Stralka, Mislove, J.D. Lawson, Carruth + Scott (April 5) + Keimel (May 10)

MEMO on a letter from Klaus Keimel (March 23, 76).

... et magno cum murmure erumpunt

Vergil

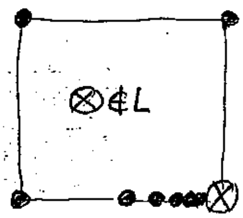
[As for the classical quotation of the day, the scene is a mountain in which Neptune keeps his winds confined which he is to unleash against Aeneas by puncturing the side of the mountain with his trident whereupon ... see above]

Our scene is ATLAS (algebraic theory of Lawson semilattices) Chapter 2 Section c. Algebraic ...characterisation of Lawson semilattices. Keimel (with his trident in his hand) particularly contemplates Corollary 2.20 which says that a lattice L carries a compact Lawson topology iff

(8) For each $x \in L$ there is a smallest lattice ideal J such that $x = \sup J$.

He then proceeds to consider (essentially) the following example:

EXAMPLE (KEIMEL) Part 1. Let L be the following (inf) subsemilattice of of the square $L = \{(1,1),(0,1);(1-\frac{1}{n},0): n=1,2,...\}$.



Note that L is a complete lattice satisfying (8). But there is no way to endow L with a compact Lawson topology.

EXAMPLE Part 2. Let T be the compact subsemilattice of the square given by $T = L \cup \{(1,0)\}$. Then $L = K(T)$. The inclusion function $j: K(T) \longrightarrow T$ has a right adjoint $c: T \longrightarrow K(T)$ called the compact closure operator, since T is a dually complete Z-object. (ATLAS, Chapter 2, Section b.) Now let us look at Proposition 2.14. The closure operator does not preserve infs: First observe that c fixes all element of L and satisfies $c(1,0) = (1,1)$. Then

$c((0,1)(1,0)) = c(0,0) = (0,0) \neq (0,1) = [c(0,1)](1,1) = c(0,1)c(1,0)$.

Thus c violates 2.14 (1), hence also (2) since (1)<=>(2) (there is nothing wrong with that). But 2.14 (3) is evidently satisfied [$(1,0) = \min\{k \in K(T): c(t) = (1,1)\}$]. The is nothing wrong with the conclusion (2) => (3) in 2.14. Let us look at this a bit more carefully. Let $g: S \longrightarrow T$ be the left adjoint of $d: T \longrightarrow S$ between complete lattices. Then $d(t) = \inf g^{-1}(\uparrow t) = \min g^{-1}(\uparrow t)$ by 1.9. Suppose that g is surjective. Then

(*) $\uparrow g^{-1}(t) = g^{-1}(\uparrow t)$

[Proof. $\subseteq$ is clear. Let $s \in g^{-1}(\uparrow t)$, then $g(s) \geq t$; but $t = gd(t)$ (by 1.12); but $g(s) \geq t \Leftrightarrow s \geq d(t)$, and $gd(t) = t$, so $s \in \uparrow g^{-1}(t)$.]

From (*) we obtain $d(t) = \min g^{-1}(\uparrow t) = \min \uparrow g^{-1}(t) = \min g^{-1}(t)$. Thus

Remark. If a ~~rig~~ ~~right~~ left adjoint is surjective then $d(t) = \min \uparrow g^{-1}(t)$.

This remark could have been made in 1.12.

The EXAMPLE Part 2 shows that in 2.14 (3) does not imply (2).
So what is correct?

I Reformulate 1.8.

PROPOSITION (ATLAS 1.8 slightly amplified). Let S and T be posets and $g: S \longrightarrow T$ and $d: T \longrightarrow S$ order morphisms. Then the following statements are equivalent:

(i) $d(t) = \min g^{-1}(\uparrow t)$ for all $t \in T$.

(ii) $g(s) = \max d^{-1}(\downarrow s)$ for all $s \in S$.

(iii) (g,d) is a Galois connection between the posets S and T.

Proof. (iii) $\Rightarrow$(i),(ii) given in ATLAS,1.8. We prove e.g. (i)$\Rightarrow$(iii) Firstly let $g(s) \geq t$, then $s \in g^{-1}(\uparrow t)$ whence $s \geq \min g^{-1}(\uparrow t) = d(t)$ (by (i)). Secondly let $s \geq d(t)$. Then $s \geq a$ where $a = \min g^{-1}(\uparrow t)$ (by i). In particular $a \in g^{-1}(\uparrow t)$, i.e. $g(a) \geq t$. Thus $g(s) \geq g(a) \geq t$. ∎

THUS PROPOSITION 2.14 AND ITS PROOF IS VALID IF (3) IS CORRECTED TO READ

(3) For a compact element $k \in K(T)$, the set of all $t \in T$ whose compact closure $c_T(T)$ dominates k has a smallest element.

~~and~~ AND IF IN THE PROOF, LAST PARAGRAPH WE READ : (3) is clearly equivalent to the existence, for each $k \in K(T)$, of $m_T(k) =$ ~~$\min c_T^{-1}(k)$~~ $\min\{t \in T: c_T(t) \geq k\}$. ... etc.

In the proof of 2.15, line 2 work with $R(\uparrow k) = \{t \in T: c_T(t) \geq k\}$

In the proof of 2.17 read

(3') For each element $k \in K(T)$ there is a smallest lattice ideal J of K(T) such that $k \leq \sup_{K(T)} J$.

In Lemma 2.18, read

(8) For each $x \in L$ there is a smallest ~~element~~ lattice ideal J such that $x \leq \sup J$.

(and adjust (8') accordingly). In the proof of 2.18 drop "Obviously, $\sup J_o \leq x$", and define $J_o = \bigcap \{I: I$ a lattice ideal with $x \leq \sup I\}$.

For your edification you may add to the formulation of Corollary 2.20:

Moreover, if (8) and (9) are satisfied, then conditions (8) and (9) still hold when = replaces $\leq$, resp. $\geq$.

You want to make some minor adjustments in the paragraph following 2.22, notably y is relatively compact under x iff for all subsets $X \subseteq L$ with $x \leq \sup X$ there is a finite subset $F \subseteq X$

with $y \leq \sup F$.

In Theorem 2.23 in conditions (iv) and (v) you replace = by $\leq$, resp. $\geq$, and for your edification you might insert after condition (v).

If these conditions hold then (iv) and (v) are true if = replaces $\leq$ resp. $\geq$.

MAGNI MURMURIS FINIS.

I think that Keimel unearthed a very subtle point. The example shows that for a surjective order morphism g it may very well happen that $\min g^{-1}(t)$ exists for all t while $\min g^{-1}(\uparrow t)$ does not exist. It took me a whole night plus a haranguing of a patient listener (J.R.Liukkonen) to figure out what had been wrong (ever so minutely, but wrong nevertheless).

Keimel says that there are preprints by

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on Convergence Lattices which supposedly provided him with the trident.

The funny thing is that once you formulate for a complete lattice an algebraic definition of a CL-object according to

(8) For each x there is a smallest lattice ideal J with $x \leq \sup J$

you can relax in your knowledge that in such a lattice you have

(8_{old}) For each x there is a smallest lattice ideal J with $x = \sup J$.

So, fortunately, you do not have to adjust your intuition on these matters substantially. It is just that (8_{old}) does not suffice to characterize CL-objects among the complete lattices.

MEMO ON PERIPHERALITY IN CL-THEORY.

PROPOSITION. Let $S \in CL$ and suppose that 1 has a basis of open neighborhoods U such that the component of 1 in U has inner points. Then every inner point p of S is way below 1 (i.e. $p \ll 1$).

Indication of Proof. Take Lawson J, and B. Madison, Peripheral and inner points, Fund. Math. 69 (1970), 253-266 and find Theorem 3.4 on p.262. Use this to show that a facial point (i.e. a point $p \not\ll 1$) cannot be inner.

I believe that the hypothesis on the local connectivity at 1 may be dropped. If $p \not\ll 1$ there are points t arbitrarily close to 1 such that $tS \not\ni p$. We apply Thm 3.4 loc. cit. with $X = S = X$, $\Gamma(t,x) = tx$. For any cohomology class h, the function $t \mapsto H^*(\Gamma_t)(h)$ is locally constant. Since W surrounds p, there is a cohomology class $h \in H^n(S, S \setminus V')$ with $H^n(i)(h) \neq 0$. Thus by taking a t with $tS \not\ni p$ very close to 1, we get $H^n(\Gamma_t)(h) \neq 0$. This suffices for the contradiction in the proof.