Sheaf Representations for Algebraic Systems

A personal historical account

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I am reporting here how I was involved in sheaf representations of various structures between roughly 1965 and 1975. During that period I was working in close collaboration with Karl Heinrich Hofmann who has greatly influenced my work.

Background

Pierre Shapira says that *sheaves on topological spaces were introduced by Jean Leray as a tool to deduce global properties from local ones.* Leray's worked on these ideas as a prisoner of war between 1940 and 1945. He published his definition of a sheaf in 1946 in the Comptes Rendus Acad. Sc. Paris 232, p.1366. Leray's definition was modified by Lazard in Exposé 14, Séminaire Cartan 1950-51. Sheaves were introduced as a methodological tool in algebraic topology, in particular in cohomology theory. The extension of certain results on cohomology from compact spaces to locally compact spaces was only possible by admitting 'variable coefficient groups depending continuously on the points of the space' as opposed to the usual constant cohomology groups. A second source for sheaves was the theory of analytic functions (sheaves of germs of analytic functions). In 1953 they were finally introduced in algebraic topology. These developments culminated in the work of Grothendieck. A major source is:

> A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique. I. Le langage des schémas, Inst. Hautes Etudes Sci. Publ. Math. No. 4 (1960), 228 pages.

The early history of the development of the notion of a sheaf with an extensive bibliography one can find in:

> J. W. Gray, Fragments of the history of sheaf theory, in: M. P. Fourman, C. J. Mulvey, and D. S. Scott (eds.), Applications of Sheaves, Lecture Notes in Mathematics 753 (1979), 1 – 79, Springer-Verlag.

Gray's contribution shows that the development in the line of 'representations of algebras by sheaves' (which I am referring to in the sequel) is a minor one compared with the main research topics.

Sheaf representations

It seems to me that Grothendieck's *affine scheme* associated with a commutative ring with unit was the motivating example for the developments that I was involved in. Whereas the developments mentioned in the previous section were driven by methodological needs of classical theories, the point of view of the community that emerged around 1965 was that of 'Representation of algebraic structures by sections in sheaves'. A typical theorem reads as follows

> *Let A be an algebraic structure. There is a sheaf $\mathcal{S}$ of algebraic structures $A_x, x \in X$, of the same type over a topological space X such that A is isomorphic to the algebra $\Gamma(\mathcal{S})$ of all global sections (alternatively: all global sections with compact support).*

The hope is that the stalks A_x are of a simpler nature than A itself and that one will gain a better understanding of the structure of A by means of the structure of the stalks A_x and the structure of the base space X. But in this community, there was no particular problem that urged the use of sheaves.

Sheaf representations of rings, the Zariski topology

At the beginning the efforts were concentrated on rings, mostly commutative rings with or without units. Later – with less success – non-commutative rings were attacked.
The base space X was generally obtained as the space of all prime ideals with the Zariski topology or as a subspace thereof.

Personally I entered the world of sheaf representations after reading:

> J. Dauns and K. H. Hofmann, The representation of biregular rings by sheaves, Math. Z. 91 (1966), 103-123.

A ring (not necessarily commutative or unital) is *biregular*, if every ideal is generated by a central idempotent. The ideals of such a ring form a generalized Boolean algebra (a distributive lattice in which every principal ideal is a Boolean algebra). The prime ideals with the Zariski topology form a locally Boolean space. The associated sheaf is a sheaf of simple rings with identity, and the given ring is isomorphic to the ring of all sections with compact support. In the commutative case the stalks are fields,

The Memoir:

> R. S. Pierce, Modules over commutative regular rings, Memoirs AMS 70 (1967), 112pp.

is a step further in the same direction and was greatly influential at the time.

I entered the scene by observing that the methods developed in the previous papers could also be applied to lattice-ordered rings:

K. Keimel, Représentation d'anneaux réticulés dans des faisceaux, C. R. Acad. Sci. Paris Sr. A-B 266 (1968), A124-A127. MR 37 6224.
K. Keimel, Anneaux réticulés quasi-réguliers et hyperarchimédiens, C. R. Acad. Sci. Paris Sr. A-B 266 (1968), A524-A525. MR 37 6225.

I had indeed learned about ordered algebraic structures during my year at Tulane University in 1964/65 from Lectures delivered by A. H. Clifford. And when I came to Paris in 1965, I joined the group around M.-L. Dubreil-Jacotin (comprising Alain Bigard and Samuel Wolfenstein) working on lattice-ordered groups.

Lattice-ordered groups and rings

I then observed that the ring structure was not needed for the sheaf representations and I extended my work to commutative lattice ordered groups (which can be viewed as lattice ordered rings with the multiplication being identically zero). This led to:

K. Keimel, Représentation de groupes et d'anneaux réticulés par des sections dans des faisceaux, Thèse d'État, Paris 1970.

On my homepage (http://www.mathematik.tu-darmstadt.de/ keimel/publications.html) one can find the contents of this thesis, partly without proofs. Chapter I deals with prime elements of lattices and the Zariski (= Stonean) topology on them. Chapter II deals with the Stone-Čech-Isbell compactification. Chapter III finally presents various sheaf representations. They are formulated for lattice-ordered *anneloïdes*: These are lattice ordered 'rings' with a non-commutative addition. In this way one also includes non-abelian lattice ordered groups as lattice-ordered anneloïdes with trivial multiplication. This latter aspect has been omitted in:

K. Keimel, The representation of lattice-ordered groups and rings by sections in sheaves, Lecture Notes in Math. 248, Springer-Verlag, Berlin and New York, 1971, pages 1–98.

This article is an improved version of Chapter III of the thesis above. It contains:
- A general representation theorem,
- Representation over the direct factor space,
- Representation over the space of minimal irreducible ℓ-ideals,
- Representation of quasi-regular and hyperarchimedean f-rings and ℓ-groups,
- Representation over the maximal ℓ-ideal space, Dominated ℓ-ideals.

Some of the results of the article just indicated are reproduced in:

A. Bigard, K. Keimel, S. Wolfenstein, Groupes et Anneaux Réticulés Lecture Notes in Mathematics, 608 (1977), Springer Verlag, xiv+334 pages.

One should not forget the following contribution along the same lines:

J. Dauns, Representation of L-groups and F-rings, Pacific J. Math. 31 (1969), 629-654: MR 41 130.

It should be noted that a research group in Roumania had already begun to work on sheaf representation of distributive lattices:

A. Brezuleanu and R. Diaconescu, Sur la duale de la catégorie des treillis, Rev. Roumaine Math. Pures Appl. 14 (1969), 311-323. MR 41 116.
A. Brezuleanu, Sur les schémas de treillis, Rev. Roumaine Math. Pures Appl. 14 (1969), 949 - 954.

Sheaf representations for universal algebra

The extension of the sheaf representation methods from rings to lattice-ordered structures lead me to generalize further and to publish a representation theorem for universal algebras, in particular, semigroups, based on a generalized Boolean lattice of commuting factor congruences:

K. Keimel, Darstellung von Halbgruppen und universellen Algebren durch Schnitte in Garben; bireguläre Halbgruppen, Math. Nachrichten 45 (1970), 81 - 96.

As this paper was published in German, it has not been cited very often. Shortly later, but independently, Comer published a paper that is more easily accessible and written in a better style based on the same ideas:

S. D. Comer, Representation by algebras of sections over Boolean spaces, Pacific J. Math. 38 (1971), 29 - 38.

In the following years these ideas were developed further in numerous papers as for example in:

K. Keimel and H. Werner, Stone duality for varieties generated by quasi-primal algebras, Memoirs Amer. Math. Soc. 148 (1974), pp. 59-85.

In this context one should mention:

B. A. Davey, Sheaf spaces and sheaves of universal algebras, Math. Z. 134 (1973), 275-290.

A student of mine wrote an interesting paper on sheaf representations of universal algebras in general (the title is misleading as it seems to point to classical algebras):

A. Wolf, Sheaf representations of arithmetical algebras, Memoirs Amer. Math. Soc. 148 (1974), pp. 85-97.

In this paper a general setting is described that allows sheaf representations of universal algebras: *The essential ingredient for a good sheaf representation is the existence of a distributive lattice of pairwise commuting congruences.* It seems to be the first time that *the Chinese Remainder Theorem is shown to be responsible for the patch property, which allows us to glue sections and which is essential for proving that the given algebra is represented by the algebra of ALL global sections (of compact support) of the associated sheaf.* But note that appropriate formulations of the Chinese Remainder Theorem for universal algebras were already available in the late 1950-ies; see for example the book on Universal Algebra by G. Grätzer.

Independently, quite similar observations have been made by Swamy, who reproves the appropriate Chinese Remainder Theorem:

I. M. Swamy, Representation of universal algebras by sheaves, Proc. Amer. Math. Soc. 45 (1974), 55-58.

When I came to Darmstadt in 1971, there was a strong group working in Universal Algebra and Lattice Theory around R. Wille. After my arrival, I introduced sheaf representations there. H. Werner and A. Wolf began to work together with me. S. Burris and B. Davey were visiting Darmstadt and contributed to the spreading of these techniques around to the community of Universal Algebra.

A Survey

The following survey contains the developments on sheaf representations described above in much more detail up to the year 1972 with a rich list of references:

K. H. Hofmann, Representation of algebras by continuous sections, Bull. Amer. math. Soc. 78, Number 3 (1972), 291-373.
—, Some bibliographical remark on "Representation of algebras by continuous sections", memoirs Amer. Math. Soc. 148 (1974), 177-182.

What do I mean by sheaf representation?

By a *sheaf* I here mean a local homeomorphism $p\colon E \to X$ of topological spaces. The *stalks* are the $E_x = p^{-1}(x); x \in X$. The n-fold fibered product is the sheaf $p^{(n)}\colon E^{(n)} \to X$, where $E^{(n)} = \{(a_1, \ldots, a_n) \mid p(a_1) = \ldots = p(a_n)\}$ with the subspace topology from E^n.
If Ω is a finitary signature, a sheaf of Ω-algebras is a sheaf $p\colon E \to X$ together with continuous maps $\omega\colon E^{(n)} \to E$ which respect fibers, where $\omega \in \Omega$ is of arity n. This means in particular that the stalks E_x are Ω-algebras.

For an Ω-algebra A, a *sheaf representation* is a an algebra isomorphism of A onto the algebra $\Gamma(p)$ of all global (continuous) sections of a sheaf $p\colon E \to X$ of Ω-algebras.

In order to find a sheaf representation of an Ω-algebra A, one may proceed roughly as follows. One chooses a set X of congruence relations c_x of A. One forms the disjoint union E of all the quotient algebras $A_x = A/c_x, x \in X$, together with the obvious projection $p\colon E \to X$. For every $a \in A$ let $\widehat{a}\colon X \to E$ be defined by $\widehat{a}(x) = a \bmod c_x$. One then chooses a topology on X.
1) If for all $a, b \in A$, the 'equalizer' $e(a,b) = \{x \in X \mid (a,b) \in c_x\}$ is open, the we lift this topology up to E by declaring the sets $\widehat{a}(e(b,c))$ to be open. We then have a sheaf of Ω-algebras $p\colon E \to X$ where the stalks are the quotients A/c_x. The map $a \mapsto \widehat{a}$ is an algebra homomorphisms of A into the algebra $\Gamma(p)$ of all global sections. This map is injective iff the intersection of the congruence $c_x, x \in X$, is the equality relation. But there is no general criterion for surjectivity. SURJECTIVITY IS THE REAL CHALLENGE.
The condition that the sets $e(a,b)$ are open can be enforced by choosing on X the coarsest topology such that all the $e(a,b)$ are open. For example the co-Zariski topology on the prime ideals of a commutative ring (commutative ℓ-group, commutative ℓ-ring) is such a topology.
2) In case that the equalizers $e(a,b)$ are not always open, one can enforce this condition by 'localizing', that is, replacing c_x by $\widetilde{c}_x = \bigcup_U \bigcap_{y \in U} c_y$, where U ranges over all neighborhoods of x.
This procedure is applied when considering the prime ideals of a commutative ring (commutative ℓ-group, commutative ℓ-ring) with the Zariski topology.

Zariski or Co-Zariski topology

As far as I know, until 1975, only the Zariski topology was used on the base spaces for the sheaf representations.

Why the Zariski topology? Firstly, the Zariski topology was used in algebraic topology. Secondly: If we start with the ring $C(X)$ of continuous real-valued functions on a compact Hausdorff space X, then the topology of X is reproduced by the Zariski topology on the set of all maximal ideals. It does not matter whether one looks at $C(X)$ as a ring or as a lattice-ordered group. See, for example:

R. Bkouche, Couples spectraux et faisceaux associés. Applications aux anneaux de fonctions, Bull. Soc. Math. France 98 (1970), 253 - 295.

Another example is the ring direct sum $\sum_{i \in I} K_i$ of fields K_i. The Zariski topology on the maximal ideals reproduces the discrete topology on I, the co-Zariski topology is the cofinite topology on I.

To the best of my knowledge, the use of the co-Zariski topology for sheaf representations first occurs in a paper by J. F. Kennison in 1976:

J. F. Kennison, Integral domain type representations in sheaves and other topoi. Math. Z. 151 (1976), 35-56.

J. F. Kennison and C. S. Ledbetter, Sheaf representations and the Dedekind reals. Applications of Sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977), pp. 500-513, Lecture Notes in Math., 753, Springer, Berlin-New York, 1979.

In the case of Boolean algebras of factor congruences the Zariski and the co-Zariski topology agree on the space of maximal ideals. So there is no difference between the two. But in other cases the Zariski topology seems to be the natural one. I would be interesting to hear arguments that for some purposes the co-Zariski topology is preferable.

Kennison's approach

Kennison has an approach that is different from others and I want to comment on it, as I understand it. (The reason is that it sheds light on the role of the co-Zariski topology for sheaf representations.) He considers the class $\mathcal{A}$ of all Ω-algebras or an equationally defined subclass thereof. In the class $\mathcal{A}$ he chooses a subclass Σ that is closed for subalgebras and ultraproducts.

Kennison asks the question: *Which algebras in $\mathcal{A}$ allow a representation as algebras of all global section of a sheaf of Σ-algebras?*
His answer is: *Precisely those algebras that are in the limit closure of Σ.*

Actually I do not know whether this statement holds in general, since Kennison suppose that the algebras in $\mathcal{A}$ have a group operation among their operations. He determines this limit closure for some examples.
1) $\mathcal{A}$ is the class of all unital commutative rings R, Σ the subclass of integral domains. Then R is representable as the ring of all global sections of a sheaf of integral domains if and only if R has no nilpotent element AND satisfies a sequence of axioms (D_n). As a consequence, not every unital commutative ring without nilpotent element is isomorphic to the ring of global sections of the sheaf constructed canonically (as above) over the set of prime ideals with the co-Zariski topology.
2) $\mathcal{A}$ is the class of all unital commutative f-rings R, Σ the subclass of integral domains. Then R is representable as the ring of all global sections of a sheaf of integral domains if and only if R has no nilpotent element AND satisfies the axioms (D_1).
3) In those early papers Kennison does not apply his general result to the class $\mathcal{A}$ of unital commutative lattice ordered groups or MV-algebras R with the subclass Σ of totally ordered ones. I am sure that in these cases the limit closure of Σ is all of $\mathcal{A}$, which explains that in these cases one has representations of R by the global sections of a sheaf of totally ordered groups (MV-algebras) over the spectrum with the co-Zariski topology.

Applications of sheaf representations

Sheaf representations over Boolean spaces, also being rephrased as Boolean products have been used extensively in **model theory**:

A. Macintyre, Model-completeness for sheaves of structures, Fundamenta Mathematicae 81 (1973), 73-85.
S. Comer, Elementary properties of structures of sections, Bol. Soc. Mat. Mexicana 19 (1974), 78-85.
L. Lipshitz, D. Saracino, The model companion of the theory of commutative rings without nilpotent elements, Proc. Amer. Math. Soc. 38 (1973), 381-387.
A. B. Carson, The model completion of the theory of commutative regular rings, J. Algebra 27 (1973), 136-146.
L. van den DRIES, Artin-Schreier theory for commutative regular rings, Annals of Mathematical Logic 12 (1977), 113-150.
S. Burris, Decidability and Boolean representations, Memoirs AMS 246 (1981).
S. Burris, H. Werner, Sheaf constructions and their elementary properties, Trans. Amer. Math. Soc. 248 (1979), 269-309.

In later chapters of the book by Burris and Sankappanavar one can find a well presented summary of model theoretic results using sheaf representations over Boolean spaces (equivalently Boolean powers):

S. Burris, H.P. Sankappanavar, A course in universal algebra, Springer-Verlag, 1981.

Sheaf representations over Boolean spaces, equivalently Boolean products, are particularly apt to applying model theoretic methods. Weispfennning has tried to extend the realm of applicability of these methods to more general sheaf representations replacing 'Boolean' by 'distributive lattice'. But the generalization was not too successful:

V. Weispfenning, Model theory of lattice products, Habilitationsschrift, Universität Heidelberg (235 pp.).
—, Lattice products, in Logic Colloquium 1978, Mons, North-Holland, Amsterdam, pp. 423-426.

These applications in model theory are the only systematic use of sheaf representations that I know of. Occasionally sheaf representations have been used for exhibiting Examples with specific properties. Already Pierce constructs *modules with almost any pathological properties that can be imagined* in the Memoir cited above. Certain 'hulls' of given algebraic structures have been constructed using sheaf representations:

Kist, J.: Compact spaces of minimal prime ideals. Math. Z. 111, 151-158 (1969).
K. Keimel, Baer extensions of rings and Stone extensions of semigroups Semigroup Forum 2 (1971), 55-63.

D. H. Adams, Semigroups with not non-zero nilpotents, Math. Z. 123 (1971), 168-176.

B. A. Davey, m-Stone lattices, Can. J. Math., Vol. XXIV, No. 6, (1972), 1027-1032.

W. Rump, Y. C. Yang, Lateral completion and structure sheaf of an archimedean l-group, J. Pure and Appl. Algebra 213 (2009), 136-143.

In his paper mentioned in a previous section, B. A. Davey has given a general explanation for the constructions given in the three papers just mentioned.

Apart from these applications sheaves are mainly used for visualizing given algebraic structures as algebras of continuous functions with values in hopefully simpler algebras. For this reason I am not sure whether the revival of sheaf representations in the last years is justified. What is a representation good for, if one is not using it as tool for obtaining new results?

Sheaves with values in other categories

In the above I was concentrating on sheaves with values in categories of algebras. The stalks are algebraic structures with the discrete topology.

But already in early times one was considering sheaves with values in topological spaces, uniform spaces, Banach spaces, C^*-algebras, etc. One can see this already from Hofmann's SURVEY mentioned above. These efforts were motivated from functional analysis. Instead of the notion of a sheaf, often the terminology 'bundles of Banach spaces' is used, although these 'bundles' need not be locally trivial.

Questions

I would like to invite every reader of these Notes to communicate to me[1] any information concerning the following questions:

- Which aspects should be modified in the above?
- Which aspects should be added to the above considerations?
- Where have sheaf representations been used or applied to obtain results that do not concern the sheaf representation itself?
- What is a justification for the use of the co-Zariski topology for sheaf representations?
- Which bibliographical items not mentioned above have influenced the developments in a significant way?

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- Which sheaf representation theorem published during the past 20 years has been used for purposes outside of sheaf representations?

Please keep in mind that the above reflects my personal experience and knowledge. There must be omissions. I have not touched to questions concerning intuitionistic logic, toposes etc. But I am willing to include any helpful comment. The bibliographical items have not been selected systematically.