \in A$ and all $b \in B$. Then there are disjoint closed neighbourhoods U and V of A and B, respectively, where U is an upper and V a lower set.*

Proof. From Lemma 2.2(b) we obtain closed neighborhoods U and V of A and B, respectively, such that $u \not\lesssim v$ for all $u \in U$ and all $v \in V$. By reflexivity, $U \subseteq \uparrow U$ and $V \subseteq \downarrow V$. The sets $\uparrow U$ and $\downarrow V$ are closed by Lemma 2.2(a). Using transitivity we obtain that $\uparrow U$ and $\downarrow V$ are disjoint. □

We now are prepared to prove the crucial generalisation of Bourbaki's lemma 2.1 above:

Main Lemma 2.4. *Let X be a locally compact, σ-compact Hausdorff space with a preorder $\lesssim$ the graph of which is closed in $X \times X$. Let $\cong$ be the equivalence relation associated with the preorder $\lesssim$, i.e., $a \cong b$ iff $a \lesssim b$ and $b \lesssim a$. Then the graph of the quotient order $\leq$ on the quotient space $X/\cong$ is closed.*

Proof. It suffices to consider arbitrary elements a, b in X with $a \not\lesssim b$ and to show that there are disjoint open neighborhoods V, W of a, b, respectively, with $V = \uparrow V$ and $W = \downarrow W$.

Thus, let $a \not\lesssim b$ in X. Using the hypotheses of local compactness and σ-compactness we may find a sequence $(U_n)_n$ of relatively compact open set covering X such that $\overline{U_n} \subseteq U_{n+1}$ (see [3, §9, Proposition 15]). We may suppose that $a, b \in U_0$. By Corollary 2.3 applied to the compact Hausdorff space $\overline{U_0}$, we may find inside $\overline{U_0}$ disjoint closed relative neighbourhoods V_0, W_0 of a, b, respectively, where V_0 is an upper and W_0 a lower subset of $\overline{U_0}$. We now form the upper set $\uparrow V_0$ and the lower set $\downarrow W_0$ in X, and we consider the sets $\uparrow V_0 \cap \overline{U_1}$ and $\downarrow W_0 \cap \overline{U_1}$. Again by Corollary 2.3 now applied to the compact Hausdorff space $\overline{U_1}$, we may find inside $\overline{U_1}$ disjoint closed relative neighbourhoods V_1, W_1 of $\uparrow V_0 \cap \overline{U_1}$ and $\downarrow W_0 \cap \overline{U_1}$, respectively, where V_1 is an upper and W_1 a lower subset of $\overline{U_1}$. Recursively, we may find

sets $V_n, W_n \subseteq \overline{U_n}$ which are closed and disjoint,
V_n is an upper and W_n a lower set in $\overline{U_n}$,
inside $\overline{U_{n+1}}$, V_{n+1} is a relative neighbourhood of $\uparrow V_n \cap \overline{U_{n+1}}$ and W_{n+1} a neighbourhood of $\downarrow W_n \cap \overline{U_{n+1}}$.

From the construction it follows that the sequences (V_n) and (W_n) are increasing. We now form the sets $V = \bigcup_n \uparrow V_n$ and $W = \bigcup_n \downarrow W_n$. Clearly, V is an upper set and W a lower set disjoint from V. It remains to show that V and W are open. Indeed, let $x \in V$. We can find an n such that both $x \in U_n$ and $x \in \uparrow V_n$. By construction, V_{n+1} is a neighbourhood of $\uparrow V_n \cap \overline{U_{n+1}}$ relative to $\overline{U_{n+1}}$. As x belongs to U_n which is open in X, V_{n+1} is a neighbourhood of x in X. This shows that V is a neighbourhood of each of its points. □

3 Cones and convex sets

In a real vector space V a subset C is understood to be a *cone*, if $x + y \in C$ and $ra \in C$ for all $a, b \in C$ and every nonnegative real number r. A subset A is *convex*, if $(1-p)a + pb \in A$ for all $a, b \in A$ and every real number p with $0 \leq p \leq 1$. We generalise to an abstract notion of a cone:

Definition . An *abstract cone* is a set C with an addition $(x, y) \mapsto x + y \colon C \times C \to C$, which is commutative and associative and admits a neutral element 0, and a multiplication by nonnegative real numbers $(r, x) \mapsto r \cdot x \colon \mathbb{R}_+ \times C \to C$ satisfying the same equational laws as vector spaces (see, e.g., [10]), i.e., for all $x, y, z \in C$ and all $r, s \in \mathbb{R}_+$:

$$
\begin{aligned}
x + (y + z) &= (x + y) + z \\
x + y &= y + x \\
x + 0 &= x \\
r \cdot (x + y) &= r \cdot x + r \cdot y \\
(r + s) \cdot x &= r \cdot x + s \cdot x \\
(rs) \cdot x &= r \cdot (s \cdot x) \\
1 \cdot x &= x \\
0 \cdot x &= 0
\end{aligned}
$$

A map $f \colon C \to D$ between cones is said to be *linear* if for all $x, y \in C$ and all $r \in \mathbb{R}_+$:

$$
\begin{aligned}
f(x + y) &= f(x) + f(y) \\
f(r \cdot x) &= r \cdot f(x)
\end{aligned}
$$

A subset A of a vector space or, more generally, of an abstract cone is *convex*, if $(1-p)a + pb \in A$ for all $a, b \in A$ and every real number p with $0 \leq p \leq 1$. They are abstract convex sets in the following sense:

Definition . An *abstract convex set* or *barycentric algebra* is a set A endowed with a binary operation $a +_p b$ for every p in the unit interval $\mathbb{I} = [0, 1]$ such that the following equational laws hold, where $p' = 1 - p$:

$$
\begin{aligned}
a +_1 b &= b \\
a +_p a &= a \\
a +_p b &= b +_{p'} a \\
(a +_q b) +_p c &= a +_{(p'q')'} (b +_{\frac{p}{(p'q')'}} c)
\end{aligned}
$$

A map $f \colon A \to B$ between barycentric algebras is *affine* if for all $a, b \in A$ and $0 \leq p \leq 1$:

$$
f(a +_p b) = f(a) +_p f(b)
$$

Cones and convex sets in vector spaces satisfy these laws with $a +_p b = (1-p)a + pb$, as one easily verifies. Not all abstract cones are embeddable in vector spaces. For example, any $\vee$-semilattice C with a smallest element 0 becomes a cone if we define $a + b =_{\text{def}} a \vee b$ and $ra =_{\text{def}} a$, if $r > 0$, and $ra =_{\text{def}} 0$, if $r = 0$. Similarly, every $\vee$-semilattice can be viewed as a barycentric algebra with $a +_p b = a \vee b$ for $0 < p < 1$.

An abstract cone C is embeddable in a real vector space V if and only if it satisfies the following *cancellation property*:

(C) $$a + b = a + c \implies b = c$$

Clearly a cone in a vector space satisfies this property. For the converse one uses the following:

Standard construction 3.1. We define a relation $\cong$ on $C \times C$ by

$$(a, b) \cong (a', b') \iff a + b' = a' + b$$

The relation $\cong$ is an equivalence relation, if we suppose the cancellation axiom (C) to hold (which is needed for transitivity). Moreover, $\cong$ is a congruence relation, i.e., compatible with addition and scalar multiplication, and the quotient $V =_{\text{def}} C \times C/\cong$ is a vector space. We have a natural linear embedding $\eta\colon C \to V$ given by $\eta(a) = \tilde{a}$, the congruence class of $(a, 0)$ mod $\cong$.

W. Neumann [16] has shown that a barycentric algebra A is embeddable in a real vector space as a convex set in such a way that $a +_p b$ becomes $(1-p)a + pb$ if and only if the following cancellation axiom holds in A:

(C′) $$\text{For every } p \text{ with } 0 < p < 1, \quad a +_p b = a +_p c \implies b = c$$

Calculations in barycentric algebras are tedious. We show that every barycentric algebra A is embeddable as a convex subset in an abstract cone C_A by the following:

Standard construction 3.2. For a given barycentric algebra A, let

$$C_A =_{\text{def}} \{0\} \cup \{(r, a) \mid 0 < r \in \mathbb{R}, a \in A\} = \{0\} \cup (]0, +\infty[\times A)$$

Define addition and multiplication with scalars $r > 0$ by:

$$(r, a) + (s, b) =_{\text{def}} (r + s, a +_{\frac{s}{r+s}} b), \qquad r(s, a) =_{\text{def}} (rs, a)$$

For $r = 0$, one puts $r(s, a) = 0$ and addition with 0 is defined in the obvious way. Simple calculations show:

C_A becomes a cone and the map $e = a \mapsto (1, a)$ is an injection of A into C_A in such a way that $e(a +_r b) = (1-r)a + rb$. The cancellation axiom (C) holds in C_A if and only if (C') holds in A.

Thus the question of embeddability of barycentric algebras in vector spaces is reduced to the embedding of cones in vector spaces. We will identify the elements $a \in A$ with the elements $(1, a) \in C_A$ thus identifying A with the convex subset $1 \times A$ of C_A. In this way A becomes a *base* of the cone C_A in the sense that A is convex and that every element $x = (r, a) \neq 0$ in C_A can be written in the form $x = ra$, where r and a are uniquely determined by x.

4 Ordered cones and ordered convex sets

Definition . (a) An *ordered abstract cone* is an abstract cone C with a partial order $\leq$ the graph of which is a subcone of $C \times C$ which is equivalent to the axiom $a \leq b \implies a + c \leq b + c$ and $ra \leq rb$ for all $r \in \mathbb{R}_+$.

(b) An *ordered barycentric algebra* is a barycentric algebra A with a partial order $\leq$ such that the barycentric operations $a +_p b$ are order preserving for every $p \in \mathbb{I}$.

As every vector space is a cone, the above definition of an ordered cone yields a notion of an ordered vector space which is the usual one (see [19, Ch. V.1]). Every subcone and every convex subset of an ordered vector space becomes an ordered cone and an ordered barycentric algebra, respectively, with respect to the induced order.

An ordered abstract cone can be embedded in an ordered vector space if and only if its satisfies the following *order cancellation axiom*:

(OC) $$a + b \leq a + c \implies b \leq c$$

This axiom is cleary satisfied in subcones of ordered cones. For the converse we continue with the standard construction for cones from Section 2:

Standard construction 4.1. For an ordered abstract cone we define a relation $\lesssim$ on $C \times C$ by:

$$(a, b) \lesssim (a', b') \iff a + b' \leq a' + b$$

Supposing order cancellation, the relation $\lesssim$ on $C \times C$ is a preorder compatible with addition and scalar multiplication. The associated equivalence relation is the relation $\cong$ from 3.1. On the vector space $V = C \times C/\cong$ the quotient order $\widetilde{(a, b)} \leq_V \widetilde{(a', b')}$ iff $(a, b) \lesssim (a', b')$ is a partial order such that V becomes an ordered vector space. The canonical injection $\eta\colon C \to V$ is not only linear but also an order embedding.

An ordered barycentric algebra A can be embedded in an ordered abstract cone:

Standard construction 4.2. We use the embedding of A in the abstract cone C_A as in 3.2 and we extend the order on A by defining an order $\leq$ on C_A by $0 \leq 0$ and:

$$ra \leq sb \iff r = s \text{ and } a \leq b \text{ in } A$$

With this order, C_A becomes an ordered abstract cone. The order cancellation axiom (OC) holds in the cone C_A if and only if the following order cancellation axiom holds in A for every $p > 0$:

$$(\mathrm{OC}') \qquad a +_p b \leq a +_p c \implies b \leq c$$

Thus, an ordered barycentric algebra can be embedded in an ordered vector space if and only if it satisfies order cancellation.

5 Topological cones and convex sets

Definition . (a) A *topological abstract cone* is an abstract cone C with a topology such that addition $(x, y) \to x+y\colon C \times C \to C$ and scalar multiplication $(r, x) \to rx\colon \mathbb{R}_+ \times C \to C$ are continuous.

(b) A *topological barycentric algebra* is a barycentric algebra A with a topology such that the map $(r, a, b) \mapsto a +_r b\colon \mathbb{I} \times A \times A \to A$ is continuous.

Cones and convex sets in topological vector spaces are topological cones and topological barycentric algebras, respectively, for the induced topology. The embeddability of topological abstract cones into topological vector spaces is a difficult question that cannot be answered in general. We will heavily rely on results due to Lawson and Madison [13].

The following lemma follows from [11, Corollary 4.2]; alternatively, it also follows from Lemma 6.1 for which we give a simple proof below:

Lemma 5.1. *Every Hausdorff topological abstract cone satisfies the cancellation axiom (C).*

The following Theorem is a slight strengthening of results due to Lawson and Madison [13, Corollary 2.4, Theorem 3.1 and Theorem 3.2] in as far as, by Lemma 5.1, we can drop their hypothesis that the cone satisfies the cancellation axiom:

Theorem 5.2. *Let C be a locally compact Hausdorff topological abstract cone. Then C is σ-compact and satisfies the cancellation property* (C). *The vector space $V = C \times C/\cong$ with the quotient topology is a topological vector space and the canonical map $\eta\colon C \to V$ is a linear topological embedding.*

Of course, one wants to know under which conditions an ordered locally compact cone can be embedded in a locally convex topological vector space. For topological cones and barycentric algebras there are various notions of local convexity. They have been proved to be equivalent for locally compact Hausdorff topological cones by Lawson [12]. Thus, we choose the formally weakest among them:

Definition . A topological abstract cone [barycentric algebra] C is called *weakly locally convex* if each of its points has a basis of convex neighbourhoods.

A result due to Lawson [12, Theorem 5.3] tells us:

Theorem 5.3. *For every weakly locally convex, locally compact Hausdorff topological cone C, the vector space $V = C \times C/\cong$ with the quotient topology is a locally convex topological vector space.*

In fact, Lawson supposed his cone to satisfy the cancellation axiom, a hypotheis which we may drop because of Lemma 5.1.

Standard construction 5.4. The standard embedding of a barycentric algebra A into a cone C_A (see 3.2) can be extended to a topological embedding of a topological barycentric algebra A into a topological cone by defining the following topology on C_A: On the points of C_A different from 0 we take the product topology of $]0, +\infty[\times A$ and as a neighborhood basis of 0 we take the sets of the form $0 \cup (]0, \varepsilon[\times A) = 0 \cup \{ra \mid 0 < r < \varepsilon, a \in A\}$. Note that A is embedded in C as a closed subset. If A is compact Hausdorff, then C_A is locally compact Hausdorff, and if A is weakly locally convex, then C_A also is (compare [12, Proposition 2.1]). From the above Theorems 5.2 and 5.3 one immediately deduces:

Corollary 5.5. *Every compact Hausdorff topological barycentric algebra A is affinely and topologically embeddable in a topological vector space V which is locally convex, if A is weakly locally convex.*

This is just a slight generalisation of Lawson's [12, Corollary 4.2], as we can omit the hypothesis that cancellation holds in A.

6 Ordered topological cones and convex sets

We now mix order and topology. It is our aim to generalise the results of the previous section to ordered topological cones and ordered topological barycentric algebras:

Definition . An *ordered topological abstract cone* [*ordered topological barycentric algebra*] is an abstract cone [a barycentric algebra] C with an order $\leq$ and a topology such that the graph

$$G_{\leq} = \{(a, b) \mid a \leq b\}$$

of the order is closed in $C \times C$ and such that C is both an ordered and a topological abstract cone [barycentric algebra, respectively].

The above definition applied in the special case of vector spaces yields the usual notion of an ordered topological vector space (see [19, Ch. V.4]). As in any ordered topological space, the topology of an ordered topological cone and of an ordered topological barycentric algebra is Hausdorff (see Nachbin [15]). The following is a consequence of [11, Proposition 4.1]; here we give a simpler independent proof:

Lemma 6.1. *Every ordered topological abstract cone satisfies the order cancellation axiom (OC).*

Proof. Let indeed $a + b \leq a + c$. Then $\frac{1}{2}(a + b) \leq \frac{1}{2}(a + b)$, whence $\frac{1}{2}a + b = \frac{1}{2}(a + b) + \frac{1}{2}b \leq \frac{1}{2}(a + c) + \frac{1}{2}b = \frac{1}{2}(a + b) + \frac{1}{2}c \leq \frac{1}{2}(a + c) + \frac{1}{2}c = \frac{1}{2}a + c$. Repeating this argument we obtain $\frac{1}{2^n}a + b \leq \frac{1}{2^n}a + c$ for every natural number n. For $n \to +\infty$ we deduce $b \leq c$ using the continuity of addition and scalar multiplication and the closedness of the graph of the order. $\square$

We now consider an ordered locally compact abstract cone C. By Lemma 6.1 the order cancellation axiom holds in C. Therefore we may consider the preorder $\lesssim$ on $C \times C$ as in 4.1 and the associated equivalence relation $\cong$:

Theorem 6.2. *For an ordered locally compact topological abstract cone C, the vector space $V = C \times C/\cong$ with the quotient topology and the quotient order $\leq_V$ is an ordered topological vector space and the canonical map $\eta\colon C \to V$ is a linear topological order embedding. If C is weakly locally convex, V is locally convex.*

Proof. From Theorem 5.2 we know that V is a topological vector space which by 5.3 is locally convex, if C is weakly locally convex. In order to prove that V is an ordered topological vector space, it suffices to show that the graph of the quotient order $\leq_V$ is closed in $V \times V$.

We first remark that the graph of the preorder $\lesssim$ on $C \times C$ is closed, as it is the preimage of the graph of the order $\leq$ on C under the continuous map $\big((a, b), (a', b')\big) \mapsto (a + b', a' + b)\colon (C \times C) \times (C \times C) \to C \times C$ and as the graph of the order on C is closed in $C \times C$. It follows that the graph of the equivalence relation $\cong$ is also closed. We form the quotient $V = C \times C/\cong$. As $C \times C$ is a locally compact Hausdorff topological cone, it is σ-compact (see 5.2). We apply the Main Lemma 2.4 to $X = C \times C$ with the preorder $\lesssim$ and we conclude that the graph of the quotient order $\leq_V$ on $V = C \times C/\cong$ is closed. Thus, V is an ordered topological vector space. $\square$

Note that in the above proof also shows that the quotient topology on V is Hausdorff, a fact that was a major step in the proof of Theorem 5.2 by Lawson and Madison.

We now turn to an ordered topological barycentric algebra A. Using the standard construction 4.2, A can be embedded in an abstract cone C_A which is also an ordered and a topological cone. As the graph of the order on C_A is easily seen to be closed, C_A is an ordered topological cone. If A is compact, then C_A is locally compact by Corollary 5.5 and we conclude:

Lemma 6.3. *Every ordered compact barycentric algebra A is embeddable as a base in an ordered locally compact topological cone C_A.*

We may embed the ordered locally compact cone C_A in an ordered topological vector space by Theorem 6.2. Combining this embedding with the embedding of A in C_A, we have:

Corollary 6.4. *Every ordered compact barycentric algebra A admits an affine topological order embedding in an ordered topological vector space V. The vector space V can be chosen to be locally compact, if A is weakly locally convex.*

We turn to a slight modification of the previous considerations. We consider the equational characterisation of convex subsets containing the origin 0 in vector spaces:

Definition . A *pointed* barycentric algebra is a barycentric algebra B together with a distinguished element 0. A map $f\colon B \to B'$ between pointed barycentric algebras is called 0-affine, if it is affine and if $f(0) = 0$.

It should be clear what we mean by pointed ordered, topological, ordered topological barycentric algebras, respectively. Our embedding theorem is as follows:

Corollary 6.5. *For every pointed ordered compact barycentric algebra B there is a 0-affine topological order embedding in a topological vector space V, which can be chosen to be locally convex, if B is weakly locally convex.*

Proof. By Corollary 6.4, there is an affine, topological order embedding of B into an ordered topological vector space V, which can be chosen to be locally convex, if B is weakly locally convex. If the distinguished element 0 of B is not mapped to zero but to an element $z \in V$, we apply the shift $x \mapsto x - z$ in V and we obtain the desired result. □

7 The monad of [sub-]probability measures over compact Hausdorff spaces

In this section we develop the monad of subprobability measures along the same lines as the monad of probability measures in [11, Section 5]. Omitting the items between square brackets yields the previous results.

For compact Hausdorff spaces X, we shall use the following notations:

$\mathcal{C}X$	the Banach space of all real valued continuous functions on X with the topology of uniform convergence,
$\mathcal{C}_+X$	the positive cone of all nonnegative functions $f \in \mathcal{C}X$,
$\mathcal{M}X$	the vector space of all signed regular Borel measures on X,
$\mathcal{M}_+X$	the cone of positive regular Borel measures,
$\mathcal{P}X$	the set of probability measures, i.e., the positive regular Borel measures φ of total mass 1,
$s\mathcal{P}X$	the set of *subprobability measures*, i.e., the positive regular Borel measures φ of total mass ≤ 1.

By $\leq$ we denote on $\mathcal{C}X$ the usual pointwise defined order with $\mathcal{C}_+X$ as positive cone and on $\mathcal{M}X$ the usual order of measures with $\mathcal{M}_+X$ as positive cone. Via the Riesz Representation Theorem we will identify $\mathcal{M}X$ with the dual vector space $(\mathcal{C}X)^*$ of all bounded linear functionals φ on $\mathcal{C}X$. For $\varphi \in \mathcal{M}X$ and $f \in \mathcal{C}X$, we will write

$$\langle \varphi, f \rangle = \int f d\varphi$$

for the natural bilinear map $\mathcal{M}X \times \mathcal{C}X \to \mathbb{R}$.

$\mathcal{M}X$ is a locally convex topological vector space with respect to the *weak* topology* also called the *vague topology*. This is the coarsest topology on $\mathcal{M}X$ for which the linear maps $\varphi \mapsto \langle \varphi, f \rangle$ are continuous for all $f \in \mathcal{C}X$. On $\mathcal{M}X, \mathcal{M}_+X, \mathcal{P}X, s\mathcal{P}X$ we will always consider the weak* topology.

Lemma 7.1. *$\mathcal{M}_+X$ is a locally compact cone in $\mathcal{M}X$. The* [*sub-*]*probability measures form a* [*pointed*] *compact convex subset* $[s]\mathcal{P}X$.

The compactness of $[s]\mathcal{P}X$ follows from the weak* compactness of the dual unit ball in $\mathcal{M}X$ in which it is closed. The local compactness of $\mathcal{M}_+X$ follows from the fact that with respect to the relative topology $s\mathcal{P}X = \{\varphi \in \mathcal{M}_+X \mid \langle\varphi, \mathbf{1}\rangle \leq 1\}$ is a compact neighbourhood of every φ with $\langle\varphi, \mathbf{1}\rangle < 1$. (Here $\mathbf{1}$ denotes the constant function on X with value 1).

Assigning the Dirac measure $\varepsilon_X(x)$ to every $x \in X$ yields a continuous embedding

$$\varepsilon_X \colon X \to \mathcal{P}X \subseteq s\mathcal{P}X \subseteq \mathcal{M}X$$

Let us specialise and choose for X a [pointed] compact Hausdorff topological barycentric algebra K. By Corollaries 6.4 and 6.5 we can suppose that K is a compact convex set [containing 0] in a topological vector space. The continuous [0-]affine real-valued functions on K form a uniformly closed linear subspace $\mathcal{A}^{[\circ]}K$ of $\mathcal{C}K$. Restricting every $\varphi \in \mathcal{M}K$ to $\mathcal{A}^{[\circ]}K$ yields a surjective linear map

$$\beta_K^{[\circ]} \quad =_{\mathrm{def}} \quad (\varphi \mapsto \varphi|_{\mathcal{A}^{[\circ]}K}) \colon \mathcal{M}K \to (\mathcal{A}^{[\circ]}K)^*$$

where $(\mathcal{A}^{[\circ]}K)^*$ is the dual vector spaces of all bounded linear functionals on $\mathcal{A}^{[\circ]}K$. The map $\beta_K^{[\circ]}$ is continuous and even a quotient maps for the respective weak* topologies.

Composing ε_K with $\beta_K^{[\circ]}$ yields a continuous map from K into $(\mathcal{A}^{[\circ]}K)^*$. A point $x \in K$ is mapped to the point evaluation $f \mapsto f(x) \colon \mathcal{A}^{[\circ]}K \to \mathbb{R}$. We denote this point evaluation in x by $\tilde{x}$. The composed map $\beta_K^{[\circ]} \circ \varepsilon_K$ is [0-]affine as $\langle\varepsilon_K(x +_p y), f\rangle = f(x +_p y) = f(x) +_p f(y) = \langle\varepsilon_K(x), f\rangle +_p \langle\varepsilon_K(y), f\rangle$ [and $\langle\varepsilon_K(0), f\rangle = f(0) = 0$] for all $f \in \mathcal{A}^{[\circ]}K$.

Suppose now that K is a compact convex set [containing 0] in a locally convex topological vector space V. Then the continuous linear functionals $g \colon V \to \mathbb{R}$ separate the points of K. Their restrictions to K are [0-]affine. This implies that the map $\beta_K^{[\circ]} \circ \varepsilon_K \colon K \to (\mathcal{A}^{[\circ]}K)^*$ is injective. Thus, K is topologically and [0-]affinely embedded into $(\mathcal{A}^{[\circ]}K)^*$. Hence-forward, we will identify K with its image $\tilde{K}$ in $(\mathcal{A}^{[\circ]}K)^*$; i.e., $x \in K$ is identified with $\tilde{x} = \beta_K^{[\circ]}(\varepsilon_K(x))$.

We now use that every probability measure on K has a barycenter:

Theorem 7.2. [1, (2.13)] *Let K be a compact convex set in a locally convex topological vector space. Then, for every probability measure $\varphi \in \mathcal{P}K$, there is a uniquely determined $x \in K$ such that*

$$\langle\varphi, f\rangle = f(x) \ \textit{for all}\ f \in \mathcal{A}K$$

The element x is called the barycenter *of φ.*

As a corollary we get:

Corollary 7.3. *Let K be a compact convex set containing 0 in a locally convex topological vector space. Then, for every subprobability measure $\varphi \in s\mathcal{P}K$, there is a uniquely determined $x \in K$ such that*

$$\langle\varphi, f\rangle = f(x) \ \textit{for all}\ f \in \mathcal{A}^\circ K$$

The element x is called the moment *of φ.*

Proof. If $\varphi = 0$, then choose $x = 0$. If $0 \neq \varphi \in s\mathcal{P}K$, let $r = \langle\varphi, \mathbf{1}\rangle$. Then $0 < r \leq 1$ and $r^{-1}\varphi$ is a probability measure. By Theorem 7.2, there is a unique $y \in K$ such that $\langle r^{-1}\varphi, f\rangle = f(y)$ for all $f \in \mathcal{A}K$. As the convex set K contains 0 and as $r \leq 1$, the element $x = ry = (1-r)\cdot 0 + ry$ belongs to K and, for all $f \in \mathcal{A}^\circ K$, we have indeed $\langle\varphi, f\rangle = rf(y) = f(ry) = f(x)$. ☐

This theorem and its corollary tell us that $\beta_K^{[\circ]}(\varphi) = \tilde{x}$, whenever φ is a [sub-]probability measure and x its [moment] barycenter. This implies that $\beta_K^{[\circ]}$ maps $[s]\mathcal{P}K$ onto the image $\tilde{K}$ of K. Thus, when restricted to $[s]\mathcal{P}K$, $\beta_K^{[\circ]}$ assigns its [moment] barycenter to every [sub-]probability measure on K.

We can apply the preceding developments to the [pointed] compact convex set $K = [s]\mathcal{P}X$ of [sub-]probability measures on a compact Hausdorff space X. We then obtain a surjective continuous 0-affine map

$$\mu_X^\circ =_{\mathrm{def}} \beta_{s\mathcal{P}X}^\circ \colon s\mathcal{P}\, s\mathcal{P}X \to s\mathcal{P}X$$

and a surjective affine continuous map

$$\mu_X =_{\mathrm{def}} \beta_{\mathcal{P}X} \colon \mathcal{P}\mathcal{P}X \to \mathcal{P}X$$

It is standard to deduce from the above:

Proposition 7.4. *$(s\mathcal{P},\mu^\circ,\varepsilon)$ and $(\mathcal{P},\mu,\varepsilon)$ define monads over the category* Comp *of compact Hausdorff spaces and continuous maps.*

The unit ε_X assigns the Dirac measure $\varepsilon_X(x)$ to every $x \in X$. The multiplication $\mu^\circ_X\colon s\mathcal{P}\, s\mathcal{P}X \to s\mathcal{P}X$ assigns to each subprobability measure Φ on $s\mathcal{P}X$ its moment $\mu_X(\Phi) \in s\mathcal{P}X$ and similarly for the probabilistic case. From the above, compact convex sets K [containing 0] in locally convex topological vector spaces are easily seen to be Eilenberg-Moore algebras of these respective monads with the [moment] barycentric map $\beta^{[\circ]}_K$ as structure map.

The following theorem is due to Swirszcz [21] in the probabilistic case. His proof as well as the proof of Semadeni [20, Section 7] use functional analytic tools. In the next section we prove a generalisation (Theorem 8.5) based on the topological methods developed in this paper.

Theorem 7.5. *The Eilenberg-Moore algebras of the monad $[s]\mathcal{P}$ over the category* Comp *of compact Hausdorff spaces are the compact convex sets K [containing 0] in locally convex topological vector spaces together with the [moment] barycentric maps $\beta^{[\circ]}_K$ as structure maps.*

8 The [sub-]probabilistic powerdomain monad on compact ordered spaces

We now consider ordered compact spaces X in the sense of Nachbin [15], that is, sets X with a compact topology and a partial order $\preccurlyeq$ the graph of which is closed in $X \times X$. Recall that the topology of a compact ordered space satisfies the Hausdorff separation axiom. We will denote by CompOrd the category of compact ordered spaces and order preserving continuous maps. Forgetting the order yields a forgetful functor from the category CompOrd to the category Comp of compact Hausdorff spaces. On the other hand, Comp may be considered to be a full subcategory of CompOrd by putting the trivial order $=$ on each compact Hausdorff space.

We denote by $\mathcal{C}^m X$ the cone of all order preserving continuous functions $f\colon X \to \mathbb{R}$. Clearly, $\mathcal{C}^m X$ is uniformly closed in $\mathcal{C}X$. Using the Stone–Weierstraß Theorem, D. E. Edwards [6] (see also [4, Lemma 19]) has shown:

Lemma 8.1. *The linear subspace of $\mathcal{C}X$ generated by the cone $\mathcal{C}^m X$ is uniformly dense in $\mathcal{C}X$.*

Following Edwards [6], we consider on $\mathcal{M}X$ the *stochastic order* $\preccurlyeq$, a second order which is weaker than the the usual order $\leq$, the positive cone of which is the dual of the cone $\mathcal{C}^m X$:

$$\begin{array}{lll} 0 \preccurlyeq \varphi & \text{if and only if} & 0 \leq \langle \varphi, f\rangle \text{ for all } f \in \mathcal{C}^m X\,, \text{ i.e.,} \\ \varphi \preccurlyeq \psi & \text{if and only if} & \langle \varphi, f\rangle \leq \langle \psi, f\rangle \text{ for all } f \in \mathcal{C}^m X\,. \end{array}$$

By its definition the positive cone for the order $\preccurlyeq$ is weak*-closed. As the linear subspace generated by $\mathcal{C}^m X$ is uniformly dense in $\mathcal{C}X$ by 8.1, this positive cone is indeed pointed. Thus, $\mathcal{M}X$ with the stochastic order is an ordered locally convex topological vector space. Restricting the stochastic order to the compact convex sets $[s]\mathcal{P}X$ of [sub-]probability measures yields a [pointed] ordered compact convex set.

In this section, $s\mathcal{P}X$ and $\mathcal{P}X$ will always be endowed with the stochastic order, although we do not express this by a new notation.

We now consider two compact ordered spaces X and Y and an order preserving continuous map $g\colon X \to Y$. It induces a positive linear map $\mathcal{C}g\colon \mathcal{C}Y \to \mathcal{C}X$ defined by $(\mathcal{C}g)(f) = f \circ g$. If f is order preserving, then $f \circ g$ is order preserving, too. Thus, $\mathcal{C}^m Y$ is mapped into $\mathcal{C}^m X$. The adjoint $\mathcal{M}g\colon \mathcal{M}X \to \mathcal{M}Y$, defined by $(\mathcal{M}g)(\varphi) = \varphi \circ \mathcal{C}g$, is linear and it preserves the orders $\leq$ and $\preccurlyeq$. As $(\mathcal{C}g)(\mathbf{1}) = \mathbf{1} \circ g = \mathbf{1}$, $\mathcal{M}g$ maps $\mathcal{M}_{\leq 1}X$ into $\mathcal{M}_{\leq 1}Y$ and $\mathcal{P}X$ into $\mathcal{P}Y$. Moreover, $\mathcal{M}g$ is continuous for the respective weak* topologies.

Thus, for every ordered compact space X, the set $[s]\mathcal{P}X$ of [sub-]probability measures with the weak* topology and the stochastic order $\preccurlyeq$ is a [pointed] ordered compact barycentric algebra. Every order preserving continuous map $g\colon X \to Y$ of compact ordered spaces induces a $\preccurlyeq$-preserving continuous [0-]affine map $s\mathcal{P}g =_{\mathrm{def}} \mathcal{M}g|_{s\mathcal{P}X}\colon s\mathcal{P}X \to s\mathcal{P}Y$ and $\mathcal{P}g =_{\mathrm{def}} \mathcal{M}g|_{\mathcal{P}X}\colon \mathcal{P}X \to \mathcal{P}Y$, respectively, and we have functors $s\mathcal{P}$ and $\mathcal{P}$ from the category CompOrd of compact ordered spaces to the categories of [pointed] ordered compact barycentric algebras and order preserving continuous [0-]affine functions.

In [11, Section 6] we have shown that $(\mathcal{P},\varepsilon,\mu)$ defines a monad over the category CompOrd. We extend this result to $s\mathcal{P}$ now:

Lemma 8.2. [11, Lemma 6.2] *The map $\varepsilon_X\colon X \to [s]\mathcal{P}X$ is not only a topological but also an order embedding.*

The proof of the following lemma can be copied from [11, Lemma 6.3], only change affine to 0-affine:

Lemma 8.3. *If K is a compact convex set [containing 0] in an ordered locally convex topological vector space, then the [moment] barycentric map $\beta_K^{[\circ]}\colon [s]\mathcal{P}K \to K$ preserves the order $\preccurlyeq$.*

We apply the preceding lemma to the [pointed] ordered compact convex set $K = [s]\mathcal{P}X$ of [sub-]probability measures with the stochastic order over a compact ordered space X and we obtain that the multiplication $\mu_X^{[}\circ] = \beta^{[\circ]}_{[s]\mathcal{P}X}\colon [s]\mathcal{P}\,[s]\mathcal{P}X \to s\mathcal{P}X$ is also $\preccurlyeq$-preserving. We summarize:

Proposition 8.4. $([s]\mathcal{P}, \varepsilon, \mu^{[\circ]})$ *is a monad over the category of compact ordered spaces and continuous order preserving maps.*

From the preceding we can infer that every compact convex set K containing 0 in a locally convex ordered topological vector space is an algebra of the monad $s\mathcal{P}$ with the moment map $\beta_K^\circ\colon s\mathcal{P}K \to K$ as structure map. Similarly, arbitrary compact convex sets in a locally convex ordered topological vector spaces are algebras of the monad $\mathcal{P}$. The converse also holds:

Theorem 8.5. *(a) The Eilenberg-Moore algebras of the subprobabilistic monad $(s\mathcal{P}, \varepsilon, \mu^\circ)$ over the category* CompOrd *of ordered compact spaces and continuous order preserving maps are the ordered compact convex sets containing 0 in locally convex ordered topological vector spaces.*

(b) The Eilenberg-Moore algebras of the probabilistic monad $(\mathcal{P}, \varepsilon, \mu)$ over the category CompOrd *are the ordered compact convex sets in ordered locally convex topological vector spaces.*

Proof. (a) Let $\alpha\colon s\mathcal{P}K \to K$ be an algebra for the monad $s\mathcal{P}$ over the category CompOrd. Then α is continuous, $\preccurlyeq$-preserving and it satisfies

$$(1)\ \ \alpha\circ s\mathcal{P}\alpha = \alpha\circ\mu_K^\circ \qquad (2)\ \ \alpha\circ\varepsilon_K = \mathrm{id}_K$$

In K we choose $0 =_{\rm def} \alpha(0)$ as a distinguished point and we define a barycentric structure $(p, a, b) \mapsto a +_p b\colon [0, 1] \times K \times K \to K$ by $a +_p b =_{\rm def} \alpha\big((1-p)\varepsilon_K(a) + p\varepsilon_K(b)\big)$ which is continuous and $\preccurlyeq$-preserving, as α and ε_K are. Equation (1) above implies that α is 0-affine, i.e., it preserves 0 and satisfies $\alpha((1-p)\varphi + p\psi) = \alpha(\varphi) +_p \alpha(\psi)$ for all $\varphi, \psi \in \mathcal{P}K$ and $0 \le p \le 1$. It follows that the equational laws of a pointed barycentric algebra, which hold in $s\mathcal{P}K$, are inherited by K; so K becomes a pointed ordered compact topological barycentric algebra.

We now show that K is weakly locally convex. Let x be any element of K and U an arbitrary neighborhood of x in K. By equantion (2) above, $x = \alpha(\varepsilon_K(x))$. Thus, by the continuity of α, the preimage $\alpha^{-1}(U)$ is a neighbourhood of $\varepsilon_K(x)$ in $s\mathcal{P}K$. As $s\mathcal{P}K$ is locally convex, there is a convex neighborhood V of $\varepsilon_K(x)$ contained in $\alpha^{-1}(U)$. The image $\alpha(V)$ is convex and contained in U. Again by equation (2) above, $\alpha(V)$ contains $\varepsilon_X^{-1}(V)$ which is a neighborhood of x. Thus $\alpha(V)$ is a convex neighborhood of x contained in U.

We now can apply Corollary 6.5 which tells us that K admits an 0-affine topological order embedding in a locally convex ordered topological vector space.

(b) is proved along the same lines using Corollary 6.4 instead of 6.5 ☐

Theorem 8.5 with its topological proof is the main result of this paper. Part (b) has already been proved in [11] with functional analytic tools.

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