Remarks on pointwise infima of lower semicontinuous functions

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Notes from 1984 complementing the paper with G. Gierz: *Halbstetige Funktionen und stetige Verbände*, Bremen Proceedings.

In their paper *semicontinuous mappings in general topology*, J.P. Penot and M. Théra (cf. Archiv der Mathematik 38 (1982)) have proved – roughly – the following:

(1) Let

   (a) X and Y be arbitrary topological spaces,

   (b) L a topological space such that every point has a neighborhood base of principal filters (with respect to the specialisation order), i.e., a C-space according to Erné,

   (c) $f\colon X \times Y \to L$ a continuous function,

   (d) $R \subseteq X \times Y$ a binary relation such that, for every neighborhood W of $R(x_0) := (\{x_0\} \times Y) \cap R$, there is a neighborhood U of x_0 such that $(U \times Y) \cap R \subseteq W$, i.e., R is graphically upper semicontinuous,

Then the function $m\colon X \to L$ defined by

$$m(x) = \inf\{f(x,y) \mid (x,y) \in R\} = \inf f(R(x))$$

is continuous, provided that the inf exists (for the specialistion order on L).

Proof. Let $x_0 \in X$. According to (b), choose a basic neighborhood $\uparrow b$ of $m(x_0)$. As $f(x_0,y) \geq m(x_0)$ for every $y \in R(x_0)$, $\uparrow b$ is a neighborhood of $f(R(x_0))$. By the continuity of f, there is a neighborhood W of $R(x_0)$ such that $f(W) \subseteq \uparrow b$. Choose U according to (d). Then $m(x) = \inf f(R(x)) \geq b$ for all $x \in U$.

(2) If $R(x)$ is compact for every $x \in X$, then (d) is equivalent to

(d') For every open set V containing $\{y \in Y \mid (x_0, y) \in R\}$, there is a neighborhood U of x_0 such that $\{y \in Y \mid (x, y) \in R\} \subseteq V$ for all $x \in U$, i.e. R is upper semicontinuous.

(3) Let L^X denote the set of all continuous functions $h\colon X \to L$ endowed with a function space topology in such a way that the evaluation map

$$\mathrm{ev}\colon (x, h) \mapsto h(x)\colon X \times L^X \to L$$

is continuous on each subset $X \times Q \subseteq X \times L^X$, where $Q \subseteq L^X$ is compact. Then (1) and (2) imply that the function $\overline{h}\colon X \to L$ defined by

$$\overline{h}(x) = \inf_{h \in Q} h(x)$$

is continuous for every compact set $Q \subseteq L^X$, provided that the infs exist.

Summary The pointwise inf of a compact family of lower semicontinuous functions is lower semicontinuous.

This follows directly from the previous. An alterbnative direct proof: Given $x_0 \in X$, show that $\overline{h}$ is continuous in x_0. For this, choose a basic neighborhood of $\overline{h}(x_0)$ of the form $\uparrow b$. For every $h \in Q$, there is a neighborhood U_h of x_0 and a neighborhood V_h of h such that $h(x) \in \uparrow b$ for all $(h, x) \in V_g \times U_g$. By compactness, a finite number of the V_h are covering Q. Let U be the intersection of the corresponding (finitely many) U_g. Then U is a neighborhood of x_0 such that $b \leq h(x)$ for all $h \in Q, x \in U$, whence $b \leq \overline{h}(x)$ for all $x \in U$.

(4) The hypotheses of (3) are satisfied if L is a continuous lattice with the Scott topology and X a locally compact space, more generally, a core compact space.

(5) **Question** Under which general conditions the property (1b) and the hypotheses of (3) are satisfied?

(6) **Question** Is there a useful caracterisation of the compact subsets Q of L^X?

(7) If X is core compact and L a continuous domain, then the evaluation map $\mathrm{ev}\colon (x,h) \mapsto h(x)\colon X \times L^X \to L$ is continuous.

Proof. Given $h_0 \in L^X$ and $x_0 \in X$, we have to show: For every $a \ll h_0(x_0)$, there is a neighborhood V of x_0 and a neigborhood W of h_0 such that

$$(*) \qquad h(x) \gg a \text{ for all } h \in W, x \in V\,.$$

For this choose $a \ll b \ll h_0(x_0)$. There is a neighborhood U of x_0 such that $b \ll h_0(x)$ for all $x \in U$. Now choose a neighborhood V of x_0 such that $V \ll U$. Let

$$W = \{h \in L^X \mid b \ll h(x) \text{ for all } x \in V\}\,.$$

Then (*) is satisfied and it only remains to show that W is a neighborhood of h_0. We firstly have indeed $h_0 \in W$. Now consider a directed family (g_i) in L^X such that $\sup g_i \in W$. We then have in particular $b \ll \sup g_i(x)$ for all $x \in U$. For every $x \in U$ we now may find an index $i(x)$ such that $b \ll g_{i(x)}$. This now is true on a whole neighborhood U_x of x. Because of $V \ll U$, finitely many $U_{x_j}, j = 1, \dots, n$ already cover V. Now choose an index $i \geq i(x_j)$ for $j = 1, \dots, n$. Then $b \ll g_i(x)$ for all $x \in V$, that is, $g_i \in W$.

For other results on the continuity of the evaluation map see the papers by Lambrinos.
Note added in 2005: If X is core compact, then the evaluation map $X \times L^X \to L$ is continuous, if L^X is endowed with the Isbell topology.

Added 2015:
The question whther the intersecio9n of a family of open sets is open is equivalent to the question whether the pointwise meet of a family of lower semicontinuous real-valued functions is lower semicontinuous.

Another paper in this line is::

L. Nachbin

Compact unions of closed subsets are closed and compact and compact intersections of open sets are open.

Portugaliae Mathematica 49 (1992), 403-409.

A recent paper along these lines and maybe the most informative is due to Martin Escardo. I ignore whether it ever has been published. I have version form his homepage dated May 27, 2009:

M. Escardo,

Intersections of compactly many open sets are open.

Draft, 15 pages.