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MORE NOTES ON SPREAD

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Notation and references consistent with "NOTES ON NOTES BY JDL". That work is referenced as H-SP (Hofmann on "Spread"). JDL should perhaps now become JDL I

LEMMA 1. Let X be an order generating set in a $\mathcal{CL}$-object T. Then

$$\mathrm{IRR}\ T \subset \overline{X}$$

Proof. Let $y \in \mathrm{IRR}\ T$, $Y = X \cap \uparrow y$. By hypothesis $y = \inf Y$. Since y is prime in the $\mathcal{CL}$-object $\uparrow y$, by "THE LEMMA", $y \in \overline{Y} \subset \overline{X}$.

THEOREM
PROP. 2. Let X be a subset of a $\mathcal{CL}$-object T. TAE

(1) $\overline{X}$ is generating
(2) $\overline{X}$ is order-generating
(3) $\mathrm{IRR}\ T \subset \overline{X}$

Proof. (1) $\Leftrightarrow$ (2) by Thm. 1.10 [H-SP], and (3) $\Rightarrow$ (2) by Prop 1.4 [H-SP]; (2) $\Rightarrow$ (3) by Lemma 1.

COROLLARY 3. $\overline{\mathrm{IRR}\ T}$ is the unique smallest closed (order) generating set if $T \in \mathcal{CL}$

A set X (in a semilattice S) is algebraically generating if S is the smallest subsemilattice containing X.

Prop.2 and Corollary 3 really wrap up the question of generation and order generation in $\underline{CL}$-objects. Also the ASSERTION on the next page settles my 1 quiry on p. 5 of (H-Sp). There remains the question: $\overline{\mathrm{Irr}}\ T = \overline{\mathrm{IRR}}\ T$ for $T \in \underline{Z}$? Be sure to understand tha. 1.12, p.4 of (H-Sp) is superseded by Corr.3 Here.
KHH

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PROP. 4. Let $T \in \mathcal{CL}$ and let $X = \bigcup_{i=1}^{n} T_i$

where each T_i is a subsemilattice containing 1 of T

and $n < \infty$. TAE

(1) $\overline{X}$ is generating

(2) $\overline{X}$ is algebraically generating

(3) $\exists$ a continuous epimorphism $m: \prod_{i=1}^{n} T_i \twoheadrightarrow T$ defined from the multiplication mapping.

Proof. (3) $\Rightarrow$ (2) Immediate since $\overline{X} = \overline{T_1} \cup \ldots \cup \overline{T_n}$

(2) $\Rightarrow$ (1) Alg. gen. $\Rightarrow$ Ord. gen. Now apply Thm. 1.10 [H-CP]

(1) $\Rightarrow$ (3) The image of m is a compact subsemilattice containing X. Hence the image is T.

ASSERTION. In [Lawson, Lattices with no interval homomorphisms] Example 1 has IRR T order generating (hence IRR T order generating $\not\Rightarrow$ $T \in \mathcal{CL}$). In Example 3 IRR T is not order generating. In fact it can be argued that this example contains a compact sublattice L for which IRR $L = \{1\}$. Hence Prop. 1.4 [H-SP] does not extend to compact semilattices.

We adopt Def. 2.8 of [H-CP] for the "spread" of a $\mathcal{CL}$-object. If $T \in \mathcal{CL}$, one formulation for spread is that $SP(T)$ is the least cardinal α for which there exists α chains in T whose union is a generating set

THEOREM 5 Let $T \in \mathcal{CL}$ and let n be a positive integer. TAE.

(1) $SP(T) \le n$;

(2) T is generated by a union of n chains;

(3) T is order-generated by a union of n closed chains;

(4) T is algebraically generated by a union of n closed chains;

(5) $\overline{\mathrm{IRR}\, T} \subset \bigcup_{i=1}^{n} C_i$ where each C_i is a compact chain;

(6) $\mathrm{IRR}\, T$ has width $\le n$ (Width $\equiv$ largest cardinality of an anti-chain)

(7) $\exists$ a continuous homomorphism $h : \prod_{i=1}^{n} K_i \twoheadrightarrow T$ where, each K_i is a compact chain and $\mathrm{Im}(h) = T$.

(8) Embedding dimension $(T, \vee) \le n$ (in category $\mathcal{S}$).

(9) If $\hat{T}$ is the dual of $(T_d, \vee)$ {i.e., the lattice of ideals of $(T, \vee)$}, then $\hat{T}$ as a $\mathcal{Z}$-object has $SP(\hat{T}) \le n$.

<u>Proof</u> (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows from definitions of spread and Lemma 2.7 [H-SP].

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) follows from Prop. 4.

4 $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) follows from Prop. 2.

(6) $\Rightarrow$ (2) by Dilworth's theorem (width $X \le n \Rightarrow X = \bigcup_{i=1}^{n} C_i$) and Prop. 1.7 [H-SP].

(5) $\Rightarrow$ (6) Immediate.

~~(1) $\Rightarrow$ (8) By Lemma 2.2 [H-SP] $\exists$~~

~~each K_i is a compact chain and $\hat{h}$ is~~

(7) $\Rightarrow$ (8) Take $\hat{h}: T \to \prod_{i=1}^{n} K_i$ where $\hat{h}(t) = \inf h^{-1}(t)$.

Then $\hat{h}$ is one-to-one and preserves sups, hence is an embedding of $(T, \vee)$.

(8) $\Rightarrow$ (9) By HMS duality $\exists\, h': \prod_{i=1}^{n} \hat{K}_i \left(= \coprod_{i=1}^{n} \hat{K}_i\right) \twoheadrightarrow \hat{T}$. K_i a chain $\Rightarrow \hat{K}_i$ is a compact chain. Since (7) $\Leftrightarrow$ (1), $Sp(\hat{T}) \leq n$.

(9) $\Rightarrow$ (7) Again since (1) $\Leftrightarrow$ (7), $\exists$ compact chains $C_1, \ldots, C_n$ and a continuous homomorphism $h: \prod_{i=1}^{n} C_i \twoheadrightarrow \hat{T}$ which is onto. By ATLAS $\exists\, p: \hat{T} \twoheadrightarrow T$, a continuous homomorphism of $\mathcal{CL}$-objects. The composition $p \circ h$ gives (7).

<u>COROLLARY 6</u> Let $T \in \mathcal{CL}$. Then

$$SP(T) = \text{Width}(\text{IRR}\, T) = \text{Embedding dim.}\,(T, \vee) = SP\,\widehat{(T,\vee)}$$

if any of these quantities is finite.

<u>COROLLARY 7</u> Let $T \in \mathcal{CL}$ and suppose T is distributive. Then $SP(T) = Br(T)$.

<u>Proof.</u> $SP(T) \leq n \Rightarrow Br(T) \leq n$ by (7) of Thm. 5 since a homomorphism cannot raise breadth. Hence $Br(T) \leq SP(T)$ always.

Conversely in the distributive case if $Br(T) \leq n$, it is not difficult to show (see (8) of JDL I or Lemma 3.2 of H-SP) that Width (PRIME T) $\leq n$. Since PRIME T = IRR T, $SP(T) \leq n$ by (6) of Thm 5. Hence $SP(T) \leq Br(T)$.

Jim Lea writes that he and K. Baker considered some related notions (mainly for distributive lattices). I summarize their definitions and Baker's results. (I do not have the proofs). Let $(L, \wedge, \vee)$ be a lattice.

(1) order dimension $(L) \equiv$ The least cardinal number of linear order relations on L whose intersection, as a subset of $L \times L$ is the given order relation on L = least cardinal for which L as a poset can be embedded in product of chains of that cardinality

(2) Embedding dimension$(L) \equiv$ The least cardinal of chains in which $(L, \vee)$ can be embedding as a subsemilattice (or a sublattice if L is distributive).

(3) Breadth (L)

(4) $\mathrm{Cov}(L)$, the covering number of $L \equiv$ Greatest number of elements which cover any single element (b covers a if $[a,b] = \{a, b\}$)

(5) width (IRR L)

(6) width $(\pi(L)) \equiv$ width of prime ideal space = width of PRIME $(L, \vee)$

~~(Check for duals?)~~

(7) subcubic dimension $(L) \equiv$ sup of all $n \ni L$ contains a sublattice isomorphic to 2^n

(8) orthodimension $(L) \equiv$ sup of all $n \ni \exists$ n distinct elements $x_1, \ldots, x_n$ and an element $c \ni x_i \wedge x_j = c$ for $i \neq j$.

(9) Width (Join irreducibles).

Results:

① For distributive lattices (1), (2), (3), (7), (8) are equivalent if all are finite

② All are equivalent for finite distributive lattices and apparently (except for 6) for dist. $\mathfrak{Z}$-objects (if all are finite).