Keimel

SUMMARY on <u>GENERATION</u> IN <u>CL</u> -objects

These notes just summarize what has been achieved on the question of generation and order generation in <u>CL</u>-objects without proofs (with the exception of Theorem 11). References to earlier notes are given instead.

1. ~~XXXXXXX~~ DEFINITION. Let S be a compact semilattice, $X \subseteq S$. Then X is said to

(1) algebraically generate

(2) order generate

(3) generate

S, provided the following conditions hold:

(1) S is algebraically generated by X, i.e. for each $s \in S$ there is a finite set $F \subseteq X$ with $s = \inf F$,

(2) S is order generated as a complete lattice by X i.e. $s = \inf(\uparrow s \cap X)$ for all $s \in S$,

(3) S is generated by X as compact semilattice, i.e. S is the smallest closed subsemilattice containing X. □

Algebraic generation will not further occur in this SUMMARY, but it did occur in JDL II)

2. NOTATION. We write

IRR $S = \{ s \in S: s = uv \Rightarrow s \in \{u,v\}$ for all $u,v \in S\}$

Irr $S = \{ s \in S: s = \inf Y \Rightarrow s \in Y$ for all $Y \subseteq S\}$

PRIME $S = \{ s \in S: s \leq uv \Rightarrow s \leq u$ or $s \leq v\}$ for all $u,v \in S\}$

Prime $S = \{ s \in S: s \leq \inf Y \Rightarrow s \in \downarrow Y$ for all $Y \subseteq S\}$

So far, Prime S does not play any noticeable role. □

Trivially,

$$\begin{tikzcd} \mathrm{Irr} \arrow[r, hook] & \mathrm{IRR} \\ \mathrm{Prime} \arrow[u, hook] \arrow[r, hook] & \mathrm{PRIME} \arrow[u, hook] \end{tikzcd}$$

3. SIMPLE CHARACTERISATIONS.

a) $s \in$ IRR S iff $\uparrow \overset{s}{\cancel{x}} \setminus \{\overset{s}{\cancel{x}}\}$ is a semigroup iff $\overset{s}{\cancel{x}} \in$ PRIME $\uparrow \overset{s}{\cancel{x}}$.

b) $s \in$ PRIMES iff $S \setminus \downarrow s$ is a semigroup.

c) $s \in$ Irr S iff $s \neq \inf(\uparrow s \setminus \{s\})$ iff $\uparrow s = \{s\} \cup \uparrow \bar{s}$ with $s < \bar{s}$, iff $s \in$ Prime $\uparrow s$. □

We now specialize to the case $S \in \underline{SL}$

4. "THE LEMMA." Let $S \in \underline{CL}$. If $f: S \longrightarrow L$ is a function into a complete lattice which is continuous from below (preserves sups of up-directed nets) then for every compact subset ~~Kx~~ $K \subseteq S$ and every $p \in$ PRIME L the relation $\inf f(K) \leq p$ ~~ip~~ implies the relation $\uparrow f(K) \ni p$.

SPECIFICALLY, if $X \subseteq S$, $p \in$ PRIME S and $\inf X \leq p$ then $p \in \uparrow \bar{X}$.

[See KG Variations on Jonsson's Lemma, or H-notes on KG] ▯

5. THEOREM. Let $S \in \underline{CL}$, and $X \subseteq S$. Then the following are equival

(1) X is generating.

(2) $\bar{X}$ is order generating.

(3) IRR $T \subseteq \bar{X}$.

[See H -SP Thm 10 and JDL II Prop.2.] ▯

6. CONSEQUENCE. EACH $\underline{CL}$-object S has a unique smallest closed order generating subset, namely $\overline{\text{IRR}}$ S.

7. EXAMPLES. THere are compact (semi)lattices S with IRR $S = \{1\}$
There are compact semilattices outside $\underline{CL}$ in which IRR T is order generating.

[See JDL II p.2]

For some examples of generating but not order generating sets in free $\underline{CL}$ -objects see H- SP ,pp.8,9. ▯

8. ZUSATZ to THEOREM 5. If $S \in \underline{Z}$, then ~~are~~ (1),(2),(3) of ~~1~~ are also equivalent to

(4) Irr $X \subseteq \bar{X}$. ▯

9. CONSEQUENCE. If $S \in \underline{Z}$, then $\overline{\text{Irr}}\ S = \overline{\text{IRR}}\ S$. ▯

Note $S = [0,1]$ to see how this fails in $\underline{CL}$.

In fact there is a bit more than 8 says:

10. PROPOSITION. Let $S \in \underline{CL}$ and $X \subseteq S$. Then among the following statements ~~are equivalent:~~ We have $(0) \Rightarrow (1) \Leftrightarrow (2) \Rightarrow (3)$

(0) IRR $S \subseteq X$,

(1) X is order generating,

(2) $\uparrow s \cap X = \uparrow t \cap X$ implies $s=t$ for all $s,t \in S$,

~~These imply~~

(3) Irr $S \subseteq X$,

and if $S \in \underline{Z}$, then (1)<=>(3). [See H-on GK , H-SP] ▯

Let T be a complete lattice. If $X \subseteq T$ let X' be the set of all $t \in T$ with $t = \inf(\uparrow t \cap X)$, i.e. the set of all t for which there is some subset $Y \subseteq X$ with $t = \inf Y$. Then X' is a complete inf subsemilattice. WE say that X <u>separates the points of</u> Y if $s \notin \uparrow t$, $s,t \in Y$ implies the existence of an $x \in X$ with $s \leq x \notin \uparrow t$.

LEMMA A. If X separates the points of Y, then it separates the points of a) the inf semilattice generated by Y, b) the sup semilattice generated by Y.

falsch

Proof. a) Let $s \notin \uparrow t$, $s = s_1 \ldots s_n$, $s_j \in Y$. Then there is an $s_k \notin \uparrow t$, whence there is an $x \in X$ with $s \leq s_k \leq x \notin \uparrow t$.

b) Let $s \notin \uparrow t$, $t = t_1 \vee \ldots \vee t_n$, $t_j \in Y$. Then there is a t_k with $s \notin \uparrow t_k$. Hence there is an $x \in X$ with $s \leq x \notin \uparrow t_k \supseteq \uparrow t$. □

LEMMA B. If X separates the points of Y, then it separates the points of the sublattice L generated by Y in T.

Proof. By induction from LEMMA A. □

LEMMA C. LEt T be a complete lattice and L the sublattice generated algebraically by X= $\{t \in T: t = \inf(\uparrow t \cap \mathrm{PRIME}\ T)\}$. Then L is distributive.

Proof. Firstly, PRIME T spearates the points of X. Hence, by LEMMA B, PRIME T separates the points of L. If $p \in \mathrm{PRIME}\ T$, then the function $f: T \longrightarrow 2$ given by $f^{-1}(0) = \downarrow p$ is a lattice morphism (preserving arbitrary sups). Thus there is a lattice morphism of T into a distributive lattice whose restriction to L is injective. Hence L is distributive. □

LEMMA D. Let T be a compact semilattice, let L be as in LEMMA C, and let T^* be the set of all $\sup Y$, $Y \subseteq L$. Then T^* is a (sup-complete) distributive sublattice.

Proof. Let $u,v,w \in T^*$. Then (u,v,w) is the limit of an up-directed net $(u_j,v_j,w_j) \in L \times L \times L$ (since L is a sup-semilattice). Then $uw = \lim u_j w_j$, whence $uw \in T^*$, thus T^* is a sublattice (closed under arbitrary sups). Now $u \vee v = \lim (u_j \vee v_j)$ because the net is up-directed. Hence $(u \vee v)w = \lim (u_j \vee v_j) w_j = \lim (u_j w_j \vee v_j w_j) = uw \vee vw$. □

LEMMA E. Let $T \in \underline{CL}$ and suppose that PRIME T is generating. Then T is distributive.

Proof. By H- SP 1.8 we have $A(T) \subseteq T^*$. Since $t = \sup(\downarrow t \cap A(T))$ for all $t \in T$, we have $T = T^*$. Lemma D then finishes the proof. □

This allows us to formulate the following conclusive theorem on distributivity in $\underline{CL}$.

11. THEOREM. LEt $T \subseteq \underline{CL}$. Then the following statemens are equivalent:

(1) T is distributive.

(2) T is Brouwerien.

(3) PRIME T is order generating.

Falsch (4) IRR T = PRIME T.

(5) $\overline{\text{IRR}}\ T = \overline{\text{PRIME}}\ T$. (5') IRR $T \subseteq \overline{\text{PRIME}}\ T$.

(6) PRIME T is generating.

Proof. (1) though (4) are known to be equivalent by H-SP . ~~(4)<=>~~ (4)=>(5) <=> (5') is trivial, and (6) <=>(5') follows from Theorem 5 above . (6) =>(1) if Lemma E. □

Recalls that in the case that $T \in \underline{Z}$ we know that these conditions are also equivalent to

(7) The chacrates semilatice of T is a distributive semilattice (in the sense of HMS -DUALITY (Grätzer)).

~~The proof of Theorem 11 has actually shown a bit more namely how a distributive.~~