Scott–Eršov domains in topology and analysis

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we have a huge advantage over todays young people:

We have been young forty years ago,

and we remember the exciting things and substantial developments that emerged at that time, often forgotten and sometimes rediscovered nowadays,

and we have experienced the profound changements of the world.

Forty years ago

Dana S. Scott

introduced certain lattices as domains for semantics in an Outline of an mathematical theory of computation

he produced his models for the untyped λ-calculus,

and he coined the notion of a continuous lattice.

Forty years ago

Yuri L. Eršov

in his investigations on the Theory of Numerations and his study of Computable functionals of higher type came up with the notion of an f-space,

and he introduced A-spaces, topological spaces with good approximations, a topological version of continuous lattices.

Two approaches: Order and Topology

Posets

Set X with a partial order $\leq$ (information order)

T_0-spaces

Set X with a topology $\mathcal{O}(X)$ such that points can be distinguished by open sets (Open sets $U \in \mathcal{O}(X)$ observable properties).

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Order $\rightarrow$ Topology: Scott topology

For a directed family $(x_i)_i$ write $x_i \nearrow x$, if x is the least upper bound of the x_i. A subset C is Scott-closed if

$$x \leq y \in C \implies x \in C$$

$$x_i \nearrow x, \; x_i \in C \implies x \in C$$

Topology $\rightarrow$ Order: Specialization order

$$x \leq y \iff (x \in U \implies y \in U) \iff x \in closure(y)$$

Semantic Domains

- **D**irected **c**omplete **p**artially **o**rdered set
  continuous dcpos (Scott, Plotkin).
- f-spaces, A-spaces, α-spaces, ... (Eršov).
- Stably compact spaces (Jung).
- **Q**uotients of **c**ountably **b**ased spaces (Schröder, Simpson).

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All these spaces are T_0 but far from being Hausdorff. Order and topology are linked. Continuous maps are order preserving.
Turning the order upside down is not continuous.
Slogan: asymmetric topology.

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A map $f : X \to Y$ of dcpos is Scott-continuous if it is

1. monotone: $x \leq y \Rightarrow f(x) \leq f(y)$
2. lower semicontinuous: $x_i \nearrow x \Rightarrow f(x_i) \nearrow f(x)$

This notion of continuity is equivalent to continuity with respect to the Scott topology.

Continuous domains order theoretically

For elements u and v of a dcpo P we say
$u \ll v$ (u is way-below v or u is relatively compact in v) if

$$x_i \nearrow x \geq v \implies \exists i.\ x_i \geq u$$

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A good approximation of v is a directed family (x_i) such that

$$x_i \nearrow v \text{ and } \forall i.\ x_i \ll v$$

$B \subseteq P$ is a basis if every $v \in P$ has a good approximation by a directed family of elements in B. A dcpo P is called continuous, if it has a basis.

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Example: $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a continuous dcpo;
$r \ll s$ iff $r < s$; the rational numbers form a basis;
the Scott-open sets are the intervals $]r, +\infty]$.

Locally compact spaces

X a locally compact T_0-space (every point has a neighborhood basis of compact neighborhoods),
$\mathcal{O}(X)$, the lattice of open subsets, is a continuous dcpo,
$U \ll V$ iff there is a compact set K such that $U \subseteq K \subseteq V$.

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Most compact subsets are not closed.
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Hofmann Mislove Theorem

In a sober space the intersection of any filter basis of nonempty compact saturated sets is nonempty.

The collection $\mathcal{K}(X)$ of nonempty saturated compact sets is a dcpo for the order $\supseteq$, even a continuous one, provided that X is locally compact, in addition; $K \gg K'$ iff $interior(K) \supseteq K'$.

Continuous domains topologically

In a T_0-space B define the topological way-below relation $u \prec v$ if $\uparrow u = \{x \in B \mid u \leq x\}$ is a neighborhood of v.
Say that B is an α-space if, for each of its points v and each neighborhood U of v, there is an element $u \in U$ such that $u \prec v$.

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In an α-space, every element v has a neighborhood basis of sets of the form $\uparrow u$.
Every α-space is locally compact.
Every continuous dcpo is an α-space in its Scott topology, the topological way-below relation $\prec$ and the order theoretical $\ll$ agree.

The interplay

Theorem (Eršov, Erné)

When B is a basis of a continuous dcpo P, then B is an α-space for the topology induced by the Scott topology on P.
Conversely, every α-space B is the basis of a uniquely determined continuous dcpo $P = \widetilde{B}$ in such a way that the topological way-below relation on B is induced by the way-below relation on P

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The Basic Extension Lemma

Let B be a basis of a continuous dcpo P (or an α-space), $f: B \to Q$ a continuous function into a dcpo Q.
Then f has a unique continuous extension $\widetilde{f}: P = \widetilde{B} \to Q$ defined by

$$\widetilde{f}(v) = \sup_{u \in B, u \ll v} f(u)$$

Case study I: Semantics of Probabilistic Features

In semantics of programming languages one interprets objects of (data) type σ (or τ) by elements of a semantic domain S(or T). A program P with objects of type σ as inputs and objects of type τ as outputs is interpreted by a continuous function $f\colon S \to T$.

Case study I: Semantics of Probabilistic Features

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If the language admits probabilistic choice, the output will be described by a probability distribution depending on the input, i.e., a function from S to the set of probability distributions on T.

We need a construction of a 'domain of probabilities (more generally measures)' over our semantic domains.

Measures and Valuations

Classically Borel measures on Hausdorff spaces X, that is, maps $m\colon \mathcal{B}(X) \to \overline{\mathbb{R}}_+$ defined on the σ-algebra $\mathcal{B}(X)$ generated by the the collection $\mathcal{O}(X)$ of open sets satisfying

finite additivity: $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$

countable continuity: $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$

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For non-Hausdorff spaces replace Borel measures by the simpler

notion of valuations, i.e., maps $m\colon \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ satisfying

strictness	$m(\emptyset) = 0$
modularity	$m(A \cup B) + m(A \cap B) = m(A) + m(B)$
monotonicity	$A \subseteq B \Rightarrow m(A) \leq m(B)$
Scott-continuity:	$A_i \nearrow A \Rightarrow m(A_i) \nearrow m(A)$

Measures and Valuations

The restriction of a Borel measure on X to the open sets is a valuation on $\mathcal{O}(X)$, but not necessarily Scott-continuous, only countably continuous.

But: A regular Borel measure on a locally compact Hausdorff space restricts to a Scott-continuous valuation on the opens. (A Borel measure is inner regular if $m(K_i) \nearrow m(B)$ when the K_i range over the compact subsets of the Borel set B.)

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Conversely, one may ask, whether a given Scott-continuous valuation on $\mathcal{O}(X)$ can be extended to a Borel mesure.

For example, on a locally compact sober space, every Scott-continuous valuation can be extended to a Borel measure (Keimel-Lawson 2005).

The Domain of Valuations

X a T_0-space,
$\mathcal{V}(X)$ the set of all valuations

$\mathcal{V}(X)$ ordered by $\mu \leq \nu$ iff $\mu(U) \leq \nu(U)$ for all $U \in \mathcal{O}(X)$ is a dcpo. Addition and multiplication by reals $r \geq 0$ are Scott-continuous. We say: $\mathcal{V}(X)$ is a dcpo-cone.

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X may be considered to be a subset of $\mathcal{V}(X)$ by identifying $x \in X$ with the Dirac measure δ_x which is a valuation.

Theorem

If X is a continuous domain with the Scott topology (or an α-space), then $\mathcal{V}(X)$ is a continuous dcpo-cone and it has the universal property: For every dcpo-cone C and every Scott-continuous map $f\colon X \to C$, there is a unique Scott-continuous linear map $\widetilde{f}\colon \mathcal{V}(X) \to C$ extending f.

Sketch of Proof

The simple valuations $\sum_{i=1}^{n} r_i \delta_{x_i}$ with $r_i \in \mathbb{R}_+$, $x_i \in X$ form a basis of $\mathcal{V}(X)$.

Given $f \colon X \to C$, extend it linearly to the basis of simple valuations: $\widetilde{f}(\sum_{i=1}^{n} r_i \delta_{x_i}) = \sum_{i=1}^{n} r_i f(x_i)$, then us the Basic Extension Lemma to extend it to all of $\mathcal{V}(X)$.

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Slogan: dcpo-cones are asymmetric vector spaces and 'asymmetric topology' leads to asymmetric analysis.

Case Study II: Distribution functions

Fact: For a (positive bounded Borel) measure μ on $\mathbb{R}$ its distribution function $F\colon \mathbb{R} \to \mathbb{R}_+$ defined by:

$$F(x) = \mu(]-\infty, x[)$$

has the following properties:

1. F is strict: $\inf_{x\in\mathbb{R}} F(x) = 0$
2. F is monotone: $x \leq y \Rightarrow F(x) \leq F(y)$
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And every function with (1), (2), (3) is the distribution of a unique measure on $\mathbb{R}$.

Usefulness: Riemann Stieltjes integral
$\int f d\mu = \int f dF = \lim_{\Delta x\to 0} \sum_i f(x_i)\big(F(x_i + \Delta x) - F(x_i)\big)$

Choquet's Theorem 1954

X a locally compact Hausdorff space,
$\mathcal{O}(X)$ the lattice of all open subsets $U \subseteq X$,
$\mathcal{K}(X)$ the space of all nonempty compact subsets $K \subseteq X$ with the Vietoris topology generated by

$$\Box U = \{K \mid K \subseteq U\} \text{ and } \Diamond U = \{K \mid K \cap U \neq \emptyset\}$$

For a measure μ on the hyperspace $\mathcal{K}(X)$ its distribution function

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And every such function is the distribution of a uniquely determined measure on $\mathcal{K}(X)$.

Problem

For which spaces L can we characterize measures by somthing like distribution functions?

There is a long paper by A. Revuz in the Annales de l'Institut Fourier 1956 dealing with this problem. The spaces that Revuz is coming up with look very much like continuous lattices.

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Claim The setting of continuous lattices is appropriate to deal with the above problem.

This is part of joint work with Jean Goubault-Larrecq, ENS Cachan, France.

Choquet's Theorem Domain Theoretically

B an α-space which is a $\wedge$-semilattice for the specialization order, $\mathcal{H}(B)$ the lattice of Scott-closed subsets.
$m\colon \mathcal{H}(B) \to \mathbb{R}_+$ a Scott-continuous valuation.

The distribution function $F\colon B \to \mathbb{R}_+$ defined by

$$F(x) = m(closure(\{x\}))$$

has the following properties:

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has the following properties:

1. F is strict: $\inf_{x \in B} F(x) = 0$
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3. F is lower semicontinuous.

And every F with these properties is the distribution of a uniquely determined valuation on $\mathcal{H}(B)$.

Proof: 1. Discrete step

The closures of finite subsets $E \subseteq B$ form a lattice $\mathcal{B}$:
$cl(E) \cup cl(E') = cl(E \cup E')$, $cl(E) \cap cl(E') = cl(E \wedge E')$

Given $F\colon B \to \mathbb{R}$ define $m\colon \mathcal{B} \to \mathbb{R}$ by:
$m\big(cl(\{u\})\big) = F(u)$
$m\big(cl(\{u_1, \ldots .u_n\})\big) =$

$\sum_i F(u_i) \; - \; \sum_{i<j} F(u_i \wedge u_j) \; + \; \sum_{i<j<k} F(u_i \wedge u_j \wedge u_k) \; -+ \; \ldots$

Then m is a strict and modular. It is monotone if and only if F is totally monotone, that is, iff

$$u \geq u_1, \ldots, u_n \implies F(u) \geq m\big(cl(\{u_1, \ldots, u_n\})\big)$$

Proof: 1. Discrete step

$\mathcal{X}_A$ denotes the characteristic function of $A \subseteq B$,
V the real vector space generated by the $\mathcal{X}_{cl(\{x\})}, x \in B$. These characteristic functions are linearly independent, hence a basis of the vector space V. Claim: $\mathcal{X}_{cl(E)} \in V$:

$$\begin{aligned} cl(E) &= \bigcup_{u \in E} cl(\{u\}) = X \setminus \bigcap_{u \in E} (X \setminus cl(\{u\}) \\ \mathcal{X}_{cl(E)} &= 1 - \prod_{u \in E} (1 - \mathcal{X}_{cl(\{u\})}) \\ &= \sum_i \mathcal{X}_{cl(\{u_i\})} - \sum_{i<j} \mathcal{X}_{cl(\{u_i \wedge u_j\})} + - \ldots \end{aligned}$$

The function $x \mapsto F(x)$ has a unique linear extension $F^* \colon V \to \mathbb{R}$. Define $m(cl(E)) = F^*(\mathcal{X}_{cl(E)})$. Then m is a strict, modular and monotone on the lattice $\mathcal{B}$ (because F^* is linear), and

$m(cl(E)) = F^*(\mathcal{X}_{cl(E)}) = \sum_i F(u_i) - \sum_{i<j} F(u_i \wedge u_j) + - \ldots$.

Proof: 2. Continuous step

Lemma

For an α-space B the set $\mathcal{H}(B)$ of all closed subsets is a continuous lattice; the lattice $\mathcal{B}$ of closures $cl(E)$ of finite subsets E is a basis.

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Technical Lemma

If $F\colon B \to \mathbb{R}_+$ is continuous, then its extension $m\colon \mathcal{B} \to \mathbb{R}_+$ is continuous, too.

Applying the Basic Extension Lemma we obtain a Scott-continuous $m\colon \mathcal{H}(B) \to \mathbb{R}_+$ by putting

$$m(C) = \sup\{m(cl(E)) \mid E \subseteq B \text{ finite }, cl(E) \ll C\}$$

A continuity argument shows that m is a valuation.
This finishes the proof.

Back to Choquet

X a locally compact space,
$B = \mathcal{O}(X)$. Our theorem yields a bijection between strict, totally monotone, Scott-continuous $F\colon \mathcal{O}(X) \to \mathbb{R}$ and valuations on $\mathcal{H}(\mathcal{O}(X))$.

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As $\mathcal{H}(\mathcal{O}(X)) \cong \mathcal{O}(\mathcal{K}(X))$ (not difficult to see), we get back Choquet's theorem, but more generally for locally compact spaces that need not be Hausdorff.

Conclusion

I am sure that I have convinced you that domain theoretical ideas going back to Scott and Eršov forty years ago are useful to deal with classical arguments in analysis as far as they are based on compactness arguments and to extend them to non Hausdorff situations as they occur in semantics and in the theory of computability.

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The construction above contains more potential to be exploited:

- Integrals of lower semicontinuous functions should be directly definable by a completion of the vector space V generated by the characteristic functions.