Healthiness conditions for predicate transformers

Klaus Keimel
Department of Mathematics
Technische Universität Darmstadt

www.mathematik.tu-darmstadt.de/~keimel

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Basic intuition: Two complementary approaches to semantics of a program

Consider finite sets X and Y,
a program that accepts inputs from X
and calculates elements of Y as outputs.

State transformer semantics: A function $t \colon X \to Y$ that describes the input output behaviour of the program.

Predicate transformer semantics: A function $p \colon 2^Y \to 2^X$ that for every predicate $A \subseteq Y$ concerning the output yields the weakest precondition $p(A) \subseteq X$ on the input that guarantees property A for the output.

Not all predicate transformers $p \colon 2^Y \to 2^X$ correspond to state transformers. Those that occur are called healthy.

Basic intuition (ctd.)

The healthy predicate transformers are the Boolean homomorphisms:

Given $t\colon X \to Y$ define $p\colon 2^Y \to 2^X$ by
$p(A) = t^{-1}(A)$ for every $A \subseteq 2^Y$.
Clearly, p is a homomorphism of Boolean algebras.

Given a Boolean homomorphism $p\colon 2^Y \to 2^X$, define $t\colon X \to Y$ by
$t(x) = y$ if $x \in p(\{y\})$

Directed complete posets (dcpos)

For semantic domains we choose the category DCPO.

Objects: Directed complete posets (= dcpos), that is, posets X such that each directed subset D has a least upper bound $\sup D$.

Morphisms: (Scott-)Continuous maps $f\colon X \to Y$ between dcpos, that is, maps preserving the order and suprema of directed sets.

Examples: $\mathbf{2} = \{0,1\}$ with $0 < 1$.
$\overline{\mathbb{R}}_+ = \{r \in \mathbb{R} \mid r \geq 0\} \cup \{+\infty\}$, usual order.
$[0,1]$, usual order.

Scott topology: $A \subseteq X$ Scott-closed if
(1) $b \leq a \in A$ implies $b \in A$ and
(2) D directed $\subseteq A$ implies $\sup D \in A$.

The category DCPO is cartesian closed

Products: $\prod_i X_i$ is the set theoretical product of dcpos X_i, ordered pointwise. Suprema of directed families are formed pointwise.

Exponentials: $R^X = [X \to R]$ denotes the set of all continuous $f: X \to R$ ordered pointwise. Suprema of directed families of continuous functions are formed pointwise.

Consequence: all functions definable in a natural way are continuous ('Definable in a natural way' means: definable by a λ-expression.) All functions and maps will be tacitly supposed to be continuous.

We can replace DCPO by any cartesian closed category with equalizers as, for example, SET or POSET or QCB.

Example: Angelic nondeterminism

State transformers $t \colon X \to \mathcal{H}Y$
where $\mathcal{H}Y$ is the Hoare powerdomain of all Scott-closed subsets of Y in which the nondeterministic choice operator is modelled by $\cup$.

Predicate transformers $p \colon \mathcal{O}Y \to \mathcal{O}X$
where $\mathcal{O}X$ is the set of Scott-open subsets (which represent the 'observable' predicates (Smyth) on X).

Predicate transformer p associated to a state transformer t:

$$p(U) = \{x \in X \mid t(x) \cap U \neq \emptyset\}$$

Healthiness conditions For $p \colon \mathcal{O}Y \to \mathcal{O}X$ to be the predicate transformer associated with a state transformer $t \colon X \to \mathcal{H}Y$ it is necessary and sufficient that

$$p(\emptyset) = \emptyset \text{ and } p(U \cup V) = p(U) \cup p(V)$$

Key observation: Translate into a functional representation

Consider $\mathbf{2}$ as a join-semilattice with the constant 0 and the binary operation $x \vee y = \max(x, y)$. The function spaces $\mathbf{2}^Y$ are also join semilattices with the pointwise defined operations.

(1) $\mathcal{O}X \cong \mathbf{2}^X$, predicate transformers: $p\colon \mathbf{2}^Y \to \mathbf{2}^X$.

(2) $\mathcal{H}X$ is (isomorphic to) the subdcpo join-subsemilattice of $\mathbf{2}^{\mathbf{2}^X}$ generated by the point evaluations $\widehat{x} = (f \mapsto f(x))\colon \mathbf{2}^X \to \mathbf{2}$.

The isomorphism is given by embedding $\mathcal{H}X$ into $\mathbf{2}^{\mathbf{2}^X}$ by assigning to A the map φ_A defined by $\varphi_A(f) = \sup_{x\in A} f(x)$ for $f \in \mathbf{2}^X$.

(3) $\mathcal{H}X$ agrees with the subdcpo $[\mathbf{2}^X \xrightarrow{\vee,0} \mathbf{2}] \subseteq [\mathbf{2}^X \to \mathbf{2}]$ of all join semilattice homomorphisms $\varphi\colon \mathbf{2}^X \to \mathbf{2}$.

Healthiness conditions The predicate transformers p corresponding to the state transformers $t\colon X \to \mathcal{H}Y \subseteq \mathbf{2}^{\mathbf{2}^Y}$ are the join semilattice homomorphisms $p\colon \mathbf{2}^Y \to \mathbf{2}^X$.

In the background: Continuation monads

Let R be a fixed dcpo (of observations).
R-valued predicates (prevision, expectation, ...) on a dcpo X: functions $f\colon X \to R$. Space of predicates: R^X.
Predicate transformers: maps $p\colon R^Y \to R^X$.

The output of a program with inputs from X is interpreted as an element of $R^{R^Y} = [R^Y \to R]$ (valuations, distributions,...).
State transformers: Maps $t\colon X \to R^{R^Y}$.

$$[X \to R^{R^Y}] = [X \to [R^Y \to R]] \cong [X \times R^Y \to R] \\ \cong [R^Y \times X \to R] \cong [R^Y \to [X \to R]] = [R^Y \to R^X]$$

Equivalence Lemma

$$P\colon [R^Y \to R]^X \quad \longleftrightarrow \quad [R^Y \to R^X]\colon Q$$

$$P(t)(g)(x) = t(x)(g), \quad Q(p)(x)(g) = p(g)(x)$$

The continuation monad

Let be given a fixed dcpo R.

Assigning to every dcpo X the dcpo $R^{R^X} = [R^X \to R]$ gives rise to a monad, the continuation monad:

Unit: $\delta_X : X \to [R^X \to R]$ defined for every $x \in X$ by

$$\delta_X(x)(f) = \widehat{x}(f) = f(x)$$

the projection on the x-th coordinate.

Kleisli lifting: For $t : X \to R^{R^Y}$ define $t^\dagger : R^{R^X} \to R^{R^Y}$ by

$$t^\dagger(\varphi)(g) = \varphi\big(x \mapsto t(x)(g)\big) \quad (= \varphi(p(g)))$$

Monads over DCPO

The state transformer semantics of progams with algebraic effects is described by a monad $\mathcal{T}$ over DCPO : If X is the input domain and Y the output domain, the (state transformer) semantics of a program is a map $t\colon X \to \mathcal{T}Y$.

Continuation monads are the mothers of all our monads $\mathcal{T}$ in the sense that, for an object R of 'observations'

a. $\mathcal{T}X$ is a subdcpo of R^{R^X},
b. For every $t\colon X \to \mathcal{T}Y$, $t^\dagger$ maps $\mathcal{T}X$ into $\mathcal{T}Y$,
c. $\delta_X(x) \in \mathcal{T}X$ for all $x \in X$.

Algebraic background

Consider a signature Ω consisting of operation symbols ω, each of some finite arity $n = 0, 1, 2, \ldots$ (e.g. $+$ of arity 2).

An Ω-algebra will be a dcpo A endowed with continuous operations $\omega\colon A^n \to A$ for each $\omega \in \Omega$ of arity n.

An Ω-homomorphism (of Ω-algebras) is a continuous map $h\colon A \to B$ such that $h(\omega(a_1, \ldots, a_n)) = \omega(h(a_1), \ldots, h(a_n))$, that is, h commutes with all $\omega \in \Omega$ in the sense that

$$h \circ \omega = \omega \circ h^n$$

(e.g.,$h(a_1 + a_2) = h(a_1) + h(a_2)$).

The collection $[A \xrightarrow{\Omega} B]$ of all Ω-homomorphisms $h\colon A \to B$ is a sub-dcpo of $[A \to B]$.

Monad I: the monad of homomorphism

From now on, fix an Ω-algebra R of 'observations'.

R^X becomes an Ω-algebra for every dcpo X, the operations being defined pointwise:

e.g., $(f_1 + f_2)(x) = f_1(x) + f_2(x)$.

Similarly, $R^{R^X} = [R^X \to R]$ becomes an Ω-algebra.

Properties:

a. $[R^X \xrightarrow{\Omega} R]$ is a subdcpo of $[R^X \to R]$,

b. for every $t \colon X \to [R^Y \to R]$, $t^\dagger$ maps $[R^X \stackrel{\Omega}{\to} R]$ to $[R^Y \stackrel{\Omega}{\to} R]$,

c. $\widehat{x} = \delta_X(x) \in [R^X \xrightarrow{\Omega} R]$ for every $x \in X$.

The monad of homomorphisms

Result: A

Assigning to each dcpo X the dcpo $[R^X \xrightarrow{\Omega} R]$ of all Ω-homomorphisms $\varphi \colon R^X \to R$ yields a monad 'subordinate' to the continuation monad R^{R^X}.

This monad is not of interest in itself, but it behaves well with respect ot predicate transformers:

Result B

The predicate transformers corresponding to the state transformers $t \colon X \to [R^Y \xrightarrow{\Omega} R]$ are the Ω-homomorphisms $p \colon R^Y \to R^X$:

$$P \colon [R^Y \xrightarrow{\Omega} R]^X \quad \longleftrightarrow \quad [R^Y \xrightarrow{\Omega} R^X] \colon Q$$

In general, the Ω-homomorphisms do not form an Ω-algebra.

Monad II: 'Free' Ω-algebras

For any dcpo X, let $\mathcal{F}_R X$ be the Ω-subalgebra of $[R^X \to R]$ generated by the projections $\widehat{x}, x \in X$. Then:

a. $\mathcal{F}_R X$ is a subdcpo of $[R^X \to R]$,
b. $t^\dagger$ maps $\mathcal{F}_R X$ to $\mathcal{F}_R Y$ for every $t\colon X \to \mathcal{F}_R Y$,
c. $\widehat{x} = \delta_X(x) \in \mathcal{F}_R X$ for every $x \in X$.

Result C

Assigning to every dcpo X the Ω-algebra $\mathcal{F}_R X$ yields a monad with unit δ_X and Kleisli lifting $t^\dagger$.

Freeness property

In classical universal algebra, a theorem due to G. Birkhoff says: $\mathcal{F}_R X$ is the free algebra over the set X in the 'variety' HSP(R) of algebras which are homomorphic images of subalgebras of powers of R.

Result D

$\mathcal{F}_R X$ is the free Ω-algebra over X in the class SP(R) of Ω-algebras isomorphic to an Ω-subalgebra of some R^Y.

An Ω-algebra A belongs to SP(R) if and only if it satisfies the order separation property, that is, if for any $a \not\leq a'$ in A, there is an Ω-homomorphism $h\colon A \to R$ such that $h(a) \not\leq h(a')$.

Questions

How to characterize the predicate transformers $p\colon R^Y \to R^X$ that correspond to the state transformers $t\colon X \to \mathcal{F}_R X$.

Under what conditions do we have $[R^X \xrightarrow{\Omega} R] = \mathcal{F}_R X$ or, at least $\mathcal{F}_R Y \subseteq [R^X \xrightarrow{\Omega} R]$?

Commuting operations

Given two operations σ of arity m and ω of arity n on a A, we say that σ and ω commute if for all 'matrices' $(x_{ij})_{i=1,\dots,m,\ j=1,\dots,n}$ of elements in A, we have:

$$\begin{array}{rl} & \omega\big(\sigma(x_{11},\dots,x_{m1}),\ \dots\ ,\sigma(x_{1n},\dots,x_{mn})\big) \\ = & \sigma\big(\omega(x_{11},\dots,x_{1n}),\ \dots\ ,\omega(x_{m1},\dots,x_{mn})\big) \end{array}$$

which means that $\sigma \circ \omega^m \cong_i \omega \circ \sigma^n$:

$$\begin{array}{ccccc} (A^m)^n & \cong_i & (A^n)^m & \xrightarrow{\ \omega^m\ } & A^m \\ \Big\downarrow \sigma^n & & & & \Big\downarrow \sigma \\ A^n & & \xrightarrow{\qquad \omega \qquad} & & A \end{array}$$

Commuting operations: Examples

- A constant c commutes with an n-ary operation ω if and only if $\omega(c, \ldots, c) = c$. Two commuting constants have to agree.
- Two unary operations ρ and σ commute if they commute as functions: $\rho \circ \sigma = \sigma \circ \rho$.
- A unary operation ρ commutes with a binary operation $+$ if

$$\rho(x + y) = \rho(x) + \rho(y) \tag{1}$$

- Two binary operation $+$ and $*$ commute if

$$(x_1 * x_2) + (x_3 * x_4) = (x_1 + x_3) * (x_2 + x_4) \tag{2}$$

- In particular, a binary relation $*$ commutes with itself if

$$(x_1 * x_2) * (x_3 * x_4) = (x_1 * x_3) * (x_2 * x_4) \tag{3}$$

A commutative, associative binary operation commutes with itself.

Healthiness conditions

Let R be an Ω-algebra as before.
Let Σ be a collection of operations $\sigma\colon R^m \to R$ that commute with all $\omega \in \Omega$. We may adopt Σ as another signature and R can be viewed also as a Σ-algebra, and similarly R^X and R^{R^X}.

The dcpo $[R^X \xrightarrow{\Sigma} R]$ is an Ω-subalgebra of $[R^X \to R]$ containing the R-free Ω-algebra $\mathcal{F}_R X$

The predicate transformers $p\colon R^Y \to R^X$ corresponding to the state transformers $t\colon X \to \mathcal{F}_R Y$ are necessarily Σ-homomorphisms.

If $\mathcal{F}_R Y = [R^Y \xrightarrow{\Sigma} R]$, then the Σ-homomorphisms $p\colon R^Y \to R^X$ are are precisely the predicate transformers corresponding to state transformers $t\colon X \to \mathcal{F}_R Y$:

$$[X \to \mathcal{F}_R Y] \cong [R^Y \xrightarrow{\Sigma} R^X].$$

How to apply the preceding result

For Σ the maximal choice would be
Σ_0 the dcpo of all elemnts of R that form a singleton Ω-subalgebra
$\Sigma_1 = [R \xrightarrow{\Omega} R]$, the dcpo of Ω-endomorphisms
$\Sigma_2 = [R^2 \xrightarrow{\Omega} R]$,
...
...

This choice may not be optimal because this Σ is difficult to determine and may be much bigger than necessary. The art will be to find a small set of representatives that have the same power than this maximal Σ.

Special case: Entropic algebras

The Ω-algebra R is called entropic if any two operations $\omega_1, \omega_2 \in \Omega$ commute.
Since entropicity is defined by a set of equational laws, one for each pair of operation symbols. Thus, entropicity is inherited by products, subalgebras and homomorphic images.

Examples:

- In an entropic algebra, there is at most one constant, and it forms a subalgebra.
- Commutative monoids are entropic, semilattices are entropic.
- Vector spaces are entropic, cones and convex sets are entropic,
- effect modules are entropic.
- Rings, lattices, maxplus-algebras are not entropic.

Entropicity (ctd.)

Suppose that the Ω-algebra R is entropic.

Result F

$[R^Y \xrightarrow{\Omega} R]$ is an Ω-subalgebra of $[R^Y \to R]$ containing $\mathcal{F}_R Y$.

The predicate transformers $p\colon R^Y \to R^X$ corresponding to the state transformers $t\colon X \to \mathcal{F}_R Y$ are necessarily Ω-homomorphisms.

If $\mathcal{F}_R Y = [R^Y \xrightarrow{\Omega} R]$, then the Ω-homomorphisms $p\colon R^Y \to R^X$ are are precisely the predicate transformers corresponding to state transformers $t\colon X \to \mathcal{F}_R Y$:

$$[X \to \mathcal{F}_R Y] \cong [R^Y \xrightarrow{\Omega} R^X].$$

Thus, in the ergodic case it remains to check in every special case whether the equality $\mathcal{F}_R X = [R^X \xrightarrow{\Omega} R]$ holds.

Example: The angelic powerdomain $\mathcal{H}X$

Algebra of observations $\mathbf{2} = \{0 < 1\}$, signature $\Omega = \{\vee, 0\}$, is a join-semilattice with bottom, hence entropic.

$\mathcal{H}X = [\mathbf{2}^X \xrightarrow{\vee,0} \mathbf{2}]$ is a $\vee$-semilattice with bottom $\emptyset$.

The predicate transformers corresponding to the state transformers $t\colon X \to \mathcal{H}Y$ are those $s\colon \mathbf{2}^Y \to \mathbf{2}^X$ preserving arbitrary joins.

For $\varphi \in [\mathbf{2}^X \xrightarrow{\vee,0} \mathbf{2}]$ we have $\varphi = \sup\{\widehat{x} \mid \widehat{x} \le \varphi\}$. Hence $\mathcal{F}_{\mathbf{2}}X = [\mathbf{2}^X \xrightarrow{\vee,0} \mathbf{2}] = \mathcal{H}X$.

For a d-join-semilattice S with bottom the continuous join-semilattice homomorphisms into $\mathbf{2}$ preserving bottom are order separating. Hence $S \in \mathrm{SP}(\mathbf{2})$.

Thus:

$\mathcal{H}X$ is the free d-join-semilattice with bottom over X.

(Extended) probabilistic choice

Algebra of observations: $R = \mathbb{R}_+ \cup \{+\infty\}$ of signature $\Omega = (+, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, is entropic.

$[R^X \xrightarrow{lin} R]$ is a cone. The predicate transformers corresponding to the state transformers $t \colon X \to [R^Y \xrightarrow{lin} R]$ are the linear maps $s \colon R^Y \to R^X$.

The probabilistic powerdomain $\mathcal{F}_R Y$ (generated by the point measures $\widehat{x}$) is a subcone of $[R^Y \xrightarrow{lin} R]$

The predicate transformers $p \colon R^Y \to R^X$ corresponding to state transformers $t \colon X \to \mathcal{F}_R Y$ are necessarily linear.

For a continuous dcpos, $\mathcal{F}_R Y = [R^Y \xrightarrow{lin} R]$ so that the predicate transformers corresponding to state transformers $t \colon X \to \mathcal{F}_R Y$ are precisely the linear maps $p \colon R^Y \to R^X$.

Combining angelic nondeterministic and probabilistic choice

Algebra of observations: $\overline{\mathbb{R}}_+$ of signature
$\Omega = (+, \vee, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, is not entropic.

$[\overline{\mathbb{R}}_+^X \xrightarrow{\Omega} \mathbb{R}_+]$ is not an Ω-algebra.

But instead of equality we have still the inequality

$$(a + c) \vee (b + d) \leq (a \vee b) + (c \vee d)$$

This incites us to introduce a relaxed notion of entropicity.

Relaxed Ω-morphisms

A relaxed Ω-morphism between two Ω-algebras A, B of signature $\Omega = \Omega_{\leq} \cup \Omega_{\geq}$: $h\colon A \to B$ such that

$$h(\omega(a_1, \dots, a_n)) \leq \omega(h(a_1), \dots, h(a_n))$$

for every $\omega \in \Omega_{\leq}$ of arity n, and the other way around for $\omega \in \Omega_{\geq}$.

Example: Consider Ω-algebras A, B of signature $\Omega = (+, \vee, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, with $+$ in $\Omega_{\geq}$ and $\vee$ in $\Omega_{\geq}$.
A relaxed Ω-morphism is a map $h\colon A \to B$ satisfying:

$$\begin{array}{rcl} h(0) & = & 0 \\ h(ra) & = & rh(a) \text{ for all } r \in \overline{\mathbb{R}}_+ \\ h(a+b) & \leq & h(a) + h(b) \\ h(a \vee b) & \geq & h(a) \vee h(b) \end{array}$$

Since the last requirement is satisfied for order preserving maps anyway, it can be omitted. Thus:

relaxed Ω-morphism = sublinear map.

Relax

Fix an Ω-algebra R of signature $\Omega = \Omega_{\leq} \cup \Omega_{\geq}$.

For every dcpo X, the set $[R^X \overset{\Omega lax}{\longrightarrow} R]$ of relaxed Ω-morphisms $h\colon R^X \to R$ is a sub-dcpo of $[R^X \to R]$.

For every state transformer $t\colon X \to [R^Y \to R]$, the Kleisli lifting $t^\dagger$ maps $[R^X \overset{\Omega lax}{\longrightarrow} R]$ to $[R^Y \overset{\Omega lax}{\longrightarrow} R]$. Thus, $X \mapsto [R^X \overset{\Omega lax}{\longrightarrow} R]$ yields a monad with unit δ and the Kleisli lifting $t^\dagger$ inherited from the continuation monad.

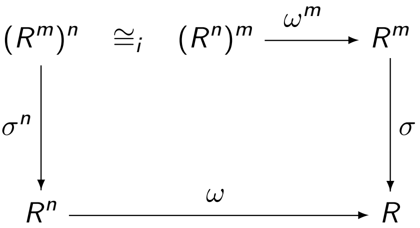
The predicate transformers corresponding to state transformers $t\colon X \to [R^Y \overset{\Omega lax}{\longrightarrow} R]$ are the relaxed Ω-morphisms $p\colon R^Y \to R^X$:

$$[R^Y \overset{\Omega lax}{\longrightarrow} R]^X \cong [R^Y \overset{\Omega lax}{\longrightarrow} R^X]$$

Healthiness conditions in the relaxed case

Let R be an Ω-algebra R of signature $\Omega = \Omega_{\leq} \cup \Omega_{\geq}$.

$\omega \colon R^n \to R$ subcommutes with $\sigma \colon R^m \to R$
(and σ supercommutes with ω), if $\omega \circ \sigma^n \leq \sigma \circ \omega^m \circ i$:



On R, consider collections $\Sigma^{\geq}$ and $\Sigma^{\leq}$ of maps $\sigma \colon R^m \to R$ that subcommute, resp., supercommute, with all $\omega \in \Omega$. These give rise to a second signature $\Sigma = \Sigma^{\leq} \cup \Sigma^{\geq}$. The relaxed Σ-morphism $h \colon R^X \to R$ form an Ω-subalgebra of $[R^X \to R]$.

Healthiness conditions in the relaxed case

Theorem

The Ω-algebra $\mathcal{F}_R Y$ generated by the projections is an Ω-subalgebra of $[R^Y \xrightarrow{\Sigma lax} R]$.

The predicate transformers corresponding to state transformers $t\colon X \to \mathcal{F}_R Y$ are relaxed Σ-morphisms $p\colon R^Y \to R^X$.

If $\mathcal{F}_R Y = [R^Y \xrightarrow{\Sigma lax} R]$, then these predicate transformers are precisely the relaxed Σ-morphisms.

Relaxed entropic algebras

Suppose that the signature Ω is the union of two subsignatures $\Omega_{\leq}$ and $\Omega_{\geq}$ which need not be disloint.

An Ω-algebra of signature $\Omega = \Omega_{\leq} \cup \Omega_{\geq}$ is relaxed entropic if every $\omega \in \Omega_{\leq}$ subcommutes with every $\sigma \in \Omega$ and every $\omega \in \Omega_{\geq}$ supercommutes with every $\sigma \in \Omega$.

Example: $\overline{\mathbb{R}}_+$ of signature $\Omega = (+, \vee, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, is relaxed entropic if we put $+$ in $\Omega_{\leq}$ and $\vee$ in $\Omega_{\geq}$; the constant 0 and the unary operations of multiplication with scalars are in both $\Omega_{\leq}$ and $\Omega_{\geq}$.

Healthiness conditions in the relaxed entropic case

Suppose that R is a relaxed entropic Ω-algebra of signature $\Omega = \Omega_{\leq} \cap \Omega_{\geq}$. Then $[R^Y \xrightarrow{\Sigma lax} R]$ is an Ω-subalgebra of $[R^X \to R]$. Applying the Theorem for $\Sigma = \Omega$, we obtain:

Corollary

$\mathcal{F}_R X$ is an Ω-subalgebra of $[R^X \xrightarrow{\Omega lax} R]$.

The predicate transformers corresponding to state transformers $t\colon X \to \mathcal{F}_R Y$ are relaxed Ω-morphisms $p\colon R^Y \to R^X$.

If $\mathcal{F}_R Y = [R^Y \xrightarrow{\Sigma lax} R]$, then these predicate transformers are precisely the relaxed Σ-morphisms.

Example: Combining angelic and probabilistic choice

Ω-Algebra of observations: $\overline{\mathbb{R}}_+$, signature $\Omega = (+, \vee, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, with $+$ in $\Omega_\leq$ and $\vee$ in $\Omega_\geq$, is relaxed entropic.

Equational Theory: Cone and join-semilattice axioms connected by the distributivity laws

$$\begin{aligned} x + (y \vee z) &= (x + y) \vee (x + z) \\ r \cdot (y \vee z) &= (r \cdot y) \vee (r \cdot z) \end{aligned}$$

$[\overline{\mathbb{R}}_+^X \xrightarrow{\Omega lax} X]$, the set of sublinear functionals on $\overline{\mathbb{R}}_+^X$ is an Ω-subalgebra of $[\overline{\mathbb{R}}_+^X \to \overline{\mathbb{R}}_+]$.

The predicate transformers p corresponding to the state transformers $t \colon X \to [\overline{\mathbb{R}}_+^X \xrightarrow{\Omega lax} X]$ are the sublinear maps $p \colon \overline{\mathbb{R}}_+^Y \to \overline{\mathbb{R}}_+^X$.

Example: Combining angelic and probabilistic choice (ctd.)

If X is a continuous dcpo, then $\mathcal{HV}X = \mathcal{F}_{\overline{\mathbb{R}}_+} X$. (For the proof one needs a Hahn-Banach type argument, that every continuous sublinear functional is the join of a family of continuous linear functionals.) The predicate transformers p corresponding to the state transformers $t \colon X \to \mathcal{HV}Y$ are the sublinear maps $p \colon \overline{\mathbb{R}}_+^Y \to \overline{\mathbb{R}}_+^X$:

$$(\mathcal{HV}Y)^X \cong [\overline{\mathbb{R}}_+^Y \xrightarrow{sublin} \overline{\mathbb{R}}_+^X]$$

For every continuous dcpo X, $\mathcal{HV}X$ is the free cone join-semilattice over X.

Concluding remarks

1. The previous developments can be carried through in any cartesian closed category, like POSET, SET, QCB. The relaxed seting of course needs an order.

2. As only functions spaces R^X for a fixed R occur, we do really need a cartesian closed category. It suffices that for our fixed R, the exponentials R^X exist.

3. The signature need not be finitasry. One may allow operations of infinite arity.

4. The signature may be internalized. Thus, in DCPO, the signature Ω may be a sequence of dcpos $\Omega_0, \Omega_1, \Omega_2, \dots$.

Concluding remarks (ctd.)

5. Also arities may be internalized. That is the elements of Ω may be dcpos. For example, an operation on R of arity $\mathbf{2} = \{0 < 1\}$ will be a map defined not on all of R^2 but only on the subdcpo $R^{\mathbf{2}} = \{(a_0, a_1) \in R \times R \mid a_0 \leq a_1\}$.

6. For an equationally defined class of entropic algebras, the monad given by the free algebras is commutative in the sense of A. Kock. Thus, there should be a category theoretical extension our results.

7. Referees always ask for new examples. But our results are of a negative nature: Entropicity is such a strong property that I do not expect that other natural situations occur in which one can find necessary and sufficient healthiness conditions for the predicate transformers corresonding to the state transformers $t \colon X \to \mathcal{T}Y$.

Please, let me knw any situation in which the above methods might be applied.