SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC On the shloop

REFERENCE For Keimel only. On handwritten notes from the Darmstadt seminar. Keimel to elaborate. Further reference: Scott 3-30-76, pp6,7

DEFINITION 1. Let S be a sup-semilattice. A shloop $\prec$ is a transitive, antisymmetric relation on S satisfying the following axioms:

AXIOM 0. $0 \prec 0$.

AXIOM 1. $(\forall a,b)\ a \prec b \Rightarrow a \leq b$.

AXIOM 2. $(\forall a,b)\ (a \leq b \prec c \ \text{ or } \ a \prec b \leq c) \Rightarrow a \prec c$.

falsch ~~AXIOM 3. $(\forall a,b,x)\ a \prec b \Rightarrow a \vee x \prec b \vee x$.~~

AXIOM 4. (INTERPOL) $(\forall a,b)\ a \prec b \Rightarrow (\exists x)\ a \prec x \prec b$.

NOTATION 2. For $X \subseteq S$ write $\twoheaddownarrow X = \{s \in S:$ there is an $x \in X$ with $s \prec x\}$. Write $\twoheaddownarrow x = \twoheaddownarrow\{x\}$.

LEMMA 3. a) AXIOM 3 ~~is equivalent to~~ implies each of the following

AXIOM 3' $(\forall a,b,x,y)\ (a \prec b \text{ and } x \prec y) \Rightarrow a \vee x \prec b \vee y$.

AXIOM 3'' $(\forall a)\ \twoheaddownarrow a$ is a cofilter.

(If S is a lattice, then a cofilter is a lattice ideal.)

Example: $\leq$ is a shloop.

LEMMA 4 (Expanded Gierz-Keimel). Let $L \in \underline{CL}$ and $k: L \to L$ a kernel function, i.e. a function satisfying

(i) $(\forall x,y)\ x \leq y \Rightarrow k(x) \leq k(y)$, (ii) $(\forall x)\ k(x) \leq x$, (i.e. $k \leq 1$).

(iii) $k^2 = k$.

Then

(I) $T = k(L)$ is a complete lattice and k is left adjoint to the inclusion function $T \hookrightarrow L$.

(II) The following conditions are equivalent:

(1) $(\forall D)$ D updirected in L $\Rightarrow \sup_L k(D) = k(\sup_L D)$.

(2) $(\forall t)\ t \in T \Rightarrow t = \sup_L\{s \in T : s \ll t\} = \sup_L(\twoheaddownarrow t \cap T)$.

(3) $\ll_T = \ll_L | (T \times T)$.

? (4) $T \in \underline{CL}$

(5) The inclusion $T \longrightarrow L$ is in $\underline{CL}^{op}$.

(6) $k \in \underline{CL}$.

DEFINIT$ON 5. For a sup-semilattice S and a shloop $\prec$ set

$$P_{\prec} S = \{ I \leq S: \text{ I is a cofilter such that } (\forall a)\ a \in I \Rightarrow (\exists b)(b \in I \text{ and } a \prec b) \}$$

Write $P_{\leq} S = PS$.

Note $PS = (S^{op})^{\wedge}$ by HMS-Duality, hence is a $\underline{Z}$ - and thus a $\underline{CL}$-object.

PROPOSITION. 6. For each shloop, $P_{\prec} S \in \underline{CL}$, and $I \mapsto \downarrow I: PS \longrightarrow P_{\prec} S$ is a kernel operator.

Note. In PS we have $I \ll J$ iff $(\exists\ a \in J)\ I \subseteq \downarrow a$ iff $(\exists a \in J)\ I \subseteq \downarrow a$
(Observe $J = \bigcup \{ \downarrow a: a \in J \}$ for $J \in PS$)

We now introduce the shloop category

DEFINITION 7. We define a category $\underline{INF}_{\prec}$ as follows:

(a) Objects: Pairs $(L, \prec)$ of a complete lattice together with a shloop.

(b) Morphisms: $g: (L, \prec) \longrightarrow (L', \prec')$ are $\underline{INF}$-morphisms $g: L \longrightarrow L'$ whose right adjoint $d: L' \longrightarrow L$ satisfies

$$(\forall x,y)\quad x \prec' y \Rightarrow d(x) \prec d(y).$$

~~Note~~
REMARK 8. There ~~are~~ is a forgetful functor~~s~~

$| \ |: \underline{CL} \longrightarrow \underline{INF}_{\prec}$ given by $S \mapsto (S, \ll)$.

(We use the following Lemma: Let $g: S \longrightarrow T$ be in $\underline{INF}$ and d the right adjoint. Then (1) below implies (2):

(1) g preserves sups of updirected sets.

(2) $(\forall x,y)\quad x \ll y \Rightarrow d(x) \ll d(y)$.

If, however, $T \in \underline{CL}$, then (1) and (2) are equivalent.

Remark. ATLAS contains a parallel statement with $\underline{CS}$ in place of INF and $y \in \text{int} \uparrow x$ in place of $x \ll y$ (see ATLAS 1.19))

THEOREM 9. The assignment $(L, \prec) \mapsto P_{\prec} L: \underline{INF}_{\prec} \longrightarrow \underline{CL}$ is functorial and is in fact the left adjoint of $| \ | : \underline{CL} \dashrightarrow INF_{\prec}$. The front adjunction is $s \mapsto \downarrow s: (L, \prec) \longrightarrow (P_{\prec} L, \ll)$. If $g: (L, \prec) \longrightarrow (S, \ll)$, $S \in \underline{CL}$ is an INF$_{\prec}$ -morphism, then the unique $g': P_{\prec} L \longrightarrow S$ determined by the adjunction is given by $g'(I) = \sup g(I)$.

Proof. It suffices to verify the universal property. (*) If the fill-in g' exists, it must have the form described in the theorem, and that function indeed satisfies $g(x) = g'(\ \ x)$. One must show that $g' \in \underline{CL}$.

wie sieht man das so schnell

We show a sublemma

SUBLEMMA. For $I \in P\not{S}_L$ and $g:(L,\prec) \longrightarrow (S,\ll)$, $S \in \underline{CL}$ one has

$$\sup g(I) = \sup g(\downarrow I). \text{ (Alex)}$$

Proof. $\geq$ clear. $\leq$: Show $(\forall t)\ t \ll \sup g(I) \Rightarrow t \leq \sup g(\downarrow I)$. But if $t \ll \sup g(I)$ then there is an $s \in I$ with $t \ll g(s)$, whence $d(t) \prec dg(s) \leq s \in I$, and so $d(t) \in \downarrow I$, whence $t \leq gd(t) \in g(\downarrow I)$. Thus $t \leq \sup g(\downarrow I)$.

Show that g' preserves infs: Let $\mathcal{J} \subseteq P_\prec S$. Show $g'(\inf \mathcal{J}) \geq \inf g'(\mathcal{J})$. Let $t \ll \inf g'(\mathcal{J})$, then ~~there~~ for all $I \in \mathcal{J}$ we have $t \ll g'(I) = \sup g(I)$. Hence there is an $s_I \in I$ with $t \leq g(s_I)$, whence $d(t) \leq s_I \in I$. Thus $d(t) \in \bigcap \mathcal{J}$, whence $t \leq gd(t) \in g(\bigcap \mathcal{J})$. So $t \leq \sup g(\bigcap \mathcal{J}) = \sup g(\downarrow \bigcap \mathcal{J}) = g'(\inf \mathcal{J})$.

Show that g' preserves sups of updirected sets: Let $\mathcal{D} \subseteq P_\prec S$ be up-directed. Then $\sup \mathcal{D} = \bigcup \mathcal{D}$. Now $g'(\sup \mathcal{D}) = g'(\bigcup \mathcal{D}) = \sup g(\bigcup \mathcal{D}) = \sup g(\bigcup \{I : I \in \mathcal{D}\}) = \sup \bigcup \{g(I) : I \in \mathcal{D}\} = \sup_{I \in \mathcal{D}} \sup g(I) = \sup_{I \in \mathcal{D}} g'(I)$.

(*) First we must show that $s \mapsto \downarrow s : (L,\prec) \dashrightarrow (P_\prec S, \ll)$ is in $\underline{INF}_\prec$.

We first note that this map has the right adjoint $I \mapsto \sup I$. Then we recall $I \ll J$ iff $I \leq \downarrow a$ for some $a \in J$ which implies $\sup I \leq a$. By the definition of $P_\prec S$ we find a $b \in J$ with $a \prec b$, whence $a \prec b \leq \sup J$, and so $\sup I \prec \sup J$. $\square$

(The following is new; Darmstadt check!)

THEOREM 10. Let L be a complete lattice. The assignment which assigns to each shloop $\prec$ on L the kernel operator $I \longrightarrow \downarrow I$ of PL onto $P_\prec L$ is ~~a bijection~~ an order isomorphism from the set of all shloops on L onto the set of all kernel operators on PL satisfying the equivalent conditions of Lemma 4. The shloop belonging to a kernel operator k is given by $x \prec y$ iff $x \in k(\downarrow y)$. Moreover, if $x \prec_2 y \Rightarrow x \prec_1 y$, then there is a kernel morphism $P_{\prec_1} L \longrightarrow P_{\prec_2} L$ given by $I \mapsto \downarrow_2 I$.

Proof. If $x \prec_2 y \Rightarrow x \prec_1 y$, then $I \in P_{\prec_2} L$ by definition implies $I \in P_{\prec_1} L$. Moreover, the identity map $(L, \prec_1) \longrightarrow (L, \prec_2)$ is a morphism in $\underline{INF}_\prec$, hence there is a unique CL-morphism $f: P_{\prec_1} L \longrightarrow P_{\prec_2} L$ such that $f(\downarrow_1 s) = \downarrow_2 s$. Use $I = \bigcup \{\downarrow_1 s : s \in I\}$ for $I \in P_{\prec_1} L$ and the fact that f preserves up-directed sups to show that $f(I) = \downarrow_2 I$. "Order isomorphism into" should not pose any problems. If k is a kernel operator on PL satisfying the conditions of Lemma 4, then define $x \prec y$ iff $x \in k(\downarrow y)$. Verify the Axioms in Definition 1. E.g. INTERPOL: Let $a \in k(\downarrow b)$. By 4(II)(2) we have $k(\downarrow b) = \bigcup \{I \in \operatorname{im} k : I \subseteq \downarrow x,\ x \in k(\underbrace{\ \ }_{\downarrow b})\}$ for some x. Hence $a \in I = k(I) \subseteq k(\downarrow x)$ for some I and x.

We should now be prepared to inspect for a CL-object S the totality ker(S) of kernel operators satisfying the equivalent conditions of Lemma 4. If $k \in \ker(S)$, then 4-(II)-(1) means that k is continuous in the sense of Scott. If Cont denotes the category of ~~complete~~ continuous lattices with Scott continuous functions, then $\ker(S) \subseteq \mathrm{Cont}(S,S) = [S\to S]$ (where we use Scott's terminology). We know from Scott that $[S\to S] \in CL$ .Furthermore we observe that the inclusion $\ker(S) \subseteq [S\to S]$ preserves sups where sups of functions are calculated pointwise. (To check e.g. ~~xx(i~~ ~~x~~ 4 -(iii) let $A \subseteq \ker(S)$, $k = \sup A$. Then $k(k(s)) = \sup_{f\in A} f(\sup_{g\in A} g(s)) \geq \sup_{f\in A} f(f(s)) = \sup_{f\in A} f(s) = k(s) \geq k(k(s))$.)The inclusion map $\ker(S) \longrightarrow S\to S$ thus has a left adjoint

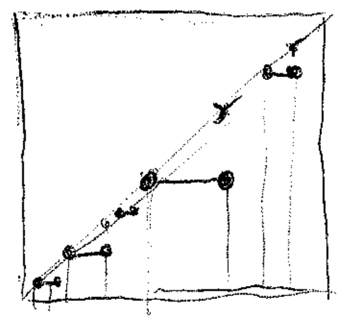
$\varkappa : [S\to S] \longrightarrow \ker(S)$, $\varkappa(f) = \sup \{k \in \ker(S) : k \leq f\}$.

It is my impression that this function is not generally in Cont so that it is not likely that one could show $\ker(S) \in CL$ via Lemma 4. Judging from Scott p.103,line 1 and p.111 ,Definition 3 I think that $f \ll g$ in $[S\to S]$ iff there is a finite set $F \subseteq S$ sucht that $f(s) = 0$ for $s \notin F$ and $f(s) \ll g(s)$ for $s \in F$. This seems to be rare for anything having more properties than just being Scott continuous.

It is of course possible that the internal $\ll$ -relation of ker(S) is not the induced one (and indeed if ker(S) is not in CL ,this must be the case) Typical samples of elements in ker (S) are the following:

(a) For $s \in S$ set $k(t) = st$. (b) For $c \in K(S)$ set $k(\uparrow c) = \{c\}$, $k(I(c)) = \{0\}$.

(c)



~~xxxxxxxxxxxxxxxxx~~ For every collection of disjoint intervals on the interval one may fabricate an element of ker (I). (Use e.g. the components of the complement of the Cantor set.) One may also read the picture as the graph of an element in ker (C). Notice that $k(C) = I$ in that case.

It seems impossible to approximate this example from below by type (b) kernel operators.

In any case ker(S) is a complete lattice and is sup-closed in $[S \to S]$

COROLLARY 11 . The shloops on a complete lattice form themselves a complete lattice. $\square$

It remains open whether or under what circumstances this lattice is continuous.

Now let L be a complete lattice and $\prec$ a relation satisfying AXIOMS 0- 2.

DEFINITION 12. Define $\prec^*$ as follows: For $x,y \in L$ we have $x \prec^* y$ iff there is a subset $C \subseteq L$ with the following properties:

(i) C is $\preccurlyeq$- totally ordered.

(ii) C is $\prec$- order dense (if $a \prec b$ in C, there is an $x \in C$ with $a \prec x \prec b$).

(iii) min C = x , max C = y . $\square$

LEMMA 13. Suppose that $\prec$ satisfies AXIOMS 0 -2 . Then the following are equivvalent

(1) $x \prec^* y$.

(2) There is a function $f: [0,1] \cap \mathbb{Q} \longrightarrow L$ such that $p < q$ implies $f(p) \prec f(q)$ and $f(0) = x$ and $f(1) = y$. $\square$

REMARK, We have $\prec = \prec^*$ iff $\prec$ satisfies INTERPOL ,

PROPOSITION 14. Let L be a complete lattice and $\prec$ a relation satisfying AXIOMS 0,1,2. Then $\prec^*$ also satisfies these axioms plus AXIOM 4 (INTERPOL). Further, $\prec^*$ satisfies AXIOMS 3,3',3" if $\prec$ does ~~simultaneously~~, respectively. In particular, if $\prec$ satisfies 0,1,2,3, then $\prec^*$ is a shloop. $\square$

Noch ne Kategorie:

~~PROPOSITION 15~~ DEFINITION 15 . Let Compl be the category of complete lattices with inf-morphisms (arbitrary infs!) preserving sups of updirected sets.

Note Compl $\subseteq$ INF.

PROPOSITION 16. There is a functor W: Compl $\longrightarrow$ $\text{INF}_\prec$ given by $W(L) = (L, \ll^*)$.

Proof. In each $L \in$ Compl the relation $\ll$ satisfies AXIOMS 0-3, hence $\ll^*$ is a shloop. If $g: L \to L'$ is in Compl then $x \ll y$ in L' implies $d(x) \ll d(y)$ in L , where d is the right adjoint of g(we recall the Lemma mentioned in REMARK 8!) Thus $x \ll^* y$ means the existence of a function $f: [0,1] \cap \mathbb{Q} \longrightarrow L'$ as in 13-(2). Then $fd: [0,1] \cap \mathbb{Q} \longrightarrow L$ is a function as in 13-(2), hence $d(x) \ll^* d(y)$. Thus f is an $\text{INF}_\prec$ morphism $(L, \ll^*) \longrightarrow (L', \ll^*)$. $\square$

THEOREM 17. $P_\prec \circ W$: Compl $\longrightarrow$ CL is the left adjoint of the grounding functor U : CL $\longrightarrow$ Compl.

Proof. By the Lemma in REMARK 8 , for $S \in$ CL and $L \in$ Compl we have

$\underline{Compl}(L, U(S)) = \underline{INF}_{\prec}(W(L), |S|)$. By Theorem 9 we know $\underline{INF}_{\prec}(W(L), |S|) \cong \underline{CL}(P_{\prec}(W(L)), S)$ (naturally). Hence $\underline{Compl}(L, U(S)) \cong \underline{CL}(P_{\prec}(W(L)), S)$, naturally. □

NOW to $\underline{CS}$ the category of compact semilattices. There is an obvious grounding functor $J: \underline{CS} \longrightarrow \underline{Compl}$. If we recall $\underline{CL} \subseteq \underline{CS}$ and note $\underline{CS}(T,S) = \underline{Compl}(J(T), U(S))$ for $T \in \underline{CS}$, $S \in \underline{CL}$, and if we recall $\underline{Compl}(J(T), U(S)) \cong \underline{CL}(P_{\prec}(W(J(T))), S)$ from Theorem 17 we note

COROLLARY 18. $P_{\prec} \circ W \circ J : \underline{CS} \dashrightarrow \underline{CL}$ is the left reflection. □

But on a $\underline{CS}$ object T we have the relation $\lessdot$ given by $x \lessdot y$ iff $y \in \operatorname{int} \uparrow x$. This relation satisfies AXIOMS 0,1,2,3',3".

(Proof of 3': if $b \in \operatorname{int}\uparrow a$, $y \in \operatorname{int}\uparrow x$ then $b \vee y \in \operatorname{int}\uparrow a \cap \operatorname{int}\uparrow x \subseteq \operatorname{int}(\uparrow a \cap \uparrow x) = \operatorname{int}\uparrow(a \vee x)$. Proof of 3" If $a \in \operatorname{int}\uparrow x$ and $a \in \operatorname{int}\uparrow y$, then $a \in \operatorname{int}\uparrow x \cap \operatorname{int}\uparrow y \subseteq \operatorname{int}\uparrow(x \vee y)$.)

I cannot prove AXIOM 3 for $\lessdot$ PERHAPS AXIOM 3 IS TOO STRONG FOR OUR THEORY AND SHOULD BE REPLACED BY AXIOM 3'+3".

If that is the case then we can claim that $\lessdot^{*}$ is a shloop (in this more general sense).

For the remainder I prodeed under this assumption.

PROPOSITION 17. There is a functor $V: \underline{CS} \longrightarrow \underline{INF}_{\prec}$ given by $V(L) = (L, \lessdot^{*})$.

Proof. As Proposition 16, use ATLAS 1.19. □

COROLLARY ~~THEOREM~~ 18. $P_{\prec} \circ V: \underline{CS} \longrightarrow \underline{CL}$ is the left reflection.

Proof. By ATLAS 1.19, if $T \in \underline{CS}$ and $S \in \underline{CL}$ then $\underline{INF}_{\prec}(V(T), |S|) = \underline{CS}(T, S)$. Then proceed as in Theorem 17. □

If $T \in \underline{CS}$, then both $P_{\prec}WJ(T) = P(T, \ll^{*})$ and $PV(T) = P(T, \lessdot^{*})$ are left reflection into $\underline{CL}$ with front adjunctions $t \mapsto \Downarrow_{*} t$ and $t \mapsto \downarrow^{*} t$ respectively. Since $x \lessdot y$ implies $x \ll y$, whence $x \lessdot^{*} y$ implies $x \ll^{*} y$ there is a uniqe map $f: P_{\prec}(T, \ll^{*}) \longrightarrow P_{\prec}(T, \lessdot^{*})$, $f(I) = \downarrow_{*} I$ which is the fill in in the universal property since $f(\Downarrow_{*} s) = \downarrow_{*} \Downarrow_{*} s = \downarrow_{*} s$. By the uniquenes of left adjoints, f is an isomorphism. But f is a kernel operator, and a kernel operator which is an isomorphism must be the identity. Hence

THEOREM 19. If $S \in \underline{CS}$, then $\lessdot^{*}$ and $\ll^{*}$ agree on S. □

Note that $\lessdot^{*} = \lessdot$ (resp $\ll^{*} = \ll$) iff $\lessdot$ (resp $\ll$) satisfies INTERPOL.

COROLLARY 20. If $\lessdot$ and $\ll$ both satisfy the interpolation axiom on a compact semilattice, then they agree. □

(Conversel, if one could find an example where one of the two has the interpolation property whereas the other not, one would know that the two relations disagree.

Further remarks.

There is some evidence that the interpolation axiom should be strengthened as follows

AXIOM 4'. $(\forall a,b)\ a\prec b \Rightarrow (\exists x)\ a\prec x\prec b$ and $a\neq x$. □

In any CL -object the relation $\ll$ satisfies this stronger interpolation.

DEFINITION 21. Let $(L,\prec)\in INF_\prec$. A ~~chain~~ set C in L is a strict chain, if $x,y\in C$ implies that $x\prec y$ or $x=y$ or $y\prec x$.
By Zorn's Lemma, eahh strict chain is contained in a maximal one.
Examples of strict chains are $\{0\}$, $\{0, x\}$ . .

THEOREM 22. Let $(L,\prec)\in INF_\prec$ and suppose that $\prec$ satisfies AXIOM 4'. (We do not need AXIOM 3, or 3' or 3".) If $C\subseteq L$ is a maximal strict chain, then C is complete (hence in CL) and there is a surjective $INF_\prec$ morphism $\psi:(L,\prec)\longrightarrow(C,\ll_C)$ whose right adjoint is given by $c\longmapsto \sup_L\{d\in C:\ d\prec c\}$. For $c\in C$ we have $\psi(c)=c$, i.e. $\psi^2=\psi$.

Proof. Memo Hofmann 4-19-76 (on chains ...) and memo Carruth 5-28-76 □
This applies in particular to any $S\in$ CS with $\prec\ =\ <\!\cdot^{*}\ =\ \ll^{*}$ provided this relation satisfies AXIOM 4'.

AXIOM 5. $(\forall x,a_1,\dots,a_n)\ x\prec a_1\vee\dots\vee a_n \Rightarrow (\exists a_1',\dots,a_n')\ x\le a_1'\vee\dots\vee a_n'$ and $a_j'\prec a_j$, $j=1,\dots,n$. □

PROPOSITION 23. Let $(L,\prec)\in INF_\prec$. Let $f:(L,\prec)\longrightarrow(S,\ll)$ be the left reflection into CL. Then $x\prec y$ implies $f(x)\ll f(y)$, and if $(\forall x,y)\ x\prec y\Rightarrow x\ll y$, and $\prec$ satisfies AXIOM 5, then f preserves sups, and thus is a lattice morphism. ~~Its left adjoint (!) is an $INF_\prec$ -morphism. Thus, if f is surjective, then it is a retraction in $INF_\prec$.~~

Proof. We may assume that $S=P_\prec(L,\prec)$ and $f(s)=\Downarrow s$. Let $x\prec y$ in L. Then there is an $a\in\Downarrow y$ (namely, $a=x$) such that $\Downarrow x\subseteq\downarrow a$. This means $f(x)=\Downarrow x\ll\Downarrow y=f(y)$ in $P_\prec(L,\prec)$. Now suppose that $\prec$ is stronger than $\ll$ and that AXIOM 5 holds. Let $X\subseteq L$ be arbitrary, $x=\sup X$ in L. Trivially $\sup f(X)\le f(x)$; we must show the converse. For this purpose we take an arbitrary $I\ll f(x)=\Downarrow x$; we must show $I\ll\sup f(X)$. But $I\ll\Downarrow x$ means the existence of some $a\prec x$ with $I\subseteq\Downarrow a$. By hypothesis, $a\prec x$ implies $a\ll x=\sup X$. Hence there is a finite set $F\subseteq X$ with $a\le\sup F$. Now take any $u\prec a$. Then $u\prec a_1\vee\dots\vee a_n$, $F=\{a_1,\dots,a_n\}$. By AXIOM 5 there are $a_j'\prec a_j$ ~~and~~ such that $u\le a_1'\vee\dots\vee a_n'$, i.e. $u\in\Downarrow a_1\vee\dots\vee\Downarrow a_n\subseteq\sup_{x\in X}\Downarrow x$ (in P_L) whence $I\subseteq\Downarrow a\subseteq\sup f(X)$. □

Remark. AXIOM 5 is satisfied in CL.