# VARIATIONEN ÜBER JONSSON'S LEMMA 

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## 1. The lemma

Let $L$ be a lattice.

1.1. FACT. For elements $c, x \in L$ the following are equivalent:

(i) For every ideal $I$ of $L$ with $\sup I = x$, one has $c \in I$.

(ii) For every increasing net $(x_i)_i$ with $\sup_i x_i = x$ there is an $i_0$ with $c \leq x_{i_0}$

(iii) Every covering of $x$ contains a finite covering of $c$, i.e. every family $(a_i)_{i\in I}$ with $\sup_{i\in I} a_i = x$ contains a finite subfamily $(a_i)_{i\in F}$ with $\sup_{i\in F} a_i \geq c$.

1.2 DEFINITION. An element $c$ is called <u>relatively compact in</u> $x$ if the equivalent conditions above are satisfied; we also write $c \ll x$. Notation: $C_x := \{c \in L \mid c \ll x\}$; $C_x$ is the intersection of all ideals $I$ with $\sup I = x$.

1.3 A compact element $k$ of $L$ is relatively compact in every $x \geq k$.

1.4 Let $X$ be a topological space. Denote by $\mathcal{O}(X)$ the lattice of all open subsets of $X$ (ordered by inclusion). For $C, U \in \mathcal{O}(X)$ one has that $C$ is relatively compact in $U$ iff the following condition holds: Every open cover of the relative closure $cl_U(C)$ of $C$ in $U$ contains a finite subcovering of $C$.

In particular: If $U$ contains a quasicompact set $K$ which in turn contains $C$, then $C \ll U$.

1.5 DEFINITION. A <u>continuous lattice</u> (cf. D. Scott) is a complete lattice in which $x = \sup C_x$ for all $x \in L$.

1.6 Every algebraic lattice is continuous.

1.7 If $X$ is a topological space with a basis of ~~comp~~ quasicompact open sets, then $\mathcal{O}(X)$ is algebraic and consequently continuous.

1.8 If $X$ is a locally compact space, then $\mathcal{O}(X)$ is continuous. In fact, for $C, U \in \mathcal{O}(X)$ one has $C \ll U$ iff $\overline{C} \subseteq U$ and $\overline{C}$ is compact.

HOFMANN and STRALKA have proved:

1.9. THEOREM. A continuous lattice admits a unique compact topology for which the operation meet is continuous; endowed with this topology, $L$ is a Lawson semilattice, i.e. every point has a neighborhoodbasis of meet sub-semilattices. Conversely, every compact Lawson semilattice is continuous, and $c \ll x$ iff $\uparrow c$ is a neighborhood of $x$.

Henceforward, a continuous lattice will always be endowed with the compact Lawson topology. Recall that $p \in L$ is <u>prime</u> iff ($F \subseteq L$ finite, $\wedge F \leq p \Rightarrow x \leq p$ some $x \in F$)

1.10. $\boxed{\text{THE LEMMA}}$. Let $L$ be a continuous lattice and $L'$ a complete lattice. Let $i : L \to L'$ be a map preserving joins of increasing nets. Let $K$ be a compact subset of $L$ and $p$ a prime element of $L'$ such that $p \geq \inf i(K)$. Then there is an element $x \in K$ such that $p \geq i(x)$.

Proof. Suppose that $p \not\geq i(x)$ for all $x \in K$. As $x$ is the join of the ideal $C_x$ (cf. 1.2) and as $i(x)$ is the join of $i(C_x)$ there is some element $c_x \in C_x$ such that $p \not\geq i(c_x)$. As $\uparrow c_x$ is a neighborhood of $x$ and as $K$ is compact, there is a finite subset $F$ of $K$ such that $K \subseteq \bigcup_{x \in F} \uparrow c_x$. We conclude that $i(K) \subseteq \bigcup_{x \in F} \uparrow i(c_x)$ and consequently $p \geq \inf i(K) \geq \inf_{x \in F} c_x$. As $p$ is prime, this implies $p \geq c_x$ for some $x \in F$: a contradiction.

1.11. COROLLARY. Let $L$ be a continuous lattice, let $A$ be any subset of $L$ and $p$ a prime element of $L$ such that $p \geq \inf A$. Then there is an ultrafilter $\mathfrak{U}$ on $A$ such that $p \geq \lim \mathfrak{U}$.

Proof: Choose $L = L'$, $i = id_L$, $K = \overline{A}$ in 1.10.

## 2. Application to universal algebra : The JÓNSSON LEMMA.

The famous Jónsson lemma of universal algebra can be considered to be a special case of THE LEMMA above. Our proof is not shorter than the original one, but it seems to reveal something about the nature of the JÓNSSON LEMMA.

2.1. LEMMA. Let $L$ be an algebraic lattice and $\mathfrak{u}$ an ultrafilter on $L$. Then

$$\lim \mathfrak{u} = \sup \{ k \in L \mid k \text{ compact}, \uparrow k \in \mathfrak{u} \}.$$

Proof. In every continuous lattice one has $\lim \mathfrak{u} = \sup_{U \in \mathfrak{u}} (\inf U)$ by HOFMANN-STRALKA or SCOTT. Now $\inf U$ is the join of the compact elements below $\inf U$.

Now let $A$ be a universal algebra represented as a subdirect product $A \subseteq \prod_{i \in I} A_i$. For every ultrafilter $\mathfrak{u}$ on $I$ one defines a congruence $[\mathfrak{u}]$ in the following way:

$$(a,b) \in [\mathfrak{u}] \text{ iff } \{ i \in I \mid a_i = b_i \} \in \mathfrak{u}$$

Let $L = \mathcal{C}(A)$ be the lattice of all congruences of $A$ and $\lambda_i = \{ (a,b) \mid a_i = b_i \}$. Let $\lambda_{\mathfrak{u}}$ be the ultrafilter on $L$ generated by $\mathfrak{u}$. Then:

2.2. LEMMA. $[\mathfrak{u}] = \lim \lambda_{\mathfrak{u}}$.

Proof. $\lim \lambda_{\mathfrak{u}} \overset{(2.1)}{=} \sup \{ \lambda(a,b) \mid \uparrow \lambda(a,b) \in \lambda_{\mathfrak{u}} \}$

($\lambda(a,b)$ is the principal congruence generated by $(a,b)$.)

$$= \sup \{ \lambda(a,b) \mid \{ i \in I \mid \lambda(a,b) \subseteq \lambda_i \} \in \mathfrak{u} \}$$
$$= \{ (a,b) \in L \times L \mid \{ i \in I \mid a_i = b_i \} \in \mathfrak{u} \}.$$
$$= [\mathfrak{u}].$$

2.3 JÓNSSON'S LEMMA. If $\pi$ is a prime congruence on a subdirect product $A \subseteq \prod_{i \in I} A_i$, then there is an ultrafilter $\check{u}$ on $I$ such that $\pi \supseteq [u]$.

Proof. By (1.11) there is an ultrafilter $\check{u}$ on $I$ such that $\pi \supseteq \lim \lambda_{\check{u}}$, but $\lim \lambda_{\check{u}} = [u]$ by 2.2.

2.4 Let us add another remark. Let $\pi : E \to X$ be a sheaf of universal algebras. Consider the lattice $L$ of all ~~ideals~~ filters of $\Theta(X)$ which surely is algebraic. Define

$$i : L \to \mathcal{C}(\Gamma\pi)$$

in the following way: If $\mathcal{F}$ is an ideal of $\Theta(X)$ let

$$(\sigma, \tau) \in i(\mathcal{F}) \text{ iff } \{x \in X \mid \sigma(x) = \tau(x)\} \in \mathcal{F}$$

Then $i$ preserves joins of increasing nets. Thus, we may apply THE LEMMA:

<u>For every prime congruence $p$ on $\Gamma\pi$ there is an ultrafilter $\check{u}$ on $P$ such that $p \supseteq i(\lim \check{u})$, where $P$ is the set of all prime filters of $\Theta(X)$ of the form</u>

$$\mathcal{J}_x = \{ U \in \Theta(X) \mid x \in U \}, \quad x \in X$$

# 4. Application to topological representations of certain continuous lattices.

4.1. EXAMPLE. Let $X$ be a compact partially ordered space. NACHBIN has proved that such a space is normal in the following sense: Let $A$ be a closed lower section and $B$ a closed upper section of $X$. Then there are disjoint open lower and upper sections $\mathcal{U}$ and $\mathcal{V}$ containing $A$ and $B$, respectively.

Let $\mathcal{L}(X)$ be the set of open lower sections of $X$, ordered by inclusion, $\mathcal{L}(X)$ is a lattice. Nachbin's result implies, that this lattice is continuous. Further, $X$ is homeomorphic to the set of prime elements of $L$.

4.2 Let $L$ be a continuous lattice with the following properties: The set $\Pi$ of prime elements is closed; every element in $L$ is a meet of prime ones.

$\Pi$ is ~~ord~~ partially ordered as a subset of $L$. We claim: $\boxed{L \text{ is isomorphic to } \mathcal{L}(\Pi)}$.

Proof. For every element $a \in L$ let $S(a) = \{p \in \Pi \mid p \not\geq a\}$. The map $a \mapsto S(a) : L \to \mathcal{L}(\Pi)$ is clearly increasing and injective. For the surjectivity let $\mathcal{U}$ be an open lower section of $\Pi$. Then $A = \Pi \setminus \mathcal{U}$ is a closed upper section of $\Pi$. Let $a = \inf A$. Clearly $\mathcal{U} \supseteq S(a)$. For the converse inclusion let $p \in \Pi \setminus S(a)$; then $p \geq a = \inf A$. By THE LEMMA, $p \geq p'$ for some $p' \in A$. As $A$ is an upper section, we conclude $p \in A$, i.e. $p \notin \mathcal{U}$.

Questions

1. For topological spaces $X$, characterise nicely the relation $C \ll U$ (i.e. $C$ is relatively compact in $U$ in the sense of section 1.)
   Give special cases where this relation can be caracterised nicely.

2. Characterise the topological spaces $X$, for which the lattice $O(X)$ is continuous. Find ~~ex~~ examples other than 1.7, 1.8.

3. Characterise the topological spaces $X$, for which the following holds: If a dense open subset $C$ is relatively compact in $X$, then $X$ is compact (or quasicompact).

4. For every lattice $L$, the set of ideals $\{C_x \mid x \in L\}$ should be a continuous lattice.

5. Study lattices which are continuous and dually continuous. These are topological lattices. Algebraic characterisation of Lawson lattices.

6. Deduce Theorem 3.2 directly from the classical JÓNBSON LEMMA.

7. Extend the results of section 3 in such a way that one includes the following: Let $X$ be a completely regular space and $P$ a closed prime ideal of the ring $\mathcal{C}_b(X)$ of bounded real or complex valued functions on $X$. Then there is an ultrafilter $\tilde{u}$ on $X$ such that
   $P = \{f \in \mathcal{C}_b(X) \mid \lim(f(\tilde{u})) = 0\}$.

8. What does remark 2.4 really mean.

9. If $X$ is a locally compact p.o. space, is the lattice of open lower sections of $X$ continuous?

10. Extend the representation theory of section 6 to continuous lattices, where every element is a meet of primes.