The Monad of Probability Measures over Compact Ordered Spaces and its Eilenberg-Moore Algebras

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Abstract

The probability measures on compact Hausdorff spaces K form a compact convex subset $\mathcal{P}K$ of the space of measures with the vague topology. Every continuous map $f\colon K \to L$ of compact Hausdorff spaces induces a continuous affine map $\mathcal{P}f\colon \mathcal{P}K \to \mathcal{P}L$ extending $\mathcal{P}$. Together with the canonical embedding $\varepsilon\colon K \to \mathcal{P}K$ associating to every point its Dirac measure and the barycentric map β associating to every probability measure on $\mathcal{P}K$ its barycenter, we obtain a monad $(\mathcal{P}, \varepsilon, \beta)$. The Eilenberg-Moore algebras of this monad have been characterised to be the compact convex sets embeddable in locally convex topological vector spaces by Swirszcz [31].

We generalise this result to compact ordered spaces in the sense of Nachbin [23]. The probability measures form again a compact ordered space when endowed with the stochastic order. The maps ε and β are shown to preserve the stochastic orders. Thus, we obtain a monad over the category of compact ordered spaces and order preserving continuous maps. The algebras of this monad are shown to be the compact convex ordered sets embeddable in locally convex ordered topological vector spaces.

This result can be seen as a step towards the characterisation of the algebras of the monad of probability measures on the category of stably compact spaces (see [12, Section VI-6]).

Keywords: Compact ordered spaces, probability measures, stochastic order, monads, Eilenberg–Moore algebras

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1 Introduction

In order to reason about programs and programming languages in computer science mathematical models have been developed. Topological structures combined with order theoretical ones have proved to be useful for these purposes. Every feature of the language has to be modelled by a construction on the spaces under consideration. This construction has to be free in the sense that the model does not become more restrictive than required by the language and its operational structure.

Free constructions for equational theories are well known in algebra, for example, free monoids, free groups, etc. Category theory provides an abstract pattern for free constructions: the notion of a monad (see, e.g., [21]). This approach that has proved to be extremely useful in semantics (see, e.g., [22]), as it allows to go beyond equational theories. Having a monad, one wants to know the structures or theories that it is free for: These are the *Eilenberg-Moore algebras* of the monad. Their concrete characterisation turns out to be non-obvious for some straightforward monads. For example, the algebras of the ultra-filter monad (= Stone-Cech compactification of discrete spaces) are the compact Hausdorff spaces (cf. [21]), the algebras of the filter

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monad (space of all filters on a set) turn out to be the *continuous lattices* (cf. A. Day [6]) introduced by Dana Scott in [29] precisely for the purpose of constructing models for untyped λ-calculus.

The background for this paper is the modelling of nondeterminism. Depending on the type of nondeterminism various powerdomain constructions have been introduced. As we are interested in probabilistic choice, we consider probabilistic powerdomain constructions. Over compact Hausdorff spaces the natural construction is the classical space of probability measures with the vague topology. The algebras of this monad have been shown to be the compact convex sets in locally convex topological vector spaces by Swirzcz [31]. Turning to Polish spaces, the algebras of the same monad have only recently been characterised by Doberkat [7, 8]. The algebraic theory of these algebras is that of convex combinations (= barycentric operations).

Typically, spaces used in semantics are far from being Hausdorff spaces, for example, dcpos (= directed complete partially ordered sets) with the Scott topology (see [12]), stably compact spaces (see [16]), qcb-spaces (see [4, 28]). Probabilistic power domains have been proposed for all of these spaces. In a few cases only the algebras of the respective monads are known as, for example, for the subprobabilistic powerdomain monad over continuous domains (see [15]).

Stably compact spaces have a mediating role between compact Hausdorff spaces and continuous domains. In fact, all compact Hausdorff spaces and most continuous domains are stably compact. One would hope to be able to combine the results known from both sides for a unifying characterisation of the algebras of the probabilistic powerdomain monad over stably compact spaces, thus reconciling the classical with the non-classical theory.

In this paper we make a step towards an answer. Every stably compact space has an intrinsic order and a refined topology making it into a compact ordered space in the sense of Nachbin [23]. We characterise the algebras of the probabilistic powerdomain monad over compact ordered spaces: *These are the compact ordered convex spaces embeddable in locally convex ordered topological vector spaces.* This result does not yet solve the problem for stably compact spaces although, at the level of objects, there is no substantial difference between compact ordered and stably compact spaces. But there are a lot more continuous maps between stably compact spaces than between their associated compact ordered spaces with their refined topologies.

Notations. We will denote the set of nonnegative real numbers by $\mathbb{R}_+$, and p, q, r, s, t will always stand for nonnegative reals.

2 Barycentric algebras

In a real vector space a *convex* subset C is characterised by the property that for all r with $0 \leq r \leq 1$:

$$a, b \in C \implies (1-r)a + rb \in C$$

There is an extended literature on the axiomatisation of convex subsets of vector spaces by axioms for the barycentric operations $a +_r b =_{def} (1-r)a + rb$. As far as I know the first axiomatisation is due to Kneser [17]. He has an axiom system close to the second axiom system below including the cancellation property. It is remarkable that his setting is quite general in the sense that he deals with convex subsets of vector spaces over totally ordered skew fields.

W. Neumann [24] seems to be the first to have looked at the equational theory of the barycentric operations. He defines an *abstract convex set* to be a set A endowed with a binary operation $+_r$ for every real number r with $0 \leq r \leq 1$ obeying the following three equational laws:

(B1) $$a +_0 b = a, \quad a +_1 b = b,$$

(B2) $$a +_r a = a$$

Whenever $0 < r < 1$ and $0 \leq p, q \leq 1$,

(B3) $$(a +_p b) +_r (a +_q c) = a +_s (b +_t c)$$

where $s = p +_r q$ and $ts = rq$. Swirscz [31] has given a simpler but equivalent axiomatisation which has been reproduced by Romanowska and Smith in their book [26, Section 5.8]. They use the term *barycentric algebra* instead of *abstract convex set* which they characterise by the following equational laws for the barycentric operations $+_r$ for $0 \leq r \leq 1$:

(B1) $$a +_1 b = b$$

(B2) $$a +_r a = a$$

(SC) $$a +_r b = b +_{r'} a$$

(SA) $$a +_r (b +_p c) = (a +_{\frac{rp'}{(rp)'}} b) +_{rp} c \quad \text{for } 0 < r, p < 1$$

where $r' = 1 - r$ for every r with $0 \leq r \leq 1$. SC stands for *skew commutativity* and SA for *skew associativity*. These axioms were also used by Graham [13] and by Jones and Plotkin [14, 15] when they introduced the notion of an *abstract probabilistic powerdomain*.[1]

The axiom (SA) can be replaced by the equivalent one:

(SA′) $$(a +_s b) +_q c = a +_{(s'q')'} (b +_{\frac{q}{(q's')'}} c) \quad \text{for } 0 < s, q < 1$$

The above laws are easily seen to hold for the barycentric operations $a +_r b =_{def} (1 - r)a + rb$ in convex subsets of vector spaces. But barycentric algebras may be very different from convex sets in vector spaces:

Example 2.1. Every $\vee$-semilattice A becomes a barycentric algebra if one defines $a +_r b = a \vee b$ for $0 < r < 1$ and $a +_1 b = b, \quad a +_0 b = a$. W. Neumann has shown that the $\vee$-semilattices form the only proper nontrivial equationally definable subclass of the class of all barycentric algebras.

A map $\alpha\colon A \to B$ of barycentric algebras is said to be *affine* if $\alpha(a +_r b) = \alpha(a) +_r \alpha(b)$ for all $a, b \in A$ and $0 \leq r \leq 1$.

The question of embeddabilty in a vector space has been answered by W. Neumann:

Lemma 2.2. *A barycentric algebra can be affinely embedded in a real vector space if and only if it satisfies the following cancellation axiom:*

(C1) *For every r with $0 < r < 1$,* $\quad a +_r c = b +_r c \implies a = b$

The following surprising lemma is due to W. Neumann [24, Lemma 2]. It has inspired an order theoretical generalisation which we present and prove in the next section (see Lemma 3.4).

Lemma 2.3. *If $a +_r b = a +_r c$ holds in a barycentric algebra for some r with $0 < r < 1$, then $a +_r b = a +_r c$ holds for all such r.*

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[1] Axiom (SA) is quite hard to use for somebody not working with it all the time. For this reason it is useful to notice that every barycentric algebra is embeddable in a cone in the following sense: A *cone* is understood to be a set C together with an addition $(x, y) \mapsto x + y$ which is commutative, associative and has a neutral element 0 and a scalar multiplication $(r, x) \mapsto rx$ for nonnegative real numbers r which satisfies the same laws as in vector spaces, namely $0 \cdot x = 0$, $1 \cdot x = x$, $r(sx) = (rs)x$, $(r + s)x = rx + sx$, $r(x + y) = rx + ry$. Thus, in a cone one can argue as in a vector space, as long as one does not use negative scalars or subtraction. This helps a lot in order to verify the claims in the sequel of this paper.

One way to embed a barycentric algebra A in a cone is as follows: Let $C =_{def} \{0\} \cup \{(r, a) \mid 0 < r \in \mathbb{R}, a \in A\}$. Define addition by $(r, a) + (s, b) =_{def} (r + s, a +_{\frac{s}{r+s}} b)$ and scalar multiplication by $r(s, a) =_{def} (rs, a)$. (For $r = 0$, one puts $r(s, a) = 0$ and addition with 0 is defined in the obvious way.) Then C becomes a cone and the map $e = (a \mapsto (1, a))$ is an injection of A into C in such a way that $e(a +_r b) = (1 - r)a + rb$.

3 Ordered barycentric algebras

An *ordered* barycentric algebra is defined to be a barycentric algebra with a partial order $\leq$ such that the barycentric operations $x +_r y$ are monotone in x and y, that is, if

(B4) $$a \leq b,\ a' \leq b' \ \Rightarrow\ a +_r a' \leq b +_r b'$$

holds for all r.

We ask the question under which additional hypothesis an ordered barycentric algebra A is embeddable into an ordered (real) vector space V with respect to its order and barycentric structure.

Let us start with an ordered barycentric algebra A. We may embed it affinely into a real vector space V iff it satisfies the cancellation axiom (C1). If this is the case, we want to endow V with an order which extends the given order on A such that V becomes an ordered vector space. An *ordered vector space* is a real vector space with a partial order such that addition and multiplication with nonnegative reals are order preserving. In an ordered vector space, one has $a \leq b$ iff $0 \leq b - a$, that is, the order is characterised by its *positive cone* P, that is, a subset of V which is a *cone*, that is:

$$P + P \subseteq P \ \text{ and } \ rP \subseteq P \text{ for all } r \in \mathbb{R}_+$$

and *pointed*, that is:

$$P \cap -P = \{0\}$$

Thus, we want to find a positive cone P in V such that, for elements $x, y \in A$, one has $x \leq y$ iff $y - x \in P$. For this, let P be the set of all elements in V which can be represented in the form $r(y - x)$ with $r \geq 0$ and $x \leq y$ in A. We then have indeed:

Lemma 3.1. *P is a positive cone in V.*

Proof. Let $a, b \in P$. Clearly $ra \in P$ for all $r \geq 0$. For proving that $a + b \in P$ it suffices to consider the case that a, b are both different from 0. Then $a = r(y - x)$ and $b = s(y' - x')$ for some $r, s > 0$ and $x \leq y, x' \leq y'$ in A. Let $p = \frac{s}{r+s}$. Then $a + b = r(y - x) + s(y' - x') = (r + s)\big((1 - p)(y - x) + p(y' - x')\big) = (r + s)\big((1 - p)y + py' - ((1 - p)x + px')\big)$. As $(1 - p)y + py' \in A$ and $(1 - p)x + px' \in A$ and as $(1 - p)x + px' \leq (1 - p)y + py'$ by (B4), we conclude $a + b \in A$. Thus, $P + P \subseteq P$.

Suppose that $0 \neq a \in P \cap -P$. Then $a = r(y - x) = -s(y' - x')$ for some $r, s > 0$ and $x < y, x' < y'$ in A. Let $p = \frac{s}{r+s}$ as above. Then $x +_p x' \leq y +_p x' \leq y +_p y'$ holds in A by (B4). If we suppose cancellation (C1), equality cannot hold in either case. Hence, $x +_p x' < y +_p y'$. We conclude that $ry + sy' = (r + s)(y +_p y') \neq (r + s)(x +_p x') = rx + sx'$, i.e., $r(y - x) \neq -s(y' - x')$, a contradiction. □

If we endow V with the order defined by the positive cone P, the embedding of A into V is order preserving. The following example shows that A need not be order-embedded in V:

Example 3.2. On the unit square $[0, 1]^2$ take the coordinatewise order except that the point $(1, 1)$ dominates only the points (r, r) on the diagonal. We have an ordered barycentric algebra. It is naturally embedded into $\mathbb{R}^2$. The above construction yields the coordinatewise ordering on $\mathbb{R}^2$ the positive cone of which is $\mathbb{R}^2_+$. In this ordering of $\mathbb{R}^2$, the point $(1, 1)$ becomes the greatest element of A, which is not the case for the original order. The natural injection is not an order embedding. The problem is that the order on A is not translation invariant in V: For $a = (0, 1), b = (1, 1), c = (1, 0), d = (0, 0)$ we have indeed $d \leq a$, but $d + c = c \not\leq b = a + c$. Internally in A, we can reformulate this fact in the following form: Whenever $0 < r < 1$, then $a +_r c = (1 - r, r) \leq (1, r) = b +_r c$, but $a \not\leq b$. Thus A does not satisfy the following order cancellation property:

For $a, b, c \in A$:

(C2) $$\text{For all } r \text{ with } 0 < r < 1, \qquad a +_r c \leq b +_r c \implies a \leq b$$

The cancellation property (C1) is a consequence of this order cancellation property (C2).

If A is embeddable in an ordered vector space, then (C2) holds. Indeed, in an ordered vector space $a +_r c \leq b +_r c$ is equivalent to $(1-r)a + rc \leq (1-r)b + rc$. By adding $-rc$ we obtain $(1-r)a \leq (1-r)b$ which implies $a \leq b$ as $1 - r \neq 0$. Thus, (C2) is a necessary property for A to be embeddable into an ordered vector space. It also is sufficient:

Proposition 3.3. *An ordered barycentric algebra A is embeddable (for its barycentric and its order structure) in an ordered vector space if and only if it satisfies the order cancellation axiom (C2).*

Proof. As (C2) implies (C1), an ordered barycentric algebra satisfying (C2) is embeddable in a real vector space V with respect to its barycentric structure. We consider the positive cone P in V as in Lemma 3.1. It remains to show: If x, y are elements of A such that $y - x \in P$, then $x \leq y$ in A. Indeed if $x, y \in A$ and $y - x \in P$, then there are elements $x' \leq y'$ in A and $r \geq 0$ such that $y - x = r(y' - x')$. Then $x + ry' = y + rx'$. With $p = \frac{r}{1+r}$ we obtain $x +_p y' = y +_p x'$ internally in A. Using $x' \leq y'$ and the monotonicity of the barycentric operations in A, we conclude that

$$(x +_p y') +_q x' \leq (y +_p x') +_q y'$$

Choosing $q = \frac{p}{1+p}$ and using the axiom (SA') this inequality can be rewritten as

$$x +_s (y' +_{\frac{1}{2}} x') \leq y +_s (x' +_{\frac{1}{2}} y')$$

As $y' +_{\frac{1}{2}} x' = x' +_{\frac{1}{2}} y'$ by axiom (SC), the order cancellation axiom (C2) yields $x \leq y$ in A. □

We now come to the order theoretical generalisation of Neumann's Lemma 2.3. One should notice that the statement is closely related to the order cancellation axiom (C2). The proof is quite technical. The reason is that one is not used to calculate in barycentic algebras. But the proof is guided by natural geometric constructions. Neumann's Lemma 2.3 is a special case thereof, if one replaces all inequalities $\leq$ by equality:

Lemma 3.4. [2] *Let a, b, c be elements of an ordered barycentric algebra A. If*

$$(*) \qquad a +_r c \leq b +_r c$$

holds for some r with $0 < r < 1$, then it holds for all such r.

Proof. We first notice: If (*) holds for some r with $0 < r < 1$, then it holds for all t with $r < t < 1$. Indeed, with $u = \frac{t-r}{1-r}$ we have $a +_t c = a +_u (a +_t c) \leq (b +_r c) +_u c = b +_t c$, where we have used axiom (SA') and the monotonicity of the barycentric operations.

In order to prove our claim, it suffices now to show that there is a sequence r_n in the unit interval decreasing to 0 such that (*) holds for all r_n.

[2] Added November 10, 2008: In the previous footnote we have given a definition of a notion of abstract cones, and we have indicated how to embed any barycentric algebra in a cone. If we have an ordered barycentric algebra A we can extend its order to an order of the cone C in which it is embedded as a basis: Define $(r, a) \leq (s, b)$ iff $r = s$ and $a \leq b$ in A. This order has the property that addition and multiplication with nonegative scalars is order preserving.
In an ordered cone C we have: If $a + b \leq a + c$ then $ra + b \leq ra + c$ for every $r > 0$.
Let indeed $a + b \leq a + c$. Then $\frac{1}{2}(a + b) \leq \frac{1}{2}(a + b)$, whence $\frac{1}{2}a + b = \frac{1}{2}(a + b) + \frac{1}{2}b \leq \frac{1}{2}(a + c) + \frac{1}{2}b = \frac{1}{2}(a + b) + \frac{1}{2}c \leq \frac{1}{2}(a + c) + \frac{1}{2}c = \frac{1}{2}a + c$. Repeating this argument we obtain $\frac{1}{2^n}a + b \leq \frac{1}{2^n}a + c$ for every natural number n. Given $r > 0$ there is an n such that $\frac{1}{2^n} < r$. Hence $ra + b = \frac{1}{2^n}a + b + (r - \frac{1}{2^n})a \leq \frac{1}{2^n}a + c + (r - \frac{1}{2^n})a = ra + c$.
From this result the Lemma can easily be deduced: Let $a +_r b \leq a +_r c$ for $0 < r < 1$. In the cone C we then have $sa + rb \leq sa + rc$ for every $s > 0$. For a given q with $0 < q < 1$ apply the last inequality for $s = \frac{r(1-q)}{q}$ and one obtains $a +_q b \leq a +_q c$. This illustrates again that it is easier to prove things about barycentric algebras after embedding them into cones.

We have supposed that (*) holds for some r with $0 < r < 1$. We let $r_0 = r$. Let $u_0 = \frac{1}{2-r_0}$ and $r_1 = \frac{r_0}{2-r_0}$. Then:

$$\begin{array}{rcll} a +_{r_1} c & = & a +_{u_0} (a +_{r_0} c) & \text{using (SA)} \\ & \leq & a +_{u_0} (b +_{r_0} c) & \text{using (*) and (B4)} \\ & = & b +_{u_0} (a +_{r_0} c) & \text{using (SA) and (SC)} \\ & \leq & b +_{u_0} (b +_{r_0} c) & \text{using (*) and (B4)} \\ & = & b +_{r_1} c & \text{using (SA)} \end{array}$$

We may continue recursively by defining $r_{n+1} = \frac{r_n}{2-r_n}$ and we find that $a +_{r_n} c \leq b +_{r_n} c$ for all n. It is an elementary exercise to verify that the sequence r_n is decreasing to 0. □

4 Topological barycentric algebras

A *topological barycentric algebra* is a barycentric algebra A together with a topology such that the map

$$(r, x, y) \to x +_r y \colon [0,1] \times A \times A \to A$$

is continuous where the unit interval $[0, 1]$ is endowed with the usual Hausdorff interval topology. Note that we not only ask each operation $+_r$ to be continuous but that the operations depend continuously on r.

An *ordered topological barycentric algebra* A is defined to be both a topological and an ordered barycentric algebra for which the graph of the order is closed in $A \times A$. Thus, it is an ordered topological space in the sense of Nachbin [23]. The closedness of the order implies that the topology is Hausdorff. With respect to the trivial order $=$, a topological barycentric algebra is an ordered topological barycentric algebra if and only if it is Hausdorff. Indeed, a topological space X is a Hausdorff space if and only if the diagonal in $X \times X$ is closed.

The generalised Neumann Lemma 3.4 implies the order cancellation property (C2) in a slightly more generalised form:

Proposition 4.1. *Every ordered topological barycentric algebra satisfies the order cancellation axiom:*

(C2) *For every* r *with* $0 < r < 1$, $\quad a +_r c \leq b +_r c \implies a \leq b$

Proof. By 3.4 the hypothesis $ba +_r c \leq b +_r c$ for some r with $0 < r < 1$ implies $a +_r c \leq b +_r c$ for all r with $0 < r < 1$. As $r \mapsto a +_r c \colon [0, 1] \to A$ is continuous and as the toplogy of A is Hausdorff, $\lim_{r\to 0} a +_r c = a +_0 c = a$ and similarly $\lim_{r\to 0} b +_r c = b$. As the graph of the order is closed, we conclude $a \leq b$. □

In particular:

Corollary 4.2. *Any Hausdorff topological barycentric algebra* A *satisfies the cancellation axiom*

(C1) *For every* r *with* $0 < r < 1$, $\; a +_r c = a +_r d \implies c = d$

Thus, any Hausdorff topological barycentric algebra is algebraically embeddable in a vector space and any ordered topological barycentric algebra is algebraically and order theoretically embeddable in an ordered vector space.

Problem. 1. Under which hypotheses is a Hausdorff topological barycentric algebra (topologically and algebraically) embeddable in a topological vector space, or even in a locally convex topological vector space?

2. Under which hypotheses is an ordered topological barycentric algebra embeddable in an ordered topological vector space, or even in a locally convex one?

There is the following partial answer to Problem 1: Using our result that a Hausdorff topological barycentric algebra is algebraically embeddable in a vector space (see 2.2, 4.2), we can restate results due to J.D. Lawson [18], Lawson and Madison [20] and independently to Roberts [25]:

Proposition 4.3. *Every compact Hausdorff topological barycentric algebra A can be embedded (algebraically and topologically) in a topological vector space V. If, in addition, A is locally convex in the weak sense, that is, if every neighborhood of an $a \in A$ contains a convex neighborhood of a, then V can be chosen to be locally convex.*

5 The monad of probability measures over compact Hausdorff spaces

For compact Hausdorff spaces X, we shall use the following notations:

$\mathcal{C}X$	the Banach space of all real valued continuous functions on X with the sup-norm,
$\mathcal{C}_+X$	the positive cone of all nonnegative functions $f \in \mathcal{C}X$,
$\mathcal{M}X$	the vector space of all signed regular Borel measures on X
$\mathcal{M}_+X$	the positive cone of nonnegative regular Borel measures.
$\mathcal{P}X$	the set of regular probability measures.

By $\leq$ we denote on $\mathcal{C}X$ the usual pointwise defined order with $\mathcal{C}_+X$ as positive cone and also on $\mathcal{M}X$ the usual order of measures with $\mathcal{M}_+X$ as positive cone. Via the Riesz Representation Theorem we will identify $\mathcal{M}X$ with the dual space of all bounded linear functionals φ on $\mathcal{C}X$. For $\varphi \in \mathcal{M}X$ and $f \in \mathcal{C}X$, we will write

$$\langle \varphi, f \rangle = \int f d\varphi$$

for the natural bilinear map $\mathcal{M}X \times \mathcal{C}X \to \mathbb{R}$.

On $\mathcal{M}X$ and its subsets we will consider the weak* topology, also called the vague topology. It is the coarsest topology on $\mathcal{M}X$ for which the linear maps $\varphi \mapsto \langle \varphi, f \rangle$ are continuous for all $f \in \mathcal{C}X$. $\mathcal{P}X$ is the subset of all $\varphi \in \mathcal{M}_+X$ with $\langle \varphi, \mathbf{1} \rangle = 1$. (Here $\mathbf{1}$ denotes the constant function on X with value 1). Thus $\mathcal{P}X$ is a convex subset which is compact Hausdorff in the weak* topology.

Occasionally we will use that $\mathcal{M}X$ is a Banach space. But when we talk about topology on $\mathcal{M}X$, we always mean the weak* topology.

Assigning the Dirac measure $\varepsilon_X(x)$ to every $x \in X$ yields a continuous embedding

$$\varepsilon_X \colon X \to \mathcal{P}X \subseteq \mathcal{M}X$$

Let us specialise now to a Hausdorff compact topological barycentric algebra K. The continuous affine real-valued functions form a uniformly closed linear subspace $\mathcal{A}K$ of $\mathcal{C}K$ which contains the constant function $\mathbf{1}$. Restricting every $\varphi \in \mathcal{M}K$ to $\mathcal{A}K$ yields a linear map

$$\beta_K = (\varphi \mapsto \varphi|_{\mathcal{A}K}) \colon \mathcal{M}K \to (\mathcal{A}K)^*$$

where $(\mathcal{A}K)^*$ is the topological dual of $\mathcal{A}K$. The map β_K is continuous and even a quotient map for the respective weak* topologies.

Composing ε_K with β_K yields a continuous map from K into $(\mathcal{A}K)^*$. A point $x \in K$ is mapped to the point evaluation $f \mapsto f(x) \colon \mathcal{A}K \to \mathbb{R}$. This composed map $\beta_K \circ \varepsilon_K$ is affine as, for all $f \in \mathcal{A}K$, we have $\langle \varepsilon_K(x +_r y), f \rangle = f(x +_r y) = f(x) +_r f(y) = \langle \varepsilon_K(x), f \rangle +_r \langle \varepsilon_K(y), f \rangle$.

Suppose now that K is embeddable in a locally convex topological vector space. Then the continuous real-valued affine functions separate the points of K. This implies that the map $\beta_K \circ \varepsilon_K \colon K \to (\mathcal{A}K)^*$ is injective. Thus, K is topologically and algebraically embedded into $(\mathcal{A}K)^*$. Henceforward, we will identify K with its image in $(\mathcal{A}K)^*$; i.e., $x \in K$ is identified with $\beta_K(\varepsilon_K(x))$ and, thus:

$$\beta_K \circ \varepsilon_K = \mathrm{id}_K$$

We now use that every probability measure on K has a barycenter:

Theorem 5.1. [1, (2.13)] *Let K be a compact convex set embeddable in a locally convex topological vector space. Then, for every probability measure $\varphi \in \mathcal{P}K$, there is a uniquely determined $x \in K$ such that*

$$\langle \varphi, f \rangle = f(x) \text{ for all } f \in \mathcal{A}K$$

The element x is called the barycenter *of φ.*

This theorem tells us that $\beta_K(\varphi) = \beta_K(\varepsilon_K(x))$, whenever φ is a probability measure and x its barycenter. This implies that β_K maps $\mathcal{P}K$ onto (the image of) K. Having identified $x \in K$ with $\beta_K(\varepsilon_K(x))$, $\beta_K(\varphi)$ becomes the barycenter of φ. Thus, when restricted to $\mathcal{P}K$, $\beta_K : \mathcal{P}K \to K$ assigns its barycenter to every probability measure on K.

We can apply the preceding to the compact convex set $K = \mathcal{P}X$ of probability measures on a compact Hausdorff space X. We then obtain a continuous affine map

$$\mu_X =_{def} \beta_{\mathcal{P}X} : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$$

It is well-known (see [31, 30] and can be easily proved from the above) that:

Proposition 5.2. *$(\mathcal{P}, \mu, \varepsilon)$ defines a monad over the category* Comp *of compact Hausdorff spaces and continuous maps.*

The unit $\varepsilon_X : X \to \mathcal{P}X$ assigns the Dirac measure $\varepsilon_X(x)$ to every $x \in X$. The multiplication $\mu_X : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$ assigns to each probability measure Φ on $\mathcal{P}X$ its barycenter $\mu_X(\Phi) \in \mathcal{P}X$ which is characterised by the property that

$$\big\langle \Phi, \varphi \mapsto \varphi(f) \big\rangle = \langle \mu_X(\Phi), f \rangle \text{ for every } f \in \mathcal{C}X$$

Compact convex sets K in locally convex topological vector spaces are easily seen to be algebras of this monad with $\beta_K : \mathcal{P}K \to K$ as structure map.

E.E. Doberkat has indicated to me that Fedorchuk in his survey article [11, p. 56] cites the following Theorem due to Swirszcz [31]:

Theorem 5.3. *The algebras of the monad $\mathcal{P}$ over the category of compact Hausdorff spaces are precisely the compact convex sets K embeddable in locally convex topological vector spaces together with the barycenter maps $\beta_K : \mathcal{P}K \to K$ as structure maps.*

In his proof Swirszcz [31] uses a slight generalisation of a theorem by Linton on monadic functors and tools from functional analysis like the Krein-Šmulian Theorem. A proof avoiding category theory is due to Peter Taylor. His proof is presented by Z. Semadeni in [30, Section 7].

Let me outline an alternative proof for this theorem: Let $\alpha : \mathcal{P}K \to K$ be an algebra for the monad $\mathcal{P}$. We may define a continuous barycentric structure $(p, a, b) \mapsto a +_p b : [0,1] \times K \times K \to K$ by $a +_p b = \alpha((1-p)\varepsilon_K(a) + p\varepsilon_K(b))$. Then α is affine, i.e., $\alpha((1-p)\varphi + p\psi) = \alpha(\varphi) +_p \alpha(\psi)$ for all $\varphi, \psi \in \mathcal{P}K$ and $0 \le p \le 1$ which implies that the equational laws of a barycentric algebra, which hold in $\mathcal{P}K$, are inherited by K; so K becomes a compact Hausdorff topological barycentric algebra.

In order to show that K is embeddable into a locally convex topological vector space we use Proposition 4.3. It only remains to verify the local convexity hypothesis: Choose an $a \in K$ and any neighborhood U of a. As $\alpha(\varepsilon_K(a)) = a$, there is a neighborhood V of $\varepsilon_K(a)$ in $\mathcal{P}K$ such that $\alpha(V) \subseteq U$. As $\mathcal{P}K$ is locally convex, we may suppose V to be convex. As the map α is affine, $\alpha(V)$ is convex. Moreover, $\alpha(V)$ is a neighborhood of a; indeed as ε_K is a topological embedding of K into $\mathcal{P}K$, the set $W = \varepsilon^{-1}(V)$ is a neighborhood of a and $W \subseteq \alpha(V)$. Thus $\alpha(V)$ is a convex neighborhood of a contained in U.

6 The subprobabilistic powerdomain monad on compact ordered spaces

We consider compact ordered spaces X in the sense of Nachbin [23], that is, sets together with a compact topology and a partial order $\preccurlyeq$ the graph of which is closed in $X \times X$. Recall that the topology of a compact ordered space satisfies the Hausdorff separation axiom. We will denote by $\mathsf{CompOrd}$ the category of compact ordered spaces and order preserving continuous maps. Forgetting the order yields a forgetful functor from the category $\mathsf{CompOrd}$ to the category Comp of compact ordered spaces. On the other hand, Comp may be considered to be a full subcategory of $\mathsf{CompOrd}$ by putting the trivial order $=$ on each compact Hausdorff space.

We are going to generalise the results of the previous section to compact ordered spaces. We would have liked to prove Theorem 6.5 along the lines of the proof of 5.3 outlined at the end of the previous section. As we did not succeed, we adapt Peter Taylor's proof as presented by Semadeni to the ordered situation and we fill a gap in that proof.

We denote by $\mathcal{C}^m X$ the cone of all order preserving continuous functions $f\colon X \to \mathbb{R}$. Clearly, $\mathcal{C}^m X$ is uniformly closed in $\mathcal{C}X$. But applying the Stone–Weierstraß Theorem, D. E. Edwards [10] (see also [2, Lemma 19]) has shown:

Lemma 6.1. *The linear subspace of $\mathcal{C}X$ generated by the cone $\mathcal{C}^m X$ is uniformly dense in $\mathcal{C}X$.*

Besides the usual order $\leq$ we consider the *stochastic order* $\preccurlyeq$ on $\mathcal{M}X$ – a notion going back to Edwards [10] – the positive cone of which is the dual of the cone $\mathcal{C}^m X$:

$$\begin{array}{rll} 0 \preccurlyeq \varphi & \text{if and only if} & 0 \leq \langle \varphi, f\rangle \text{ for all } f \in \mathcal{C}^m X\,, \\ \varphi \preccurlyeq \psi & \text{if and only if} & \langle \varphi, f\rangle \leq \langle \psi, f\rangle \text{ for all } f \in \mathcal{C}^m X\,. \end{array}$$

By its definition the positive cone for the order $\preccurlyeq$ is weak*-closed. As the linear subspace generated by $\mathcal{C}^m X$ is uniformly closed in $\mathcal{C}X$ by 6.1, this positive cone is indeed pointed. Thus, $\mathcal{M}X$ with the stochastic order is a locally convex ordered topological vector space. (Recall that an *ordered topological vector space* is at the same time a topological and an ordered vector space such that the graph of the order or, equivalently, the positive cone is closed.) Restricting the stochastic order to the compact convex set $\mathcal{P}X$ of probability measures yields an order with a closed graph.

We now consider two compact ordered spaces X and Y and an order preserving continuous map $g\colon X \to Y$. This map induces a positive linear map $\mathcal{C}g\colon \mathcal{C}Y \to \mathcal{C}X$ defined by $(\mathcal{C}g)(f) = f \circ g$ which preserves the constant functions $\mathcal{C}g(\mathbf{1}) = \mathbf{1}$. Moreover $\mathcal{C}^m Y$ is mapped into $\mathcal{C}^m X$; for if f is order preserving, then $f \circ g$ is order preserving, too. The adjoint $\mathcal{M}g\colon \mathcal{M}X \to \mathcal{M}Y$, defined by $\mathcal{M}g(\varphi) = \varphi \circ \mathcal{C}g$, is linear, it preserves the orders $\leq$ and $\preccurlyeq$ and the norm. In particular, $\mathcal{M}g$ maps $\mathcal{P}X$ into $\mathcal{P}Y$. Moreover, $\mathcal{M}g$ is continuous for the respective weak* topologies.

Thus, for every compact ordered space X, the set $\mathcal{P}X$ of probability measures with the weak* topology and the stochastic order $\preccurlyeq$ is a compact ordered convex set (see also [2, Theorem 31]). Every order preserving continuous map $g\colon X \to Y$ of compact ordered spaces induces an order preserving continuous affine map $\mathcal{P}g = \mathcal{M}g|_{\mathcal{P}X}\colon \mathcal{P}X \to \mathcal{P}Y$, and we have a functor $\mathcal{P}$ from the category $\mathsf{CompOrd}$ of compact ordered spaces to the category of compact ordered convex sets and order preserving continuous affine functions.

We are going to show that $\mathcal{P}$ defines a monad:

Lemma 6.2. [2, Proposition 32] *The map $\varepsilon_X\colon X \to \mathcal{P}X$ is not only a topological but also an order embedding.*

Proof. Let x, y be elements of X. If $x \preccurlyeq y$ then, for every $f \in \mathcal{C}^m X$, we have $f(x) \leq f(y)$, that is, $\varepsilon_X(x)(f) \leq \varepsilon_X(y)(f)$, whence, $\varepsilon_X(x) \preccurlyeq \varepsilon_X(y)$. If $x \not\preccurlyeq y$ then, by [23, Theorem 1 and Corollary of Theorem 4], there is an $f \in \mathcal{C}^m X$ such that $f(x) \not\leq f(y)$, that is $\varepsilon_X(x) \not\preccurlyeq \varepsilon_X(y)$. □

Consider now the special case of a compact ordered convex set K.

Lemma 6.3. *If K is embeddable in an ordered locally convex topological vector space, then the barycenter map $\beta_K \colon \mathcal{P}K \to K$ preserves the order $\preccurlyeq$.*

Proof. Let $\varphi, \psi \in \mathcal{P}K$ with $\varphi \preccurlyeq \psi$. For all $f \in \mathcal{C}^m K$ we have $\langle \varphi, f\rangle \leq \langle \psi, f\rangle$. This then holds in particular for all continuous order preserving affine functions $f \colon K \to \mathbb{R}$. But for all such functions f one has $\langle \varphi, f\rangle = f(\beta_K(\varphi))$ and $\langle \psi, f\rangle = f(\beta_K(\psi))$, whence $f(\beta_K(\varphi)) \leq f(\beta_K(\psi))$. And this implies $\beta_K(\varphi) \preccurlyeq \beta_K(\psi)$, as we are going to show.

We claim: if x, y be elements of K such that $x \not\preccurlyeq y$, then there is a continuous order preserving affine function on K with $f(y) < f(x)$. For the proof we use that K is embeddable in a locally convex ordered topological vector space V. Let C be the positive cone of V which is closed and convex. As $x \not\preccurlyeq y$, we have $y - x \notin C$. By the Hahn-Banach Separation theorems, there is a continuous linear functional f on V such that $f(y-x) < f(c)$ for all $c \in C$. This implies $f(y) - f(x) = f(y-x) < 0$ and $f(c) \geq 0$ for all $c \in C$. Thus f is order preserving and $f(y) < f(x)$. The restriction of f to K is affine. □

We apply the preceding lemma to the the compact ordered convex set $K = \mathcal{P}X$ of probability measures with the stochastic order over a compact ordered space X and we obtain that the multiplication $\mu_X = \beta_{\mathcal{P}X} \colon \mathcal{P}\mathcal{P}X \to \mathcal{P}X$ is also order preserving. We summarize:

Proposition 6.4. $(\mathcal{P}, \varepsilon, \mu)$ *is a monad over the category of compact ordered spaces and continuous order preserving maps.*

From the preceding we may conclude that every compact ordered convex set K which is embeddable in a locally convex ordered topological vector space is an algebra of the monad $\mathcal{P}$ with the barycenter map $\beta_K \colon \mathcal{P}K \to K$ as structure map. The converse also holds:

Theorem 6.5. *The algebras of the monad $(\mathcal{P}, \varepsilon, \mu)$ over the category of compact ordered spaces and continuous order preserving maps are precisely the compact ordered convex sets embeddable in locally convex ordered topological vector spaces.*

The remainder of this section is devoted to the proof of this theorem. We adapt the proof presented by Semadeni [30] to the ordered situation.[3] Consider an algebra of the monad $\mathcal{P}$, that is, a compact ordered space K together with a continuous order preserving map

$$\alpha \colon \mathcal{P}K \to K$$

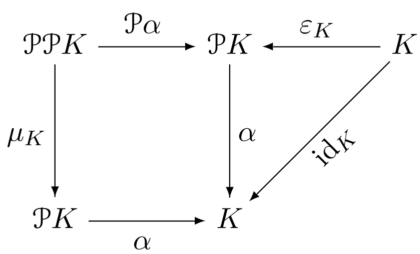
such that

(A1) $$\alpha \circ \varepsilon_K = \mathrm{id}_K$$

and

(A2) $$\alpha \circ \mathcal{P}(\alpha) = \alpha \circ \mu_K$$

i.e., the following diagram commutes



[3] It seems that Semadeni's proof has a gap, as he uses the cancellation property for algebras of his monad without justification. We deduce it from corollary 4.2. In fact, the sections 2, 3 and 4 are mainly written for the purpose of proving 4.2.

We define a convex structure on K by:

$$x +_r y =_{def} \alpha((1-r)\varepsilon_K(x) + r\varepsilon_K(y))$$

for $x, y \in K$ and $0 \leq r \leq 1$. As all maps involved in the definition of $x +_r y$ are continuous and order preserving, the operation $x +_r y$ is continuous and order preserving in x, y and r simultaneously.

Lemma 6.6. *α is an affine map, i.e., for $\varphi, \psi \in \mathcal{P}K$ and $0 \leq r \leq 1$ we have:*

$$\alpha((1-r)\varphi + r\psi) = \alpha(\varphi) +_r \alpha(\psi)$$

Proof. Consider $\nu = r\varepsilon_{\mathcal{P}K}(\varphi) + (1-r)\varepsilon_{\mathcal{P}K}(\psi)$ which is an element of $\mathcal{P}\mathcal{P}K$. Then $\alpha((\mathcal{P}\alpha)(\nu)) = \alpha(\mu_{\mathcal{P}K}(\nu))$ by (A2). On the left hand side we have $\alpha((\mathcal{P}\alpha)(\nu)) = \alpha((1-r)\varepsilon_K(\alpha(\varphi)) + r\varepsilon_K(\alpha(\psi))) = \alpha(\varphi) +_r \alpha(\psi))$ by the definition of $\mathcal{P}\alpha$ and of the convex structure on K, and on the right hand side $\alpha(\mu_{\mathcal{P}K}(\nu)) = \alpha((1-r)\varphi + r\psi)$ as $\mu_{\mathcal{P}K}$ is an affine map and as (A1) holds. □

By the previous lemma, K becomes a compact ordered convex set. Let us consider the following relation G on $\mathcal{P}K$:

$$G = \{(\varphi, \psi) \in \mathcal{P}K \times \mathcal{P}K \mid \alpha(\varphi) \preccurlyeq \alpha(\psi)\}$$

As the order $\preccurlyeq$ on K is closed and as α is continuous, G is weak*-closed in $\mathcal{P}K \times \mathcal{P}K$. As α is order preserving, the relation G is reflexive and transitive. As the barycentric operations are order preserving and as α is affine, G is convex. The associated equivalence relation

$$Q = \{(\varphi, \psi) \in \mathcal{P}K \times \mathcal{P}K \mid \alpha(\varphi) = \alpha(\psi)\}$$

is a weak*-closed convex subset of $\mathcal{P}K \times \mathcal{P}K$. As the map $(x, y) \mapsto x - y$ is affine and continuous, the sets

$$B = \{\psi - \varphi \mid (\varphi, \psi) \in G\}$$

$$P = \{\psi - \varphi \mid (\varphi, \psi) \in Q\}$$

are weak*-compact convex subset of $\mathcal{M}K$, both contain 0, and P is symmetric (i.e., $P = -P$).

Lemma 6.7. *Let φ and ψ be elements in $\mathcal{P}K$ with $\alpha(\varphi) \preccurlyeq \alpha(\psi)$. If φ' and ψ' are elements of $\mathcal{P}K$ such that $\psi' - \varphi' = r(\psi - \varphi)$ for some real number $r \geq 0$, then $\alpha(\varphi') \preccurlyeq \alpha(\psi')$.*

Proof. In the case $r = 0$ there is nothing to prove. Thus we may suppose that $r > 0$. From the hypothesis we have $\psi' + r\varphi = \varphi' + r\psi$. We multiply this equation with $\frac{1}{r+1}$ and we obtain an equality of convex combinations $(1-p)\varphi' + p\psi = (1-p)\psi' + p\varphi$ with $p = 1 - \frac{1}{r+1} = \frac{r}{r+1}$. Applying α yields $\alpha(\varphi') +_p \alpha(\psi) = \alpha(\psi') +_p \alpha(\varphi)$. Using $\alpha(\varphi) \preccurlyeq \alpha(\psi)$ and the monotonicity of the barycentric operations, we conclude that

$$(\alpha(\varphi') +_p \alpha(\psi)) +_q \alpha(\varphi) \preccurlyeq (\alpha(\psi') +_p \alpha(\varphi)) +_q \alpha(\psi)$$

Choosing $q = \frac{p}{1+p}$ and using the axiom (SA) this inequality can be rewritten as

$$\alpha(\varphi') +_s (\alpha(\psi) +_{\frac{1}{2}} \alpha(\varphi)) \preccurlyeq \alpha(\psi') +_s (\alpha(\varphi)) +_{\frac{1}{2}} \alpha(\psi))$$

As $\alpha(\psi) +_{\frac{1}{2}} \alpha(\varphi) = \alpha(\varphi)) +_{\frac{1}{2}} \alpha(\psi)$ by axiom (SC), the the last inequality allows to conclude $\alpha(\varphi') \preccurlyeq \alpha(\psi')$ with the help of Proposition 4.1. □

Consider the following two subsets of $\mathcal{M}K$:

$$C =_{def} \bigcup_{r \geq 1} rB, \quad L =_{def} \bigcup_{r \geq 1} rP$$

Lemma 6.8. *C is a cone and L is a linear subspace such that $L = C \cap -C$.*

Proof. Clearly C is a cone and L a linear subspace in $\mathcal{M}K$. As L is contained in C, we have $L \subseteq C \cap -C$. Conversely, let $\lambda \in C \cap -C$. Then $\lambda = r(\psi - \varphi) = -s(\psi' - \varphi') = s(\varphi' - \psi')$ for some $\varphi, \psi, \varphi', \psi' \in \mathcal{P}K$ with $\alpha(\varphi) \preccurlyeq \alpha(\psi)$ and $\alpha(\varphi') \preccurlyeq \alpha(\psi')$ and some $r, s \geq 1$. Lemma 6.7 allows to conclude that $\alpha(\varphi) = \alpha(\psi)$ and $\alpha(\varphi') = \alpha(\psi')$, whence $\lambda \in L$. □

Recall that $\mathcal{M}K$ also is a Banach space with the norm $||\varphi|| = \sup_{-\mathbf{1} \leq f \leq \mathbf{1}} |\langle \varphi, f \rangle||$. For $\varphi \geq 0$ one simply has $||\varphi|| = \langle \varphi, \mathbf{1} \rangle$ which implies that the norm is additive on the positive cone $\mathcal{M}_+K$. Also $\mathcal{M}K$ is a lattice-ordered vector space. We use the notations $\varphi^+ = \varphi \vee 0$, $\varphi^- = -\varphi \vee 0$, $|\varphi| = \varphi^+ + \varphi^-$ for the positive part, the negative part and the absolute value of φ, respectively. Then one has $\varphi = \varphi^+ - \varphi^-$ and $\varphi^+ \wedge \varphi^- = 0$. A measure φ and its absolute value have the same norm which by the additivity of the norm on the positive cone implies that $||\varphi|| = ||\varphi^+|| + ||\varphi^-|| = \langle \varphi^+, \mathbf{1} \rangle + \langle \varphi^-, \mathbf{1} \rangle$.

Lemma 6.9. *If $\lambda \in C$ and $||\lambda|| \leq 2$, then $\lambda \in B$. If $\lambda \in L$ and $||\lambda|| \leq 2$, then $\lambda \in P$.*

Proof. As the second claim follows from the first, let us prove the first claim. Let $\lambda \in C$ with $||\lambda|| = 2$. There are φ, ψ and $r \geq 1$ such that $\lambda = r(\psi - \varphi)$ with $\varphi, \psi \in \mathcal{P}K$ and $\alpha(\varphi) \preccurlyeq \alpha(\psi)$.

In every lattice-ordered vector space, one has $\psi - (\psi \wedge \varphi) = \psi + (-\psi \vee -\varphi) = 0 \vee (\psi - \varphi) = (\psi - \varphi)^+$, whence $r(\psi - \psi \wedge \varphi) = \lambda^+$. We conclude that

$$||\lambda^+|| = r(||\psi|| - ||\psi \wedge \varphi||) = r(1 - ||\psi \wedge \varphi||)$$

by the additivity of the norm on the positive cone. In the same way one shows that also

$$||\lambda^-|| = r(1 - ||\varphi \wedge \psi||)$$

Thus, λ^+ and λ^- have the same norm. As

$$||\lambda^+|| + ||\lambda^-|| = ||\lambda|| = 2$$

we conclude that $||\lambda^+|| = ||\lambda^-|| = 1$, i.e., λ^+ and λ^- belong to $\mathcal{P}K$. As $\lambda^+ - \lambda^- = \lambda = r(\psi - \varphi)$, Lemma 6.7 allows to conclude that $\alpha(\lambda^+) \preccurlyeq \alpha(\lambda^-)$. Hence, $\lambda = \lambda^+ - \lambda^- \in B$.

Now consider the general case of a $\lambda \in C$ with $||\lambda|| \leq 2$. Then $\lambda' = \frac{2\lambda}{||\lambda||}$ is an element in C of norm 2. By the above, $\lambda' \in B$. As B is convex and contains 0, we conclude that $\lambda \in B$. □

Corollary 6.10. *C and L are weak*-closed in $\mathcal{M}K$.*

Proof. This is a consequence of the previous Lemma: By a theorem of Krein-Šmulian (see e.g. [9, p. 429, Theorem 7]), a convex subset in the dual of a Banach space is weak*-closed if and only if its intersection with each multiple of the dual unit ball is weak*-closed. □

The following lemma proves half of our Theorem 6.5:

Lemma 6.11. *The vector space $\mathcal{M}K/L$ with the quotient topology of the weak* topology is a locally convex ordered topological vector space with C/L as a closed positive cone. The map $\varepsilon_K \colon K \to \mathcal{M}K$ composed with the quotient map $q \colon \mathcal{M}K \to \mathcal{M}K/L$ is a topological and an order embedding.*

Proof. As L is a closed linear subspace by 6.10, $\mathcal{M}K/L$ with the quotient topology of the weak* topology on $\mathcal{M}K$ is locally convex. The image cone C/L of the cone C is closed, as C is weak*-closed by 6.10 and $L \subseteq C$. As $L = C \cap -C$ by 6.8, the cone C/L is pointed. Endowed with the order $\preccurlyeq$ with C/L as positive cone, we obtain a locally convex ordered topological vector space. For $\varphi, \psi \in \mathcal{P}K$ we have $q(\varphi) \preccurlyeq q(\psi)$ if and only if $\psi - \varphi \in C$ iff $\alpha(\varphi) \preccurlyeq \alpha(\psi)$. We conclude that the map $q \circ \varepsilon_K \colon K \to \mathcal{M}K/L$ is an order embedding. As this map is continuous on the compact Hausdorff space K, it is also a topological embedding. □

For the Theorem 6.5, it remains to prove that the map $q \circ \varepsilon_K \colon K \to \mathcal{M}K/L$ is affine. For this, we will represent the quotient space $\mathcal{M}K/L$ in a more concrete form: As in Section 5, let $\mathcal{A}K$ be the set of all affine continuous functions $f \colon K \to \mathbb{R}$. The following lemma tells us that $\alpha(\varphi)$ is the barycenter of the probability measure φ on K:

Lemma 6.12. *$\mathcal{A}K$ is the set of all $f \in \mathcal{C}K$ such that $\langle \varphi, f\rangle = f(\alpha(\varphi))$ for all $\varphi \in \mathcal{P}K$.*

Proof. Let us consider the equation $\langle \varphi, f\rangle = f(\alpha(\varphi))$ in the special case were φ is the convex combination $\varepsilon_K(x) +_r \varepsilon_K(y)$ of two Dirac measures. It then tells us that $\langle \varepsilon_K(x) +_r \varepsilon_K(y), f\rangle = f(\alpha(\varepsilon_K(x) +_r \varepsilon_K(y)))$. Using the definition of the barycentric operations on K, this equation can be rewritten as $f(x) +_r f(y) = f(x +_r y)$.

Thus, if $f \in \mathcal{C}K$ satisfies $\langle \varphi, f\rangle = f(\alpha(\varphi))$ for all $\varphi \in \mathcal{P}K$, then f is affine. Conversely, let $f \in \mathcal{A}K$, that is, $f(x +_r y) = f(x) +_r f(y)$. By the considerations above, $f(\alpha(\varphi)) = \langle \varphi, f\rangle$ then holds for all convex combinations φ of two Dirac measures. By a similar argument this equation holds for all finite convex combinations φ of Dirac measures. As these are dense in $\mathcal{P}K$ for the weak* topology and as α and f are continuous, this equation holds for all $\varphi \in \mathcal{P}K$. $\square$

Let us denote by $\mathcal{A}^m K$ the set of all order preserving continuous affine functions $f \colon K \to \mathbb{R}$. Clearly, $\mathcal{A}^m K$ is a uniformly closed cone in $\mathcal{A}K$ (although not pointed).

Lemma 6.13.

$$\begin{array}{rcll} C & = & (\mathcal{A}^m K)^\perp & =_{def} \ \{\varphi \in \mathcal{M}K \mid \langle \varphi, f\rangle \geq 0 \textit{ for all } f \in \mathcal{A}^m K\} \\ L & = & (\mathcal{A}K)^\perp & =_{def} \ \{\varphi \in \mathcal{M}X \mid \langle \varphi, f\rangle = 0 \textit{ for all } f \in \mathcal{A}K\} \end{array}$$

Proof. We first prove that $\mathcal{A}^m K = C^\perp =_{def} \{f \in \mathcal{C}K \mid \langle \varphi, f\rangle \geq 0 \text{ for all } \varphi \in C\}$ and $\mathcal{A}K = L^\perp =_{def} \{f \in \mathcal{C}K \mid \langle \varphi, f\rangle = 0 \text{ for all } \varphi \in L\}$

$C^\perp \subseteq \mathcal{A}^m K$ and $L^\perp \subseteq \mathcal{A}K$: Let $f \in L^\perp$. Choose any $\varphi \in \mathcal{P}K$. As $\alpha(\varphi) = \alpha(\varepsilon_K(\alpha(\varphi)))$ by (A1), we have $\varphi - \varepsilon_K(\alpha(\varphi)) \in P \subseteq L$. Hence $\langle \varphi - \varepsilon_K(\alpha(\varphi)), f\rangle = 0$, that is, $\langle \varphi, f\rangle = \langle \varepsilon_K(\alpha(\varphi)), f\rangle = f(\alpha(\varphi))$ which shows that $f \in \mathcal{A}K$ by 6.12. If $f \in C^\perp \subseteq L^\perp$, it remains to show that f is order preserving. Take any $x, y \in K$ such that $x \preccurlyeq y$. Then $\varepsilon_K(x) \preccurlyeq \varepsilon_K(y)$ in $\mathcal{P}K$ which implies that $\varepsilon_K(y) - \varepsilon_K(x) \in C$. By hypothesis, $\langle \varepsilon_K(y) - \varepsilon_K(x), f\rangle \geq 0$, that is, $f(y) - f(x) \geq 0$ or, equivalently, $f(x) \leq f(y)$.

$\mathcal{A}^m K \subseteq C^\perp$ and $\mathcal{A}K \subseteq L^\perp$: Any $\lambda \in C$ can be written in the form $\lambda = r(\psi - \varphi)$ for some $r \geq 0$ and some $\varphi, \psi \in \mathcal{P}K$ such that $\alpha(\varphi) \leq \alpha(\psi)$. For $f \in \mathcal{A}^m K$ we then have

$$\begin{array}{rcll} \langle \lambda, f\rangle & = & r(\langle \psi, f\rangle - \langle \varphi, f\rangle) & \\ & = & r(f(\alpha(\psi)) - f(\alpha(\varphi))) & \text{by 6.12} \\ & \geq & 0 & \text{as } f \text{ is order preserving} \end{array}$$

Hence $f \in C^\perp$. If $\lambda \in L$, then $\lambda = r(\psi - \varphi)$ for some $r \geq 0$ and some $\varphi, \psi \in \mathcal{P}K$ such that $\alpha(\varphi) = \alpha(\psi)$ and the inequality above can be replaced by an equality.

We have shown that $\mathcal{A}^m K = C^\perp$ and $\mathcal{A}K = L^\perp$. As C and L are weak*-closed convex subset of $\mathcal{M}K$, the bipolar theorem yields $C = C^{\perp\perp} = (\mathcal{A}^m K)^\perp$ and $L = L^{\perp\perp} = (\mathcal{A}K)^\perp$. $\square$

As in Section 5, let β_K denote the linear map from $\mathcal{M}K$ to the dual $(\mathcal{A}K)^*$ of the Banach space $\mathcal{A}K$ defined by restriction: $\beta_K(\varphi) = \varphi|_{\mathcal{A}K}$.

Proposition 6.14. *Composing the maps $\varepsilon_K \colon K \to \mathcal{M}K$ and $\beta_K \colon \mathcal{M}K \to (\mathcal{A}K)^*$ yields an affine topological order embedding of K into the dual $(\mathcal{A}K)^*$ of the Banach space $\mathcal{A}K$ with the weak* topology and the order defined by the positive cone C' consisting of all $\varphi \in (\mathcal{A}K)^*$ such that $\langle \varphi, f\rangle \geq 0$ for all $f \in \mathcal{A}^m K$.*

Proof. By Lemma 6.13, L is the kernel of the map β_K and C is mapped onto C' by β_K. Thus $\mathcal{M}K/L$ and $(\mathcal{A}K)^*$ are canonically isomorphic ordered topological vector spaces. Thus, Lemma 6.11 implies that the map $\beta_K \circ \varepsilon_K \colon K \to (\mathcal{A}K)^*$ is a topological and an order embedding. We have already seen in Section 5 that this map is affine. □

The previous proposition terminates the proof of Theorem 6.5 and gives some more detailed information.

7 An open problem

We would like to use the results of the preceding section to characterise the algebras of the probabilistic powerdomain monad over the category StabComp of stably compact spaces and continuous maps. There is indeed a close relation between stably compact spaces and compact ordered spaces and between measures and valuations on such spaces as exposed in [2].

Let us start with a T_0-space X. It carries an intrinsic order, the specialisation order $\preccurlyeq$ which is characterised by the property that $x \preccurlyeq y$ iff x is contained in the closure of the singleton y. A subset A of X is said to be an *upper set* or also *saturated*, if $x \in A$ implies $y \in A$ for all y with $x \preccurlyeq y$.

Recall that a T_0-space is *stably compact* if it is sober, compact and locally compact and if the intersection of any two of its compact upper subsets is compact. We denote by StabComp the category of stably compact spaces and continuous maps. There is a one-to-one correspondence between compact ordered spaces and stably compact spaces:

If $X = (X, \preccurlyeq, \tau)$ is a compact ordered space, the collection $\tau^\uparrow$ of all open upper sets is a topology such that the space $X^\uparrow =_{def} (X, \tau^\uparrow)$ is stably compact. Conversely, let (X, σ) be a stably compact space with its specialisation order $\preccurlyeq$. One defines its *patch topology* σ^p to be the topology generated by σ and the the complements of the compact upper sets. Then $X^p =_{def} (X, \preccurlyeq, \sigma^p)$ is a compact ordered space. These two constructions are mutually inverse to another, more precisely, $\tau^{\uparrow p} = \tau$ and $\sigma^{p\uparrow} = \sigma$ (see [12, Theorem VI-6.18], [2]).

Thus the objects of the categories CompOrd of compact ordered spaces and StabComp of stably compact spaces are essentially the same. The morphisms in CompOrd are the continuous order preserving maps. For such a map the preimage of an open upper set is an open upper set, too. Thus such maps remain continuous for the associated stably compact topologies. But in general, there more morphisms in the second category than in the first. For example, the unit interval with its usual total order and compact Hausdorff topology is a compact ordered space. Its endomorphism in the category CompOrd are the continuous monotone increasing maps. As a stably compact space with the topology of open upper sets, the endomorphisms in the category StabComp are the lower semicontinuous monotone increasing maps. Those continuous maps between stably compact spaces that are also continuous with respect to the respective patch topologies are called *proper*.

Let (X, σ) be a topological space, not necessarily Hausdorff. A *bounded valuation* is a function $\psi \colon \sigma \to \mathbb{R}_+$ with the following properties:

$$\begin{array}{lrcl} \psi \text{ is strict:} & \psi(\emptyset) & = & 0\,, \\ \psi \text{ is modular:} & \psi(U) + \psi(V) & = & \psi(U \cup V) + \psi(U \cap V)\,, \\ \psi \text{ is monotone increasing:} & U \subseteq V & \Rightarrow & \psi(U) \subseteq \psi(V)\,. \end{array}$$

A valuation is called *(Scott-) continuous*, if

$$\psi(\bigcup_{i \in I} U_i) = \sup_{i \in I} \psi(U_i) \text{ for every directed family of open sets } U_i \in \mathcal{G}\,.$$

We denote by $\mathcal{V}X$ the set of all bounded continuous valuations on σ and by $\mathcal{V}_1 X$ the subset of all *probability* valuations, that is, those that satisfy $\psi(X) = 1$. A natural order between valuations is given by

$$\psi \preccurlyeq \psi' \;:\Longleftrightarrow\; \psi(U) \leq \psi'(U) \text{ for all } U \in \sigma\,,$$

which we call the *stochastic order* again.

On the set $\mathcal{V}X$ and on $\mathcal{V}_1X$ we consider the *weak**upper topology* which is the coarsest topology such that the maps $\psi \mapsto \psi(U)$ are lower semicontinuous (i.e., $\{\psi \mid \psi(U) > r\}$ is open for every $r \in \mathbb{R}_+$) for all open sets U in X.

The following theorem combines results obtained in [19, 3, 2]:

Theorem 7.1. *Let X be a stably compact space.*

(a) Every bounded continuous valuation ψ on X extends uniquely to a regular Borel measure $\widehat{\psi}$ on X^p.

(b) The map $\psi \mapsto \widehat{\psi}\colon \mathcal{V}X \to \mathcal{M}_+X^p$ is an order isomorphism with respect to the stochastic orders.

*(c) This map restricts to an order isomorphism from the the set $\mathcal{V}_1X$ of probability valuations on X to the set $\mathcal{P}X^p$ of probability measures on X^p. Moreover, the weak*upper open sets of $\mathcal{V}_1X$ are in one-to-one correspondence with the weak*-open upper sets of $\mathcal{P}X^p$.*

These results allow us to identify continuous probability valuations on a stably compact space X and probability measures on X^p. As the category of stably compact spaces and proper maps is equivalent to the category CompOrd we obtain the following variant of Theorem 6.5:

Corollary 7.2. *The monad $(\mathcal{P}, \mu, \varepsilon)$ on the category* CompOrd *of compact ordered spaces and order preserving continuous maps yields a monad $(\mathcal{V}_1, \mu, \varepsilon)$ with the same unit and multiplication on the the category of stably compact spaces and proper maps. The algebras of this monad are the compact convex sets in locally convex ordered topological vector spaces endowed with the topology of open upper sets.*

Of course, $(\mathcal{V}_1, \mu, \varepsilon)$ is also a monad on the larger category StabComp of stably compact spaces and all continuous mpas. The following example shows that there are more algebras in this larger category StabComp than in the category restricted to proper maps:

Example 7.3. Let $\Sigma = \{0, +\infty\}$ denote the two element $\vee$-semilattice considered as a barycentric algebra as in Example 2.1, that is, $0 +_r \infty = +\infty$ for $0 < r \leq 1$. With the singleton $\{+\infty\}$ as the only nonempty proper open subset, Σ becomes a stably compact topological barycentric algebra. Indeed, the operation $(r, x, y) \to x +_r y$ becomes continuous although it would not be continuous for the discrete topology on Σ. We are going to show that Σ is an algebra of the monad $(\mathcal{V}_1, \mu, \varepsilon)$.

The stably compact convex set $\mathcal{V}_1\Sigma$ of probabilty valuations on Σ can be identified with the unit interval $[0, 1]$ with its usual convex structure and the upper topology with the upper intervals $]r, 1], 0 \leq r \leq 1$ as only open sets. The embedding $\varepsilon_\Sigma\colon \Sigma \to [0, 1]$ maps 0 to 0 and $+\infty$ to 1. The map $\beta\colon [0, 1] \to \Sigma$ mapping 0 to 0 and all $r > 1$ to $+\infty$ is continuous and affine. In fact, β is the only continuous affine map from $[0, 1]$ to Σ such that $\beta \circ \varepsilon_\Sigma = \mathrm{id}_\Sigma$.

We want to show that Σ with the structure map β is an algebra of the monad $(\mathcal{V}_1, \mu, \varepsilon)$. The identity $\beta \circ \varepsilon_\Sigma = \mathrm{id}_\Sigma$ is clear. We have to prove the identity $\beta \circ \mu_\Sigma = \beta \circ \mathcal{V}_1\beta$. Note that $\mathcal{V}_1\mathcal{V}_1\Sigma = \mathcal{V}_1[0, 1]$. The barycenter map $\mu_\Sigma\colon \mathcal{V}_1[0, 1] \to [0, 1]$ is given by $\mu_\Sigma(\Phi) = \int x\, d\Phi$. Thus, $\beta(\mu_\Sigma(\Phi)) = +\infty$ iff $\mu_\Sigma(\Phi) = \int x\, d\Phi > 0$, and this is the case iff $\Phi(]0, 1]) > 0$. For the right hand side we have $\beta((\mathcal{V}_1\beta)(\Phi)) = +\infty$ iff $(\mathcal{V}_1\beta)(\Phi) > 0$ iff $(\mathcal{V}_1\beta)(\Phi)(\{+\infty\}) = \Phi(\beta^{-1}(\{+\infty\})) = \Phi(]0, 1]) > 0$ as before.

Problem. Characterise the algebras of the monad $(\mathcal{V}_1, \mu, \varepsilon)$ over the category StabComp of stably compact spaces and continuous maps.

The example above is an algebra without cancellation. Even the algebras satisfying cancellation are not known.

References

[1] E. M. Alfsen. *Compact Convex Sets and Boundary Integrals*. Ergebnisse der Mathematik und ihrer Grenzgebiete Band 57, Springer Verlag, 1971.

[2] M. Alvarez Manilla, A. Jung, and K. Keimel. Valuations and measures for stably compact spaces. *Theoretical Computer Science*, 328 (2004), pages 221–244.

[3] M. Alvarez Manilla. *Measure theoretic results for continuous valuations on partially ordered spaces*. PhD Thesis, Imperial College, University of London, 2001.

[4] I. Battenfeld, M. Schröder and A. Simpson. Compactly Generated Domain Theory. *Mathematical Structures in Computer Science* 16 (2006), pages 141–161.

[5] B. Cohen, M. Escardó, and K. Keimel. The extended probabilistic power domain monad over stably compact spaces. In: J.-Y. Cai, S.B. Cooper, A. Li (Eds.), Proceedings of TAMC06: *Theory and Applications of Models of Computation*, Bejing 2006. Lecture Notes in Computer Science 3959 (2006), pages 566–575. Springer Verlag.

[6] A. Day. Filter monads, continuous lattices and closure systems. *Canadian Journal of Mathematics*, 27 (1975), pages 50–59.

[7] E. E. Doberkat. Eilenberg-Moore algebras for stochastic relations. *Information and Computation*, 204 (2006), pages 1756–1781.

[8] E. E. Doberkat. Eilenberg-Moore algebras for stochastic relations. Erratum and Addendum. Submitted.

[9] N. Dunford and J. T. Schwartz. *Linear Operators. Part I: General Theory*. Interscience Publishers, New York, 1957.

[10] D. A. Edwards. On the existence of probability measures with given marginals. *Annales de l'Institut Fourier, Grenoble*, 28 (1978), pages 53–78.

[11] V. V. Fedorchuk. Probability measures in topology. *Russian Math. Surveys*, 46:1 (1991), pages 45–93.

[12] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003.

[13] S. Graham. Closure Properties of a Probabilistic Powerdomain Construction. In: M. Main and A. Melton and M. Mislove and D. Schmidt (Eds.) *Mathematical Foundations of Programming Language Semantics*, Lecture Notes in Computer Science 298 (1988), pages 213–233. Springer Verlag.

[14] C. Jones and G. Plotkin. A probabilistic powerdomain of evaluations. In: Proceedings of the *Fourth Annual Symposium on Logic in Computer Science*, IEEE Press (1989), pages 186–195.

[15] C. Jones. *Probabilistic non-determinism*. PhD Thesis, Department of Computer Science, The University of Edinburgh (1990), 201 pages (also published as Technical Report No. CST-63-90).

[16] A. Jung, M. Kegelmann and M. A. Moshier. Stably compact spaces and closed relations. In: *Seventeenth Conference on the Mathematical Foundations of Programming Semantics* (MFPS 17), Electronic Notes in Theoretical Computer Science 45 (2001), pages 209-231.

[17] H. Kneser. Konvexe Räume. *Archiv der Mathematik*, 3 (1952), pages 198–206.

[18] J. D. Lawson. Embeddings of compact convex sets and locally compact cones. *Pacific Journal of Mathematics*, 66 (1976), pages 443–453.

[19] J. D. Lawson. Valuations on continuous lattices. In Rudolf-Eberhard Hoffmann, editor, *Continuous Lattices and Related Topics*, volume 27 of *Mathematatik Arbeitspapiere*, pages 204–225. Universität Bremen, 1982.

[20] J. D. Lawson and B. Madison. On congruences and cones. *Mathematische Zeitschrift*, 120 (1971), pages 18–24.

[21] S. Mac Lane. *Categories for the Working Mathematician*, 2nd edition. Graduate Texts in Mathematics vol. 5 (1998), Springer Verlag.

[22] Ph. Mulry. Monads in Semantics. *Electronic Notes in Theoretical Computer Science*, 14 (1998), pages 275-286.

[23] L. Nachbin. *Topology and Order*. Von Nostrand, Princeton, N.J., 1965. Translated from the 1950 monograph "Topologia e Ordem" (in Portugese). Reprinted by Robert E. Kreiger Publishing Co., Huntington, NY, 1967.

[24] W. D. Neumann. On the quasivariety of convex subsets of affine spaces. *Archiv der Mathematik*, 21 (1970), pages 11–16.

[25] J. W. Roberts. The embedding of compact convex sets in locally convex spaces. *Canadian Journal of Mathematics*, 30 (1978), pages 449–454.

[26] A. Romanowska and J. D. H. Smith. *Modes*. World Scientific 2002, xi+623 pages.

[27] H. H. Schaefer. *Topological Vector Spaces*. The MacMillan Company, New York, 1966.

[28] M. Schröder and A. Simpson. Probabilistic Observations and Valuations (Extended Abstract) In: *Proceedings of the 21st Annual Conference on Mathematical Foundations of Programming Semantics* (MFPS XXI), Electronic Notes in Computer Science, 155 (2006), pages 605–615.

[29] D. S. Scott. Continuous lattices. In: F. W. Lawvere (Ed.) *Toposes, Algebraic Geometry and Logic*, Lecture Notes in Mathematics vol. 274 (1972), pages 97–136. Springer Verlag.

[30] Z. Semadeni. Monads and Their Eilenberg-Moore Algebras in Functional Analysis.. *Queen's Papers in Pure and Applied Mathematics*, 33 (1973), 98 pages.

[31] T. Swirszcz. Monadic functors and convexity. *Bulletin de l'Académie Polonaise des Sciences, Série des sciences math., astr. et phys.*, 22 (1974), pages 39–42.

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