J. Houston

Dear Jimmie:

Many thanks for your Tuesday observations. There is a really interesting example of a CL-objet! No wonder that it was so hard to ~~proofs~~ prove the elusive THEOREM 11 on GENERATION. What remains is (1)<=>(2)<=>(3)<=>(4) =>(5)<=>(5') <=>(6); your example shows the failure of the missing implication.

In some further efforts to clear up the various steps involved in my ill fated proofs, I made some observations which may be useful.

Let T be a CL-object and $L \subseteq T$ a complete inf-subsemilattice of T, i.e. $X \subseteq L$ implies $\inf X \in L$. (Thus L is a complete lattice in its own right.) Then the following statements are equivalent for $x \in T$:

(1) $x \in \bar{L}$.

(2) $x = \lim x_j$ with an up-directed net x_j in L.

(3) $x = \sup D$ with an up-directed set $D \subseteq L$.

(4) $x = \sup J$ with $J \in \mathrm{Id}\, L$.

(5) $x = \sup((\downarrow x)_0 \cap L)$, $(\downarrow x)_0$ smallest lattice ideal of T with x as sup.

~~(6)xxx()~~ Moreover, ~~if these conditions hold,~~ $(\downarrow x)_0 \cap L \in \mathrm{Id}\, L$.

Proof. ~~xx~~ We agree on (1)<=>(2)<=>(3). Note that for any up-directed set $D \subseteq L$ we have $\downarrow D \cap L \in \mathrm{Id}\, L$, whence (3)<=>(4).
(4) => (5). Let $x = \sup J$, $J \in \mathrm{Id}\, L$. Then $\downarrow J \in \mathrm{Id}\, T$ with $x = \sup \downarrow J$, whence $(\downarrow x)_0 = (\downarrow J)_0 \subseteq \downarrow J$ by ATLAS, whence $(\downarrow x)_0 \cap L \subseteq J$. By (1)<=>...(4) we know that $\bar{L}$ is a CL-object. Now let $a \ll_{\bar L} x$, $a \in A(\bar L)$. Find any $u \in \bar{L}$ with $a \ll_{\bar L} u \ll_{\bar L} x$. By (4) ~~xxxx(2)~~ there is a $j \in J$ with $u \ll j \leq x$. ~~Now let us completely work in L xxx.~~ Since $u \in \bar{L}$, then there is a $k \in L$ with $a \ll k \leq u$. From $k \leq j$ we conclude $k \in J$. Since $a \in A(\bar L)$ we have $a = \inf \{u : a \ll_{\bar L} u\}$. Hence $a = \inf \{k : a \ll_{\bar L} k \in J\}$. Thus $a \in L$ since L is inf-complete. Thus $a \in J$ because of $a \leq j$. Thus $(\downarrow x)_0 \cap A(\bar L) \subseteq J$, i.e. $(\downarrow x)_0 \cap A(\bar L) \subseteq (\downarrow x)_0 \cap J$. But $x = \sup_{\bar L}((\downarrow x)_0 \cap A(\bar L))$, whence $x = \sup_{\bar L}((\downarrow x)_0 \cap J) = \sup_{\bar L}((\downarrow x)_0 \cap L)$.

Now let $a, b \in (\downarrow x)_0 \cap J$. Then $a \vee_{\bar L} b \ll x$ and for any u with $a \vee_{\bar L} b \ll u \ll x$ there is a $j \in J$ with $u \ll j \leq x$; again for a $k \in J$ with $a \vee_{\bar L} b \ll k \leq u$ and observe $k \in (\downarrow x)_0 \cap J$. This shows that $(\downarrow x)_0 \cap J$ is up-directed. Recall that $(\downarrow x)_0 \cap L \subseteq J$, whence $(\downarrow x)_0 \cap J = (\downarrow x)_0 \cap L$. Since for up-directed sets D in a CL-object we have $\sup D = \lim D$, we conclude $\sup_{\bar L} (\downarrow x)_0 \cap L = \sup (\downarrow x)_0 \cap J$.
(5) => (4) is trivial.

Of course, the only new information I get from this is the following Proposition:

PROPOSITION. Let $T \in \underline{CL}$ and let $L \subseteq T$ be a complete inf-subsemilattice. Then

(a) $x \in \overline{L}$ iff $x = \sup((\downarrow x)_0 \cap L)$

(b) $\sup : \mathrm{Id}\, L \longrightarrow \overline{L}$ is left adjoint to $x \longmapsto (\downarrow x)_0 \cap L$ (hence preserves arbitrary ~~sup~~ infs). It also preserves sups of up-directed sets. (" sup is algebraically continuous")

Proof. (a) was just shown. (b) We saw $(\downarrow x)_0 \cap L \in \mathrm{Id}\, L$ for $x \in \overline{L}$. For $J \in \mathrm{Id}\, L$ and $t \in \overline{L}$ we clearly have $\sup J \geq t$ iff $J \supseteq (\downarrow t)_0 \cap L$. This shows that sup is left adjoint ~~to~~ as asserted. By ATLAS, it then preserves arbitrary infs. Since sups of up-directed collections in Id L are just set theoretical unions, the rest is clear.□

Consequence: If, by happenstance, L is a continuous lattice, then sup is a $\underline{CL}$ -morphism.

Let us return to PRIME T for a moment. Let $T \in \underline{CL}$, $L = \{x \in T: x = \inf(\uparrow x \cap \mathrm{PRIME}\ T)\}$. For $a,b \in L$ we note $a \leq b$ iff $\uparrow b \cap \mathrm{PRIME}\ T \subseteq \uparrow a \cap \mathrm{PRIME}\ T$. Since $a \vee_L b = \inf \{c \in L: a,b \leq c\}$ we note $a \vee_L b = \inf(\uparrow a \cap \uparrow b \cap \mathrm{PRIME}\ T) = \inf \uparrow(a \vee b) \cap \mathrm{PRIME}\ T) = a \vee_{\overline{L}} b$ provided that PRIME T order generates $\overline{L}$, which is the case iff $\overline{L}$ is distributive (and trivially, iff $L = \overline{L}$). On the other hand, if L is a sublattice of $\overline{L}$, then $\sup : \mathrm{Id}\, L \longrightarrow \overline{L}$ is a lattice morphism, since $\sup(I \vee J) = \sup \{i \vee_{\overline{L}} j : i \in I, j \in J\} = \sup_{\overline{L}} I \vee_L \sup_{\overline{L}} J = \sup I \vee_L \sup J$ (I, J being up-directed). But then, since L is distributive, ~~xxxixxid~~ whence Id L is distributive, also $\overline{L}$ is distributive. Thus (for what it is worth)

PROPOSITION. Let $T \in \underline{CL}$, $L = \{x \in T: x = \inf(\uparrow x \cap \mathrm{PRIME}\ T)\}$. Then the following statements are equivalent:

(1) $\overline{L}$ is distributive.

(2) L is a sublattice of $\overline{L}$.

(3) $\sup : \mathrm{Id}\, L \longrightarrow \overline{L}$ is a lattice morphism.

(4) L is closed.

Your examples shows that (1)-(4) are not automatic. I also observe that in your example $\mathrm{Irr} = \mathrm{IRR} =$ ~~xxxx~~ $\overline{\mathrm{PRIME}}$ which shows that it would not suffice for distributivity to have $\overline{\mathrm{PRIME}}$ order generating.