Commentary on the preprint by Gierz and Keimel on

" Topologische Darstellung von Verbänden"

Naturam Expellas ~~naturam~~ furca tamen unde recurrit

Ovid

We will use as primary references

HMS DUALITY (~~xxxx~~ Lecture Notes in Mathem. 396,1974)

ATLAS (The Algebraic Theory of compact LAwson Semilattices ,Dissertationes Math. 1976 (Hofmann-Stralka).

1. Duality,congruences,lattices.

A semilattice (lattice) is always a semilattice (lattice) with identity; morphisms of semilatttices (lattices) always preserve identities: These things hold by FIAT. The category of semilattice is called $\underline{S}$, the category of lattices $\underline{L}$, the category of compact zero dimensional topological semilattices $\underline{Z}$. We know that we may view every $\underline{Z}$ object as an algebraic lattice and vice versa (HMS DUALITY).

Lattice theoreticians rather talk about lattices than semilattices,topological algebraists, in view of the DUALITY between $\underline{S}$ and $\underline{Z}$, rather think in terms of semilattices. How is the passage from $\underline{S}$ to $\underline{L}$ handled in terms of DUALITY?

1.1. <u>DEFINITION.</u> An algebraic lattice or a $\underline{Z}$ -object ~~xx~~ T is called <u>arithmetic</u> if ~~Kx~~ $K(T)^2 \subseteq K(T)$(i.e. if $K(T)$ is a sublattice of T). A $\underline{Z}$-morphism $f:T \longrightarrow T'$ is called <u>arithmetic</u> iff its left adjoint $g:T' \longrightarrow T$ (ATLAS CHapter I) is a morphism of lattices (i.e. preserves finite infs). ~~Sixxxxxxxxxxxxxxxxxxxx~~ Let Z_L be the class of all arithmetic $\underline{Z}$ -objects together with the ~~calls~~ class of all arithmetic ~~x x~~ $\underline{Z}$-morphisms.□

Since adjoints compose by ATLAS 1.11 (easy!), the composition of arithmetic $\underline{Z}$-morphism is again one such. Thus $\underline{Z}_L$ is a subcategory of $\underline{Z}$. Recall also from ATLAS 1.21 that for the left adjoint $g: T' \longrightarrow T$ of $f: T \longrightarrow T'$ one has $g(K(T')) \subseteq K(T)$ and that $K(f) = g|K(T')$. Recall from HMS DUALTIY that $K(f)$ is equivalent to $\hat{f}: \hat{T}' \longrightarrow \hat{T}$. Since g is continuous from below (ATLAS 1.20) and since $t = \sup \downarrow t \cap K(T)$ in any $\underline{Z}$-object we have:

1.2. REMARK. If g is the left adjoint of a $\underline{Z}$-map $f: T \longrightarrow T'$, then $g(t) = \sup K(f)(\downarrow t \cap K(T'))$ for all $t \in T'$. □

We are ready for the answer to the question how lattice are to be handled in the framework of ~~inside the~~ DUALITY.

1.3. PROPOSITION. (EK 1.7) The duality between $\underline{S}$ and $\underline{Z}$ induces a duality between $\underline{L}$ and $\underline{Z}_L$.

Proof. Let $S \in \underline{S}$ and $T = \hat{S}$ be its dual in $\underline{Z}$. Then $S \in \underline{L}$ iff $T \in \underline{Z}_L$, since $S \cong (K(T), \vee)$ and $K(T) \hookrightarrow T$ preserves existing finite infs(consider $k \wedge_{K(T)} h \le kh = \sup \downarrow k \cap K(T)$!). Let $\varphi: S' \longrightarrow S$ with adjoint g. be in $\underline{S}$ and $f: T \longrightarrow T'$ its dual. Then

(i) $\varphi \in \underline{L}$ iff (ii) $f \in \underline{Z}_L$.

(i) $\Longleftrightarrow$ $K(f)(xy) = K(f)(x)K(y)$ for $x,y \in K(T')$ (by HMS DUALITY) $\Longleftrightarrow$ $g(xy) = g(x)g(y)$ for $x,y \in K(T')$ (by 1.2) $\Longleftrightarrow$ $f \in \underline{Z}_L$ (by definition). □

Duality gives an ideal tool to work with congruences in $\underline{S}$-objects. Let us consider $S \in \underline{S}$ and $T \in \underline{Z}$ its dual. Every congruence $\lambda \in \mathrm{Cong}(S)$ gives an $\underline{S}$-epic $S \twoheadrightarrow S/\lambda$ whose dual is a $\underline{Z}$-monic $(S/\lambda)^\wedge \rightarrowtail \hat{S}$ (HMS) which we describe as a subobject $T[\lambda] \subseteq T$. Conversely, if $T' \subseteq T$ is a subobject, its inclusion map gives a dual map $S \twoheadrightarrow T[\lambda]^\wedge$ whose kernel congruence we denote with $S[T']$.

Let Sub T the set of (closed!) subobjects of T.

1.4. PROPOSITION. $\lambda \mapsto T[\lambda] : (\text{Cong } S, \cap) \longrightarrow \text{Sub } T$ is an $\underline{S}$-isomorphism with inverse $T' \mapsto S[T']$, where the semigroup operation in Sub T is $(A,B) \mapsto AB$.

~~Fr~~ Moreover, If $S \in \underline{L}$, then the isomorphism maps $\text{Cong}_{\underline{L}}$ (the subsemilattice of lattice congruences) bijectively onto ~~Exp~~ $\text{Sub}_L\, T$, the semilattice of all $T' \hookrightarrow T$ for which the inclusion is a $\underline{Z}_L$-morphism.

In particular $T[\bigvee_J \lambda_j] = \bigcap_J T[\lambda_j]$ and $T[\bigcap_J \lambda_j] = \bigvee_J T[\lambda_j]$.

Proof. By DUALITY it is clear that $\lambda \mapsto T[\lambda]$ and ~~Tt~~ $A \mapsto S[A]$ are inverse of each other. Let $\rho, \lambda \in$ Cong S. We consider the dual diagrams ($\times = \amalg$ in $\underline{S}$ and $\underline{T}$!)

$$\begin{tikzcd}
S \arrow[rr, two heads] \arrow[dr] \arrow[dd] & & S/\rho \\
& S/(\lambda\cap\rho) \arrow[ur] \arrow[dl] \arrow[dr, "\varphi"] & \\
S/\lambda & & S/\lambda \times S/\rho \arrow[ll, "pr_\lambda"] \arrow[uu, "pr_\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T[\lambda] \times T[\rho] \arrow[dr, "\hat\varphi"] & & T[\rho] \arrow[ll, "copr_\rho"'] \arrow[dl] \arrow[dd] \\
& T[\rho\cap\lambda] \arrow[dr] & \\
T[\lambda] \arrow[uu, "copr_\lambda"] \arrow[ur] \arrow[rr, hook] & & T
\end{tikzcd}$$

The point is now that φ is injectve. Hence $\hat\varphi$ is surjective, and consquently is the image under the canonical map $T[\rho] \times T[\lambda] \longrightarrow T$ given by the coproduct property, which is precisely the map $(a,b) \mapsto ab$. Hence $T[\rho\cap\lambda] = T[\rho]T[\lambda]$.

The remainder is clear from 1.3. □

Now we remember from the ELEMENTS (of compact ~~xxxxxxx~~ semigroups by Hofmos, Chapter A) that for each topological compact semigroup [monoid] G the hyperspace C(S) of compact subsets is a compact semigroup [monoid] under $(A,B) \mapsto AB$, and if S is abelian, then Sub(S) is a compact subsemigroup. But every topologist knows that the hyperspace of a compact 0-dimensional space is compact zero dimensional. Consequence: For $T \in \underline{Z}$ we have ~~xxxxxxx~~ $\text{Sub}(T) \in \underline{Z}$. This means that the hyperspace topology is <u>the unique</u> $\underline{Z}$-object topology.

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1.5. LEMMA. $\mathrm{Sub}_L(T)$ is closed in $\mathrm{Sub}(T)$, hence is in particular a $\underline{Z}$-object.

Proof. By Numakura's theorem, the unique uniform structure of a compact zero dimensional monoid T has a basis of open-closed congruences R. If $X \subseteq C(T)$, then $A \in C(T)$ is in the closure $\bar{X}$ of X iff for each R there is a $B \in X$ with $A \subseteq R(B)$ and $B \subseteq R(A)$. Now let $A \in \overline{\mathrm{Sub}_L(T)}$. Let $g_A: K(T) \to K(A)$ be $K(A \hookrightarrow T)$ ~~be the left adjoint of the inclusion~~ i.e. for each $k \in K(T)$ we have $k \leq g_A(k) \in A$ and $g_A(k)$ is minimal with these properties. We have to show that g_A is multiplicative. ~~Since g_A is continuous $K(T)$.~~ Now let $k \in K(T)$. For any compact open congruence R on T we find a $B \in \mathrm{Sub}_L T$ with (i) $A \subseteq R(B)$ and (ii) $B \subseteq R(A)$. We take R so small that all congruence classes are entirely in $\uparrow k$ or its complement. Since $k \leq g_B(k)$, then $R(g_B(k)) \subseteq \uparrow k$, and by (i) there is an $a \in R(g_B(k)) \cap A$; then $k \leq a$ whence $g_A(k) \leq a\ R\ g_B(k)$. By symmetry we have a $b \in B$ with $g_B(k) \leq b\ R\ g_A(k)$. We conclude $g_A(k)\ R\ g_B(k)$. Now let $k, h \in K(T)$. Let R be so small that it is both compatible with k and h. Then we have $g_A(kh) R g_B(kh) = g_B(k)g_B(h)\ R\ g_A(k)g_A(h)$, i.e. $g_A(kh) \equiv g_A(k)g_A(h) \bmod R$ for all sufficiently small R, whence $g_A(kh) = g_A(k)g_A(h)$. □

Since $\underline{Z}$-topologies are unique,
We have shown the following result

1.6 THEOREM. Let $S \in \underline{S}$ [resp. $S \in \underline{L}$] and let $T \in \underline{Z}$ [resp. $T \in \underline{Z}_L$] its dual. Then $\mathrm{Cong}(S)$ [$\mathrm{Cong}_L(S)$] are algebraic lattices and if equipped with the unique $\underline{Z}$-topology, then the map $\lambda \longmapsto T[\lambda]$ is [resp. induces] an $\underline{Z}$-isomorphism $(\mathrm{Cong}(S), \cap) \longrightarrow (\mathrm{Sub}(T), .)$ (with $A.B = AB$) [resp. induces a Z-isomorphism $(\mathrm{Cong}_L S, \cap) \longrightarrow (\mathrm{Sub}_L T, .)$]. □

This is shown in GK in Section 3 MULTIS CUM CALCULATIONIBUS.

NOTES. The compact elements in Cong (S) and $\mathrm{Cong}_L(S)$ are the finitely generated ones. By HMS DUALITY, p.38, an element $A \in \mathrm{Sub}(T)$ is compact iff it is isolated in $\{B \in \mathrm{Sub}(T) : A \subseteq B\}$. This is the case iff there is an open closed congruence R in T such that $A \subseteq B \subseteq R(A)$ for $B \in \mathrm{Sub}(T)$ implies $A = B$. So far this holds just the same way in $\mathrm{Sub}_L(T)$ (with an open closed semilattice congruence R). For semilattices we may conclude that this is tantamount to saying that A is open: Indeed if $r \in R(A)$, set $B = A \cup rA$. Then $B \in \mathrm{Sub}(T)$ with $A \subseteq B \subseteq R(A)$, hence $A = B$, so $r = r1 \in rA \subseteq B = A$. Thus $A = R(A)$ and so A is open. Conversely if A is closed open then there is a congruence R with $R(A) \subseteq A$.

For lattices this fails at two points: Firstly we took $B = \{r,1\}A$ where $\{r,1\}$ is the smallest objects in $\mathrm{Sub}(T)$ containing r. I do not know exactly what the smallest $\mathrm{Sub}_L(T)$-object $\langle r \rangle$ is which contains r; if r is prime, $\{r,1\}$ is still ok. Secondly, even if we could describe $\langle r \rangle$ there is little reason to believe that we could get $\langle r \rangle A \subseteq R(A)$ since we do not necessarily have $\langle r \rangle \subseteq R(A)$.

As far as the $\underline{Z}$-topology is concerned, it is generated by sets $\uparrow k$ and $T \setminus \uparrow k$, $k \in K(T)$; thus a subbasis for the $\underline{Z}$-topology on Cong (S) [resp. $\mathrm{Cong}_L(S)$] is given by the sets $\{\lambda : (a,b) \in \lambda\}$, $a, b \in S$ and their complements (GK p.11). For a semigroup S, I can identify the associated object $T[\lambda]$ of a congruence λ generated by (a,b): Let $f, g : 2 \longrightarrow S$ be given by $f(0) = a$, $g(0) = b$, and let the dual of f be the characteristic function of $\uparrow h$, $h \in K(T)$ and that of g the characteristic function of $\uparrow k$, $k \in K(T)$. Now $2 \underset{g}{\overset{f}{\rightrightarrows}} S \xrightarrow{quot} S/\lambda$ is a co-equalizer, hence $T[\lambda] \xrightarrow{incl} T \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} 2$ is an equalizer; the equalizer of $\chi_{\uparrow h}$ and $\chi_{\uparrow k}$ is $\uparrow(k \vee h) \cup (I(h) \cap I(k))$ $I(k) = T \setminus \uparrow k$, i.e. the complement of the symmetric difference of $\uparrow k$ and $\uparrow h$. (I do not know what good this does, but it is another exercise on duality.)

2. Irreducibles, generating sets.

We recall that an element p in a semilattice S is called completely meet irreducible (and we will just say irreducible) if $p = \inf X$ implies $p \in X$. ~~It is important to remember~~ The set of all irreducible elements will be called $\mathrm{Irr}(S)$. It is important to remember that in any T-object ~~xx~~ we have not only the frequently invoked relation

$$\text{(I)} \quad t = \sup\left(\downarrow t \cap K(T)\right) \text{ for all } t \in T,$$

but also the relation

$$\text{(II)} \quad t = \inf\left(\uparrow t \cap \mathrm{Irr}(T)\right) \text{ for all } t \in T.$$

(HMS p.57)

2.1 <u>LEMMA.</u> Let $T \in \underline{Z}$ and $X \subseteq T$ a subset. Then the following statements are equivalent:

(1) $(\forall k,h \in K(T))\ \uparrow k \cap X = \uparrow h \cap X \Rightarrow h=k$.

(2) $(\forall k \in K(T))\ k = \inf \uparrow k \cap X$

(3) Every $k \in K(T)$ is approximated from above by elements $x_1 \ldots x_n$, $x_j \in X$.

(4) X generates T (i.e. $T = (\cup X^n)^-$).

Proof (2)$\Rightarrow$(1) is trivial.

(1)$\Rightarrow$(2): Suppose not (2), then there is a $k \in K(T)$ with $k < a = \inf \uparrow k \cap X$. There is a $h' \in K(T)$ with $h' \leq a$ and $h' \not\leq k$. Set $h = k \vee h'$, then $k < h \leq a$ and $h \in K(T)$. From $k < h$ we derive $\uparrow h \cap X \subseteq \uparrow k \cap X$, from $h \leq \inf \uparrow k \cap X$ ~~xx~~ we conclude $\uparrow k \cap X \subseteq \uparrow h \cap X$. Hence $\uparrow k \cap X = \uparrow h \cap X$. We showed "not (1)".

(2)$\Leftrightarrow$(3) is clear.

(3)$\Rightarrow$(4) : $K(T)^- = T$.

(4)$\Rightarrow$(3) : By (4), every k can be approximated by elements $x_1 \ldots x_n$, $x_j \in X$; since $\uparrow k$ is open, the approximation is finally from above.

7.

2.2. LEMMA. Let $T \in \underline{Z}$ and $X \in T$ a subset. Then the following statements are equivalent:

(1) $(\forall a,b \in T)\ \uparrow a \cap X = \uparrow b \cap X \Rightarrow a = b$

(2) $(\forall t \in T)\ t = \inf \uparrow t \cap X$

(3) Every $t \in T$ is approximated from above by elements $x_1 \ldots x_n$, $x_j \in X$.

(4) $\mathrm{Irr}\, T \subseteq X$.

Moreover these conditions imply those of 2.1.

Proof. (2)=>(1) trivial.

(1)=>(2). Suppose not (2). Then there is an $a \in T$ with $a < \inf \uparrow a \cap X$. Set $b = \inf \uparrow a \cap X$ and show $\uparrow a \cap X = \uparrow b \cap X$ as in the proof of (2)=>(1) in 2.1. This shows " not (1)".

(2)<=>(3) is clear.

(4) =>(2) by (II).

(2) =>(4) Let $p \in \mathrm{Irr}\, T$. By (2) we have $p = \inf \uparrow p \cap X$. Since p is irreducible, $p \in X$. Evidently (2) above implies (2) of 2.1. ∎

NOTATION. If $X \subseteq T \in \underline{Z}$ we say that X is a <u>generating set</u> if it satisfies the equivalent conditions of 2.1. We say that it is an <u>order generating set</u> if it satisfies the equivalent conditions of 2.2. ∎

Clearly Irr T is the unique smallest order generating set, and $\overline{\mathrm{Irr}}\, T$, the closure of Irr T is the smallest closed order generating set in T. If $S = \{1\} \cup \{1 - \frac{1}{n}: n=1,2,\ldots\}$ under min and $T = S \times S$, then $X = (S\setminus\{1\}) \times (S\setminus\{1\})$ is a generating set but no order generating set, and $X \cap \mathrm{Irr}\, T = \emptyset$.

I don't know (but would like to) whether or not a closed generating set must contain Irr T (i.e. is order generating). I would like to point out, however, that prime elements have to be in the ~~saturation of~~ closure of a generating set:

For this purpose I record the following noteworthy observation of Keimel's:

2.3. KEIMEL'S LEMMA. Let T be a CL-object (a compact Lawson semilattice) p a prime element and Y a ~~closed~~ subset of T with $\inf Y \leq p$. Then there is a $y \in \overline{Y}$ with $y \leq p$.

Proof. Fact: If a compact set Y is contained in an open subsemilattice U of a CL-object, then the closed subsemilattice $\langle Y \rangle$ generated by Y is still contained in U: Indeed cover Y with a finite number of closed subsemilattices contained in U and note that the subsemilattice generated by them is still in U. Now let p be prime. ~~Then~~ This means that $T \setminus \downarrow p$ is an open subsemilattice. If it contains $\overline{Y}$ then by the "fact" it contains $\inf \overline{Y} (= \inf Y)$. ∎

2.3.a. COROLLARY. If $\inf X = 0$, then $\mathrm{Prime}\, T \subseteq \uparrow \overline{X}$. ∎

This allows us to combine, in the distributive case, the preceding results:

2.4. PROPOSITION. (KEIMEL and GIERZ-KEIMEL)
Let T be a distributive $\underline{Z}$ object. (See HMS DUALITY pp. 65 and 71). Let X be a subset of T. Then the following statements are equivalent:

(1) $(\forall k,h \in K(T))\ \uparrow k \cap X = \uparrow h \cap X \Rightarrow h=k.$

(2) $(\forall s,t \in T)\ \uparrow s \cap \overline{X} = \uparrow t \cap \overline{X} \Rightarrow s=t.$

(3) $(\forall k \in K(T))\ k = \inf \uparrow k \cap X.$

(4) $(\forall t \in T)\ t = \inf \uparrow t \cap \overline{X}.$

(5) X generates T.

(6) $\overline{X}$ contains $\mathrm{Irr}\, T$.

(7) $\overline{X}$ contains $\mathrm{Prime}\, T$.

In other words, X is generating iff its closure is order generating iff its closure contains Prime T. In particular, $\overline{\mathrm{Prime}}\, T = \overline{\mathrm{Irr}}\, T$ is the smallest closed generating set.

Proof. ~~[struck-out text]~~ By 2.1 we have (1)<=>(3)<=>(5) and by 2.2 we know (2)<=>(4). Since T is distributive, $t = \inf \uparrow t$ Prime T by HMS DUALITY p.65, whence (7) => (4). Remains (3) => (7): Let $p \in$ Prime T. Take an arbitrary $k \in K(T)$ with $k \leq p$. By (3) we have $k = \inf \uparrow k \cap \bar{X} \leq p$. Hence there is an $x_k \in \bar{X}$ with $k \leq x_k \leq p$ by KEIMEL's LEMMA 2.3. Since p is approximated from below by k (DUALITY) then $p \in \bar{X}$. □

I do not really know how much can be said about the structure and position of the set Irr T in general. By way of challenge I make the following remarks. Recall from ATLAS 4.1 the set $A(T) = \{t : t = \inf \operatorname{int} \uparrow t\}$. These are the elements which are not "locally facial" wher an element x would be called facial if not $x \ll 1$.

REMARK. For $p \in$ Irr T set $\bar{p} = \inf(\uparrow p \setminus \{p\})$, then $p < \bar{p}$ and $\bar{p}$ is not irreducible; one has $p \in K(T)$ iff p is isolated (in which case $\bar{p} \in K(T)$). Moreover, $\mathrm{Irr}(T) \cap A(T) \subset K(T)$ consists of isolated points. □

Irr T is an a-multiplicative subset in the sense that $a, b, ab \in$ Irr T implies $a \leq b$ or $b \leq a$. Once again I do not know exactly how to describe $\overline{\mathrm{Irr}}$ T other than saying in some sense that it is approximately a-multiplicative. I would like to know more about the internal order structure of $\overline{\mathrm{Irr}}$ T or $\overline{\mathrm{Prime}}$ T even in the presence of distributivity

REMARK. ~~Even~~ If T is a Z-object then Prime T need not be order generating while being generating. In fact, Prime T is order generating iff T is distributive (HMS DUALITY p.65) whereas Prime T is generating iff the primes filters x of S (the dual of T) separate the points (loc.cit.p.65). J.C.Varlet has provided an example to show that these are not equivalent properties (On separation properties in semilattices, Semigroup Forum 10 (1975), 220-228, notably p 225 ff.)

GK are primarily interested in the case $S \in \underline{L}$ but do not wish to assume that S (hence $T = \hat{S}$) is distributive. So they look for the nearest distributive $\underline{Z}$ object in sight. They find it in $\mathrm{Cong}_L(S) \cong \mathrm{Sub}_L(T)$.
We know that $\mathrm{Irr}\ \mathrm{Cong}_L(S)$ separates the points of S and that its closure is the smallest closed generating set of $\mathrm{Cong}_L(S)$. Thus we discuss pointseparation on S and a characterisation of closures in $\mathrm{Cong}_L(S)$.

2.5. LEMMA. Let $S \in \underline{S}$, $\Lambda \subseteq \mathrm{Cong}(S)$. Let $T = \hat{S}$. Then the following statements are equivalent:

(1) Λ separates the points (i.e. S is a subdirect produc of the S/λ, $\lambda \in \Lambda$)

(2) $\coprod_\Lambda T[\lambda] \xrightarrow{m} T$ is surjective (where $\coprod$ is the coproduct in $\underline{Z}$.

(3) $\bigcup_\Lambda T[\lambda]$ is a generating set in T.

Proof. (1) means $S \longrightarrow \prod_\Lambda S/\lambda$ is injective, hence by DUALITY, (1) is equivalent to (2). Every generating set G of $\coprod_\Lambda T[\lambda]$ maps onto a generating set m(G) of im m ; if $j_\lambda : T[\lambda] \longrightarrow \coprod_\Lambda T[\lambda]$ are the coprojections then $G = \bigcup_\Lambda j_\lambda(T[\lambda])$ is a generating set, and $m(G) = \bigcup_\Lambda T[\lambda]$. Hence (2) and (3) are equivalent.□

I wonder when $\bigcup_\Lambda T[\lambda]$ is order generating. If for $x \in T$, I write $y = \inf\left(\uparrow x \cap \bigcup_\Lambda T[\lambda]\right)$, then I find $y = \inf_{\lambda \in \Lambda} g_\lambda(x)$ (where g_λ is the left adjoint of $T[\lambda] \longrightarrow T$) and, as a consequence $g_\lambda(x) = g_\lambda(y)$ for all $\lambda \in \Lambda$. So what? I am unable to show that under the conditions of the ~~theore~~ Lemma the g_λ separate the points. If Λ is finite, products and coproducts coincide and we have this information. The answer appears to be yes for finite Λ, therefore. I doubt its validity in general.

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2.6. ZUSATZ. The conditions (1),(2),(3) of 2.5 imply

Prime ? (4) Irr $\mathrm{Cong}(S) \subseteq \uparrow\bar{\Lambda}$,i.e. for each irreducible congruence ρ there is a $\lambda \in \bar{\Lambda}$ with $\lambda \subseteq \rho$.

In case $S \in \underline{L}$ and $\Lambda \subseteq \mathrm{Cong}_L(S)$, we have

(4L) Irr $\mathrm{Cong}_L(S) \subseteq \uparrow\bar{\Lambda}$

Proof. (1) $\Leftrightarrow$ inf $\Lambda = 0$. The assertion follows by 2.3.a.□

2.7. LEMMA. Let $\Lambda \subseteq \mathrm{Cong}\ S$. Then

$$(\cup\{T[\lambda]: \lambda \in \Lambda\})^{-} = \cup\{T[\lambda]: \lambda \in \bar{\Lambda}\}.$$

Proof. The right hand side is closed by 1.6 and a fact known to people working with hyperspaces. Hence $\subseteq$ follows. Now let $\lambda \in \bar{\Lambda}$, then $T[\lambda] = \lim T[\lambda_j]$ with λ_j a net on Λ . Once again the theory of hyperspaces yields that $T[\lambda]$ is contained in the left hand side.□

2.8. DEFINITION. Let S be a lattice and T its dual. The subspace

$$(\cup\{T[\lambda]: \lambda \in \mathrm{Irr}\ \mathrm{Cong}_L(S)\})^{-} =$$

$$\cup\{T[\lambda]: \lambda \in \overline{\mathrm{Irr}}\ \mathrm{Cong}_L(S)\} =$$

$$\cup\{T[\lambda]: \lambda \in \overline{\mathrm{Prime}}\ \mathrm{Cong}_L(S)\}$$

(see 2.7 and 2.4) in T together with the induced partial order is called the Gierz-Keimel spectrum of S, written $X(S)$.□

In some sense this construction is minimal:

2.9. REMARK. If Λ is any separating set of lattice congruences on $S \in \underline{L}$ then

$$X(S) \subseteq \cup\{T[\lambda]: \lambda \in \bar{\Lambda}\}.$$

Proof. W.l.g. we may assume that Λ is closed. Then $\overline{\mathrm{Irr}}\ \mathrm{Cong}_L(S) \subseteq \uparrow\Lambda$ by 2.6. Thus, if $\rho \in \overline{\mathrm{Irr}}$, there is a $\lambda \in \Lambda$ with $\lambda \subseteq \rho$, hence $T[\rho] \subseteq T[\lambda]$. The claim follows.□

2.10. PROPOSITION. If $S \in \underline{L}$ and T is the dual of S, then $\overline{\text{Prime}}\ T \subseteq X(S)$, and if S is distributive, then equality holds. In that case $X(S)$ is the unique smallest closed generating set.

Proof. If $p \in$ Prime T, then $\{p,1\} = $ ~~Km~~ $T[\lambda]$ for a two-coset congruence $\lambda \in \mathrm{Cong}_L(S)$. Thus $p \in X(S)$. If S is distributive $\mathrm{Irr}\ \mathrm{Cong}_L(S)$ consists precisely of all two-coset congruence and the ~~xxxxxxx~~ one-coset congruence, and thus its image in $\mathrm{Sub}_L(T)$ consists of all $\{p,1\}$ and $\{1\}$. The assertion follows. □

I do not know, nor is there any evidence in GK that they know, whether of not $X(S)$ is order generating in general ~~t~~ (i.e. whether or not $\mathrm{Irr}(T) \subseteq X(S)$ (cf. 22.). However, GK provide the following insight into the structure of $X(G)$:

2.11. LEMMA. If ~~xxxxx~~ $a,b \in T$ and $ab \in X(S)$, then there is a $\lambda \in \overline{\mathrm{Irr}}\ \mathrm{Cong}_L(S)$ and elements $a',b' \in T[\lambda]$

$\in X(S)$ with $a \leq a'$, $b \leq b'$ such that $ab = a'b'$.

Proof. We have $ab \in T[\lambda]$ for a suitable λ by 2.7. Let $g = g_\lambda : T \longrightarrow T[\lambda]$ be the lattice retraction according to 1.1 which is left adjoint to the inclusion. Then $ab = g(ab) = g(a)g(b)$. Set $a' = g(a)$, $b' = g(b)$. □

3. Partial semilattices.

> So habt ihr denn die Teile in der Hand!
> Fehlt leider nur das geistig Band!
>
> Mephisto als Faust.

I have never been turned on by the appearance of partial algebras in universal algebra (or for that of local groups in topological group theory,for that matter). Here they go again; but I do not see how one can avoid the issue here and GK make a convincing case for it (except for their sequence of presentation which I do not like).

3.1.DEFINITION. A partial monoid is a triple $(X,D,.)$ ~~xxxxixt~~ consisting of the following ingredients:

(1) a pointed set X with distinguished element 1,

(2) a symmetric subset D in $X \times X$ (symmetric: $(a,b) \in D$ iff $(b,a) \in D$),

(3) a function $(x,y) \longmapsto xy: D \longrightarrow X$

such that $(\{1\}\times X) \cup (X\times\{1\}) \subseteq D$ and that the following algebraic conditions are satisfied:

(i) $1x = 1x = x$ for all $x \in X$.

(ii) ~~x~~ If $(x,y),(y,z) \in D$ then (x,yz) ~~iff~~ $(xy,z) \in D$ and ~~then~~

$x(yz) = (xy)z$ for all $x,y,z \in X$.

A partial semilattice is a partial monoid such that

(4) $\operatorname{diag} X \times X \subseteq D$,

and

(iii) $xx = x^2 = x$ for all $x \in X$,

(iv) $xy = yx$ for all $x,y \in X$ with $(x,y) \in D$.

A morphism $f:(X,D,.) \longrightarrow (Y,E,.)$ of partial monoids is a function with ~~Ex~~ $(f\times f)(D) \subseteq E$ and $f(1) = 1$, $f(xy)=f(x)f(y)$ for all $x,y \in X$ with $(x,y) \in D$.

A character of a partial semilattice is a morphism $f:(X,D,.) \longrightarrow 2$.$\square$

The partial semilattice and their morphisms form a category ~~XX~~ $\underline{SP}$, and $\underline{S}$ is a full subcategory thereof.

3.2. LEMMA. Let $X = (X, D, .)$ be in $\underline{SP}$. Define $\hat{X} = \underline{SP}(X,2)$. Then $\hat{X} \subseteq 2^{|X|}$ is a closed subsemilattice of $2^{|X|}$.

Proof. (i) Let $(x,y) \in D$, $f,g \in \hat{X}$; then $(fg)(xy) = \cancel{f(g(x)g(y))}$ $= f(xy)g(xy) = f(x)f(y)g(x)g(y) = f(x)g(x)f(y)g(y) = \cancel{(fg)(x)}$ $(fg)(x)(fg)(y)$, so $fg \in \hat{X}$. If $e: X \longrightarrow 2$ is the constant morphsmsm, then $ef = fe$ for all $f \in \hat{X}$.

(ii) The topology on $2^{|X|}$ is that of pointwise convergence. If $f_j \longrightarrow f$ in $2^{|X|}$ and $f_j \in X$, and if $(x,y) \in D$, then $f(xy) = \lim f_j(xy) = \lim f_j(x)f_j(y) = \lim f_j(x) \lim f_j(y) = f(x)f(y)$. Thus $f \in \hat{X}$. $\square$

It follows immediately that $X \longmapsto \hat{X} : \underline{SP} \longrightarrow \underline{Z}^{op}$ is a functor, which extends the HMS duality functor.

3.3. PROPOSITION (Ovid). $X \longmapsto \hat{X} : \underline{SP} \longrightarrow \underline{Z}^{op}$ is left adjoint to $T \longmapsto \hat{T} : \underline{Z}^{op} \longrightarrow \underline{S} \hookrightarrow \underline{SP}$.

Proof. For $X \in \underline{SP}$, set $\alpha_X : X \longrightarrow \hat{\hat{X}}$, $\alpha(x)(\varphi) = \varphi(x)$; for each $f: X \longrightarrow \hat{T}$, $T \in \underline{Z}$, $f \in \underline{SP}$ define $f': T \longrightarrow \hat{X}$ by $f'(t)(x) = f(x)(t)$. Since $x \longmapsto f(x)(t) : X \longrightarrow 2$ is a character, ~~xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx~~ this is a good definition. Routinely one verifies that f' is a $\underline{Z}$-morphism. (using, e.g. that $t \longmapsto f(x)(t) : T \longrightarrow 2$ is continuous).

Moreover, $[\widehat{f'}(\alpha(x))](t) = [\alpha(x) \circ f'](t) = [\alpha(x)](f'(t))$
$_t$ This determines f' uniquely.
$= f'(t)(x) = f(x)(t)$. Thus we have shown the required universal property : For each $f \in \underline{SP}(X, \hat{T})$ there is a unique $f' \in \underline{Z}^{op}(\hat{X}, T)$ such that $f = \widehat{f'} \alpha_X$. $\square$

3.4. LEMMA. Let $X = (X, D, .) \in \underline{SP}$. Define

$x \leq y$ iff $(x,y) \in D$ and $xy = x$.

Then $\leq$ is a partial order.

Proof. i) Let $x \in X$. Then $(x,x) \in D$ and $xx = x$ by 3.1 (4),(iii). Hence $x \leq x$ for all $x \in X$.

ii) Let $x \leq y$ and $y \leq z$. Then $(x,y),(y,z) \in D$ and $xy = x$, $yz = y$. Then $(x,yz) \in D$. By 3.1 (ii) it then follows that $(x,z) = (xy,z) \in D$ and that $xz = (xy)z = x(yz) = xy = x$. Thus $x \leq z$.

iii) Let $x \leq y$ and $y \leq x$. Then $(x,y) \in D$ and $xy = x$, but also $yx = y$. Then $x = y$ by 3.1 (iv) ~~and~~. □

[I do not see how GK obtain a "Halbordung"(whatever this means) from their definitions. There is a subtle point in proving transitivity.]

3.5. LEMMA. Let $X \in \underline{SP}$. If $(x,y) \in D$, then $xy \leq x,y$.

Proof. If $(x,y) \in D$, then also $(xx,y) \in D$ since $xx = x$. by 3.1 (ii). Thus also $(x,xy) \in D$ and $x(xy) = x^2y = xy$. This $xy \leq x$. By commutativity, the remainder follows. □

3.6. LEMMA. Let $X \in \underline{SP}$. Then for each $x \in X$ the characteristic function p_x of $\uparrow x$ is a character.

Proof. Let $(x,y) \in D$, $u \in X$. Case 1: $p_u(x) = p_u(y) = 1$. Then $u \leq x$, $u \leq y$, i.e. $(u,x),(u,y) \in D$ and $ux = u$, $uy = u$. Thus $(ux,y) \in D$, whence $(u,xy) \in D$ and $u(xy) = (ux)y = uy = u$ (by 3.1.(ii)). Thus $u \leq xy$, whence $p_u(xy) = 1$. Case 2. $p_u(x) = 0$. Claim: $p_u(xy) = 0$. Assume not. Then $p_u(xy) = 1$ i.e. $u \leq xy$. By 3.5, we have $xy \leq x$, thus $u \leq x$ by 3.4. Hence $p_u(x) = 1$, contradiction. □

NOTATION. p_u is called the principal character generated by u.

3.7. PROPOSITION. Let ~~HMS~~ $x \in \underline{SP}$. Then $\alpha_X : X \longrightarrow \hat{\hat{X}}$ is an epic embedding (in $\underline{SP}$).

embedding $\overset{?}{=}$ injektiv

Proof. Suppose $\alpha_X(x) = \alpha_X(y)$. Then $\varphi(x) = \varphi(y)$ for all characters of X. Hence in particular $p_u(x) = p_u(y)$ for all principal characters p_u. Taking $u = x$ and $u = y$ we obtain $x \leq y$ and $y \leq x$, hence $x = y$ by 3.4. Hence α_X is an embedding. But if the front adjunction (in an adjoint situation) is monic, then it is epic. □

Note this means that X is generated by $\alpha_X(X)$, since the inclusion $S \hookrightarrow \hat{X}$ of the subsemilattice S generated in $\hat{X}$ by im α_X is epic by 3.7, hence is surjective by HMS DUALITY p.7 ~~[illegible]~~.

We may therefore consider every partial semilattice as a partial subsemilattice of a semilattice. For the ~~this~~ purpose of stating this accurately, we make the following ~~definition~~ observation:

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3.8. ~~DEFINITION~~ LEMMA. Let $S \in \underline{S}$ and $1 \in X \subseteq S$. Define $D_X = \{(x,y) \in X \times X : xy \in X\}$. Then $(X, D_X, \cdot)$ (with $\cdot$ denoting the induced multiplication) is a partial semilattice, called the partial subsemilattice induced on X by S.

Proof. Verify the conditions of 3.1. Notably (ii): Suppose $x,y,z \in X$ and $xy, yz \in X$, then $x(yz) = (xy)y$ shows that $(x,yz) \in D_X$ iff $(xy,z) \in D_X$. □

3.9. PROPOSITION (Ovid). The functor $X \longmapsto \hat{\hat{X}} : \underline{SP} \longrightarrow \underline{S}$ is a left reflector F, whose front adjunction α is an epic embedding. If we embed X in FX via α_X, then D_X contains D.

Proof. $\underline{S}(FX, S) \cong \underline{T}^{op}(X,S)$ (by ~~[illegible]~~ DUALITY and $FX = (\hat{X})^\wedge$) $\cong \underline{SP}(X, |S|)$ (by 3.3). The rest is clear from 3.7 and 3.8. □

Herrlich and his crowd would call $\underline{SP}$ an epireflective subcategory of $\underline{S}$.

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GRATUITOUS REMARKS

3.10. LEMMA. Let $(X,\leq)$ be a poset(with maximal element 1).
Define $D^{(X)} = \{(x,y) \in X \times X : \inf\{x,y\} \text{ exists}\}$. ~~and~~

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Then $(X, D^{(X)}, \inf)$ is a partial semilattice.

Proof. Verify the conditions of 3.1, notably (ii): if min (x,y), min (y,z) exist, then $\min(x,\min(y,z))$ exists iff $\min(\min(x,y),z)$ exists (and equals $\min\{x,y,z\}$). □

3.11. LEMMA. Let $(X,D,.) \in$ SP, and let $\leq$ be the associated partial order. Then let $(X, D^{(X)}, \min)$ be the SP-object associated to $(X, \leq)$ according to 3.10. Then $D \subseteq D^{(X)}$, and min extends . .

Proof. If $(x,y) \in D$ then $xy \leq x,y$ by 3.5. Now suppose $xy \leq a \leq x,y$. Then $(a,x),(a,y),$ ~~(ax,y)~~ $(xy,a) \in D$ whence also $(ax,y) \in D$ and $(xy)a = x(ya) = xa = a$, thus $a \leq xy$. Hence $a = xy$. Thus $xy = \min\{x,y\}$. Therefore $D \subseteq D^{(X)}$, and min extends . . □

3.12. EXAMPLE. Let $S = \{0=(0,0),\ d=(\frac{1}{2},0),\ x = (\frac{1}{2},1), y=(1,0), 1 = (1,1)\}$ with the semilattice structure induced by $I \times I$.
Let $X = \{0,x,y,1\}$, $D = D_X$, $\cdot$ = multiplication induced from S. Then $FX = S$ and α_X = incl, for $X = (X, D_X, .)$.
However, $D^{(X)} = X \times X$ and $(X, D^{(X)}, \min) \cong 2^2 \ncong (X, D_X, .)$.
Moreover, $F(X, D^{(X)}, \min) \cong 2^2 \ncong F(X, D_X, .) = FX$.

It is, therefore necessary to keep certain points very clearly before one's eyes: There are several ways to look at partial semilattices and to complete them. The poset approach, while on the surface being simpler, is apparently not the one we ~~xxxxx~~ use in the completion given by the adjunction. It seems to be essential that we speak on partial multiplications rather than partial orders. The difference vanishes only for those $(X,D,.)$ for which D contains all ~~xxxx~~ pairs (x,y) for which ~~min~~ inf$\{x,y\}$ exists, where $\leq$ is the partial order~~s~~ associated with $(X, D, \cdot)$.

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— END GRATUITES REMARKS

We must turn to the category of partial $\underline{Z}$-objects which really concern us in the context of the GK-theory; the $\underline{SP}$-theory was only a warm-up to clarify some fine points.

3.12. LEMMA. Let $(X,D,.)$ be a partial semilattice and X a compact space. The following statements are equivalent:

(1) D is closed in $X \times X$ and $(x,y) \mapsto xy : D \to X$ is continuous.

(2) $G = \{(x,y,z) \in X \times X \times X : (x,y) \in D$ and $z = xy\}$ is compact.

Proof. We have $D = pr_{12}\,G$ and $G = \text{graph}\,(x,y) \mapsto xy$. Thus (1)=>(2); but a function between compact spaces is continuous iff its graph is closed. Thus also (2)=>(1).∎

3.13. DEFINITION. A partial compact semilattice is a partial semilattice $(X,D,.)$ together with a compact topology on X such that (1), (2) of 3.12 are satisfied. A morphism of partical compact semilattices is a morphism of partial semilattices which is in addition continuous. A character is a morphism into 2. ~~x~~ The category of all compact zero dimensional partial semilattices is called $\underline{ZP}$.∎ Note that $\underline{ZP}$ is a full subcategory of $\underline{Z}$.
By 3.12 (2), the partial order $\leq$ associated with a p.c.sl. has closed graph.

3.14. PROPOSITION. (Ovid). For a ~~xxxxxxx~~ $\underline{ZP}$-object X the set $\hat{X} = \underline{ZP}(X,2)$ is a subsemilattice of $2^{|X|}$ and the assignement $X \mapsto \hat{X}: \underline{ZP} \to S^{op}$ is a functor which is left adjoint to the functor $S \mapsto \hat{S} : S^{op} \to \underline{Z} \hookrightarrow \underline{ZP}$. The front adjunction $\beta_X : X \to \hat{\hat{X}}$ is given in the usual way by $\beta_X(x)(\varphi) = \varphi(x)$.

Proof. As usual (see 3.3).∎

3.15. PROPOSITION. The functor $X \mapsto \hat{\hat{X}} : \underline{ZP} \to \underline{Z}$ is a left reflector, G whose front adjunction is epic in $\underline{ZP}$. Specifically $\beta_X(X)$ is a generating set in GX.
The following statements are equivalent:

(1) β_X is injective.

We must turn to the category of partial $\underline{Z}$-objects which really concern us in the context of the GK-theory; the $\underline{SP}$-theory was only a warm-up to clarify some fine points.

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By 3.12 (2), the partial order $\le$ associated with a p.c.sl. has closed graph.

3.14. PRODOSITION. (Ovid). For a ~~compact~~ $\underline{ZP}$-object X the set $\hat{X} = \underline{ZP}(X,2)$ is a subsemilattice of $2^{|X|}$ and the assignement $X \longmapsto \hat{X}: \underline{ZP} \longrightarrow S^{op}$ is a functor which is left adjoint to the functor $S \longmapsto \hat{S} : S^{op} \longrightarrow \underline{Z} \hookrightarrow \underline{ZP}$. The front adjunction $\beta_X : X \longrightarrow \hat{\hat{X}}$ is given in the usual way by $\beta_X(x)(\varphi) = \varphi(x)$.

Proof. As usual (see 3.3). □

3.15. PROPOSITION. The functor $X \longmapsto \hat{\hat{X}} : \underline{ZP} \longrightarrow \underline{Z}$ is a left reflector. G whose front adjunction is epic in $\underline{ZP}$.
Specifically $\beta_X(X)$ is a generating set in GX.
The following statements are equivalent:

(1) β_X is injective.

(2) β_X is an embedding

(3) If $x \not\leq y$ in X (w.r.t. the partial order associated with X by 3.4) then there is a character φ on X with $\varphi(x) = 1$ and $\varphi(y) = 0$.

Proof. The adjunction machinery works as usual. We have to prove that β_X is epic independently: ~~Let $T = G(X)$, then $\hat{X}$ may be identified with $(K(T),\vee)$ by HMS so that every $\varphi \in X$ determines a unique $k_\varphi \in K(T)$. Then $\varphi(x) = 1$ is equivalent to $k_\varphi \leq \beta(x)$.~~

let $G_o(X)$ be the closed subsemilattice generated in $G(X)$ by $\beta_X(X)$. The $\underline{ZP}$-morphism $X \longrightarrow G_oX$ then factors uniquely through β_X with a surjection into G_oX. The commuting diagram

$$\begin{tikzcd} & GX \arrow[d] \arrow[dd, bend left=60, "1_{GX}"] \\ X \arrow[ur, "\beta_X"] \arrow[r] \arrow[dr, "\beta_X"'] & G_oX \arrow[d, hook] \\ & GX \end{tikzcd}$$

then shows that $G_oX = GX$. Thus $\beta_X(X)$ is a generating set. By compactness, (1)<=>(2) , and (1) is equivalent to the condition

(3') The bharacters of ($\mathbb{Z}$ X,D,.) separate the points.

But (3') is clearly equivalent to (3).□

3.16. REMARK. If $T \in \underline{Z}$ 1∈ and $X \subseteq T$, $X = \bar{X}$, set $D_X = \{(x,y) \in X \times X : xy \in X\}$ and let $(x,y) \longmapsto xy : D \longrightarrow X$ be the restriction of the given multiplication . Then $(X, D_X, \cdot)$ is a $\underline{ZP}$-object on which the characters separate, i.e. $\beta_X : X \longrightarrow GX$ is an embedding. The induced map $GX \longrightarrow T$ is injective iff every character of X is induced by one of T. xxxx Its image is always the closed subsemiâattice of T generated by X.

Proof. The first assertions are clear,as is the claim that $GX \longrightarrow T$ maps onto the closed subsemigroup generated by X. But $GX = \hat{\hat{X}} \longrightarrow T$ is injective iff $\hat{T} \longrightarrow \hat{X}$ is surjective by DUALITY, and $\hat{T} \longrightarrow \hat{X}$ is the restriction of characters.□

It appears that the $\underline{ZP}$-situation is more complicated than the $\underline{SP}$-situation. One would like to see an example for which β_X is not an embedding and an example of ~~a~~ partial subsemilattice X of a $\underline{Z}$-object for which $GX \longrightarrow T$ is not injective. But GK leave us alone in the quest for such examples. However, they do have the following cle~~a~~ver observation which we indicated earlier in 2.11 (in an unnecessarily special version).

3.16. PROPOSITION. Let $T \in \underline{Z}_L$ and let $X = \bar{X} \subseteq T$ be a union of a collection $T[\lambda]$ of subobjects in $\mathrm{Sub}'_L(T)$. Then a order preserving continuous function $\varphi: X \longrightarrow 2$ is a character of $(X, D_X, \cdot)$ iff $\varphi | T[\lambda]$ is a character for all λ.

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Proof. The necessity is clear. Sufficiency: Let $(x,y) \in D_X$, i.e. $xy \in X$. Suppose $\varphi(xy) = 0$. Let $xy \in$ ~~T~~ $T[\rho]$ for some ρ and let $g_\rho: T \longrightarrow T[\rho]$ be the lattice retraction ~~of~~ according to 1.1 (which is left adjoint to the inclusion). Then $0 = \varphi(xy) = \varphi(g_\rho(xy)) =$ $\varphi(g_\rho(x)g_\rho(y)) = \varphi(g_\rho(x)g_\rho(y)) \geq \varphi(x)\varphi(y) \geq 0$, since $\varphi | T[\rho]$ is a character and $g_\rho(z) \geq z$ for all z. Thus $\varphi(xy) = \varphi(x)\varphi(y)$.

Suppose $\varphi(xy) = 1$. Claim: $\varphi(x) = \varphi(y) = 1$. ~~If not, then e.g. $\varphi(x) = 0$,~~ Now ~~Then~~ $x \in T[\lambda]$ for some λ. Then $1 = \varphi(xy) \leq \varphi(g_\lambda(xy)) =$ $\varphi(x g_\lambda(y)) = \varphi(x)\varphi(g_\lambda(y))$, which shows that $\varphi(x) = 1$. Similarly $\varphi(y) = 1$. □

3.17. COROLLARY. Let $T \in \underline{Z}_L$, $X = \bar{X} = \bigcup\{T[\lambda] : \lambda \in \Lambda\}$ for some separating collection $\Lambda \subseteq \mathrm{Cong}_L(S)$ (S = dual of T). Then every character of $(X, D_X, \cdot)$ is induced by a unique character of T, and $GX \longrightarrow T$ is an isomorphism. ~~and then~~

Proof. Every character φ of X induces a character $\varphi_\lambda : T[\lambda] \longrightarrow 2$ by 3.16. This yields a family $\hat{\varphi}_\lambda : S/\lambda \xrightarrow{a} 2$, and since Λ separates, there is a unique $\psi: 2 \longrightarrow S$ with $\hat{\varphi}_\lambda = (2 \xrightarrow{\psi} S \longrightarrow S/\lambda)$. The dual $\psi: T \longrightarrow 2$ is the unique character with $\hat{\psi} | T[\lambda] = \varphi_\lambda$, hence with $\hat{\psi} | X = \varphi$. Thus $GX \longrightarrow T$ is an isomorphism onto the closed subsemilattice generated by X (by 3.16). But X is generating by 2.5. Hence $GX \longrightarrow T$ is an isomorphism. □

falsch??

We note in passing how partial compact semilattices arise:

NOTE. If $(X,\leq)$ is a poset and a compact space and $G=\{(x,y,z) \in X \times X \times X : z = \min\{x,y\}\}$ is closed in $X \times X \times X$, then $(X, pr_{12}G, \min)$ is a partical compact semilattice. If X is zero dimensional, we obtain a ZP – object. Every partial subsemilattice of a Z-object T is of this form.

3.18. CONVENTION. We consider the Gierz-Keimel spectrum $X(S)$ of a lattice S (2.8) to carry the structure of a partial compact zero dimensional semilattice induced by T, the dual of S, i.e. we write $X(S) = (X(S), D_{X(S)}, \cdot)$.

3.19. THEOREM (GK). Every lattice is isomorphic to the character (semi)lattice of ~~the~~ its GK ~~spectrum~~ spectrum.

Proof. Let $S \in \underline{S}_L$ and T its dual. Then $S \cong \hat{T} \cong \widehat{X(S)}$ by 3.17. □

The deficiency of the theorem in the general situation is that it cannot be spelled out intrinsically in what sense $X(G)$ is minimal. E.g. S is isomorphic to the character semilattice $\hat{T}$ of its full character semilattice, which is perhaps too big for some people's taste. But then there may be closed generating sets $X \subseteq T$ for which $\hat{T} \longrightarrow \hat{X}$ is surjective and which are genuinely smaller than $X(S)$. But in the absence of any systematic knowledge of such generating sets, GK offer the most economic ZP-object associated with a lattice whose character(semi-)lattice is isomorphic to the given lattice. □

GK offer a sort of converse to 3.19 which emerges out of the adjoint situation 3.15.

3.20. LEMMA. a) Let $T \in \underline{Z}$ and let X be a closed generating set. We define a collection $\mathcal{T} = \mathcal{T}(X) \subseteq \mathrm{Sub}(T)$ by letting $A \in \mathcal{T}$ iff $A \times A \subseteq D_X$. Then $\mathcal{T}$ is closed in $\mathrm{Sub}(T)$ and $\bigcup \mathcal{T} \subseteq X$ is a closed subspace of T.

b) Let $T \in \underline{Z}_L$ and let X be a closed generating set. Define $\mathcal{T}_L \subseteq \mathcal{T}(X)$ by requireing $A \in \mathcal{T}_L$ iff $g_A\colon T \longrightarrow A$, the left adjoint of the inclusion, is a lattice morphism. In other words we set

$$\mathcal{T}_L = \mathrm{Sub}_L(T) \cap \mathcal{T}(X).$$

Then $\mathcal{T}_L$ is a closed collection and $X_L = \bigcup \mathcal{T}_L$ is a closed subspace of X.

Proof. Not much has to be proved here: In a) if A_j is a net on $\mathcal{T}$ converging to A in $\mathrm{Sub}(T)$ one needs to observe that $A_j \times A_j$ converges to $A \times A$ which is in D_X since D_X is closed. The rest is elementary hyperspace theory. b) is an immediate consequence in view of 1.5. []

Now let us assume that $(X,D,.) \in \underline{ZP}$. Let $GX = \hat{\hat{X}} = T \in \underline{Z}$. Assume the following postulates:

(Postulate I) $T \in \underline{Z}_L$ (equivalently $\hat{X} (= \hat{T})$ is a lattice)

(Postulate II) The characters of X separate the points.

Then we may assume $X \subseteq T$ (under β_X) by 3.15, that and X is a generating set. We form X_L according to 3.20. Then $X_L \subseteq X$.

(Postulate III) $X_L = X$ (i.e. for each $x \in X$ there is an $A \in \mathrm{Sub}_L(T)$ with $x \in A \subseteq X$).

By 3.17 we know that $\hat{T}$ may be identified with $\hat{X}$. Set $S = \hat{T} = \hat{X}$. Then $X(S) \subseteq T$; the GK-spectrum is a generating set, and by 2.9 (in view of 2.5) we know that

$X(S) \subseteq X_L = X$. This is where it stands. If one wants equality here one might just as well postulate it; this postulate amounts to something like this:

(Postulate IV) $\cup\{T[\lambda]: \lambda \in \mathrm{Irr\,Cong}_L(X)$ and $T[\lambda] \subseteq X\}$

is dense in X.

All ~~of~~ of this shows the following:

3.21. THEOREM (GK). Let $(X,D,.)$ be a partial compact zero-dimensional semilattice. If it satisfies postulates (I, II, III) then there is a lattice S (namely, $\hat{X}$) such that, upon embedding of X into T ~~XxxxS~~ $\hat{\hat{X}} \cong \hat{S}$ we have $X(S) \subseteq X$. If Postulate IV is also satisfied, then $X(S) = X$. ▯

The theorem says that ~~a~~ partial compact zero dimensional semilattice is indeed the Gierz -Keimel spectrum of a lattice provided that it is the Gierz -Keimel spectrum of a lattice (or, to put it a bit less tautologically if Postulates I, II, III, and IV are satisfied). The interest in the Theorem is that one can indeed pinpoint the necessary and sufficient conditions for X to be an X(S).

So the over-all result of the paper is that a shrewd application of ~~xxxxxxx~~ duality plus some rather original ideas allow the representation of an arbitrary lattice in terms of the character semilattice of certain compact zero dimensional partial semilattices and to what extent such a representation is typical. GK test the validity of this theory by specializing it to distributive lattices where it emerges to yield the Priestley duality and by specializing it to finite lattices where it becomes a theory developed by Wille. In the meantime what the general theory will do for arbitrary lattices is a promise for the future.